# QUARK MASSES AND MASSIVE TADPOLES

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SFB TR9

# QUARK MASSES AND MASSIVE TADPOLES

from relativistic 4 loop moments

1. Why

2. Theory

3. Results, from experiment and from lattice

in collaboration with

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# 1. WHY precise masses?

## B-decays:

$\Gamma(B \to X_u l \bar{\nu}) \sim G_F^2 \, m_b^5 \, |V_{ub}|^2$

$\Gamma(B \to X_c l \bar{\nu}) \sim G_F^2 \, m_b^5 \, f(m_c^2/m_b^2) \, |V_{cb}|^2$

$B \to X_s \gamma$

## comparison with $\Upsilon$-spectroscopy:

$M\,(\Upsilon(1s)) = 2M_b - \left(\frac{4}{3}\alpha_s\right)^2 \frac{M_b}{4} + ... +$ excitations

(Penin & Zerf,... $\delta m_b \sim 9$ MeV)

# H decay (ILC, TLEP)

$H \to b\bar{b}$ dominant decay mode, all branching ratios are affected!

status: $\Gamma_b = \frac{G_F M_H^2}{4\sqrt{2}\pi} m_b^2(M_H) R^S(M_H)$

$$\begin{aligned} R^S(M_H) &= 1 + 5.667\left(\frac{\alpha_s}{\pi}\right) + 29.147\left(\frac{\alpha_s}{\pi}\right)^2 + 41.758\left(\frac{\alpha_s}{\pi}\right)^3 - 825.7\left(\frac{\alpha_s}{\pi}\right)^4 \\ &= 1 + 0.19551 + 0.03469 + 0.00171 - 0.00117 \Lsh \qquad \uparrow \end{aligned}$$

(Chetyrkin, Baikov, JK, 2006)

Theory uncertainty $(M_H/3 < \mu < 3M_H)$ : 5‰ (four loop)
reduced to 1.5‰ (five loop)

present uncertainties from $m_b$

$m_b(10 \text{ GeV}) = 3610 - \frac{\alpha_s - 0.1189}{0.002} 12 \pm 11$ MeV (Karlsruhe, arXiv:0907.2110)

running from 10 GeV to $M_H$ depends on
anomalous mass dimension, $\beta$-function and $\alpha_s$

$m_b(M_H) = 2759 \pm 8|_{m_b} \pm 27|_{\alpha_s}$ MeV aim $\pm 4$ MeV ($\widehat{=} 1.5 \times 10^{-3}$)

$\gamma_4$ (five loop): Baikov + Chetyrkin, 2013

$\beta_4$ under construction

$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.4 \times 10^{-4} (b_4 = 0) \mid -4.3 \times 10^{-4} (b_4 = 100) \mid -7.3 \times 10^{-4} (b_4 = 200)$

to be compared with $\delta\Gamma/\Gamma = 2.0 \times 10^{-3}$ (TLEP)

# Yukawa Unification

$\lambda_\tau \sim \lambda_\mathrm{b}$ or $\lambda_\tau \sim \lambda_\mathrm{b} \sim \lambda_\mathrm{t}$ at GUT scale

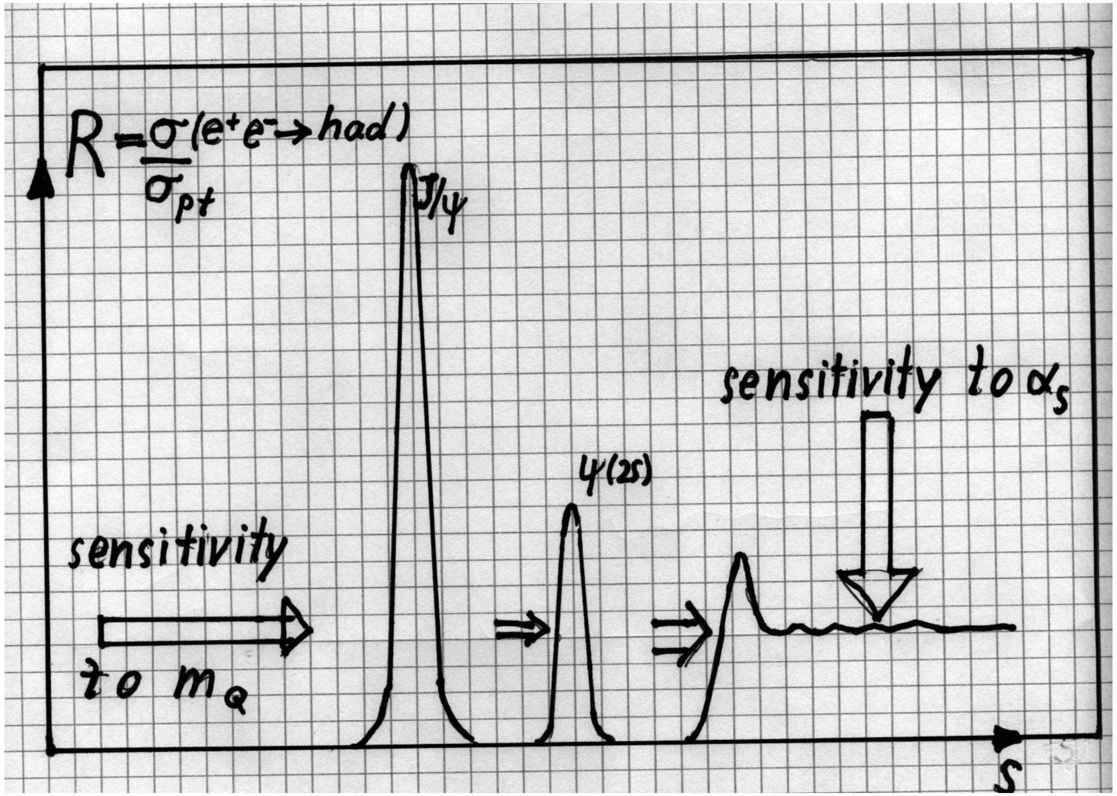
top-bottom $\rightarrow m_\mathrm{t} \big/ m_\mathrm{b} \sim$ ratio of vacuum expectation values

request $\frac{\delta m_\mathrm{b}}{m_\mathrm{b}} \sim \frac{\delta m_\mathrm{t}}{m_\mathrm{t}} \Rightarrow \delta m_\mathrm{t} \approx 0.5\,\mathrm{GeV} \Rightarrow \delta m_\mathrm{b} \approx 12\,\mathrm{MeV}$

# 2. Theory

## $m_Q$ from SVZ Sum Rules, Moments and Tadpoles

### Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu\, q_\nu\right)\, \Pi(q^2) \;\equiv\; i \int \mathrm{d}x\, e^{iqx} \langle T j_\mu(x) j_\nu(0)\rangle$$

with the electromagnetic current $j_\mu$.

$$R(s) \;=\; 12\pi\, \mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

Taylor expansion: $\Pi_Q(q^2) \;=\; Q_Q^2\, \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n\, z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi} \bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \ldots$$

generic form

$$
\begin{aligned}
\bar{C}_n = \; & \bar{C}_n^{(0)} \\
& + \frac{\alpha_s}{\pi}\left( \bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c} \right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \left( \bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2 \right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 \left( \bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3 \right) \\
& + \ldots
\end{aligned}
$$

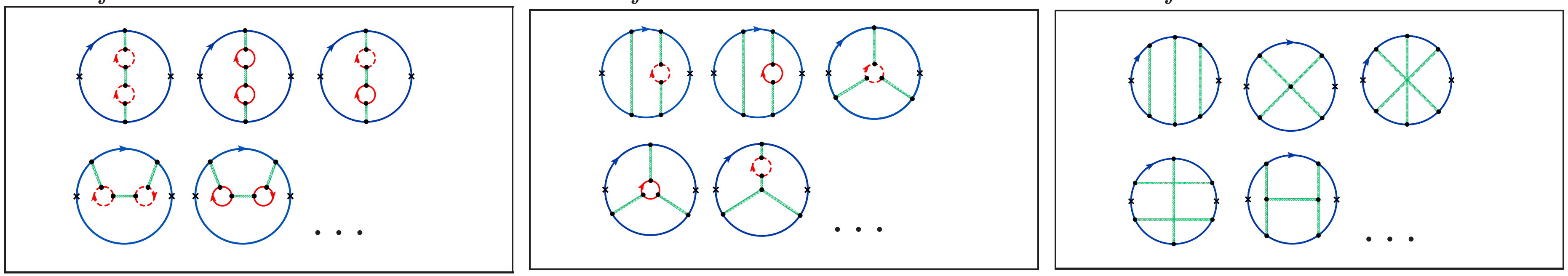
## Analysis in NNLO

- FORM program MATAD
  Coefficients $\bar{C}_n$ up to $n = 8$
  (also for axial, scalar and pseudoscalar correlators)
  (Chetyrkin, JK, Steinhauser, 1996)

# Analysis in N$^3$LO

Algebraic reduction to 13 master integrals (Laporta algorithm);
numerical and analytical evaluation of master integrals

$n_f^2$-contributions | $n_f^1$-contributions | $n_f^0$-contributions




 : heavy quarks,  : light quarks,

$n_f$: number of active quarks

$\Rightarrow$ About 700 Feynman-diagrams

➪ Reduction to master integrals

| $\bar{C}_0$ and $\bar{C}_1$ in order $\alpha_s^3$ (four loops!) |
|---|

Program "Sturman" (Sturm) (2006)

(Chetyrkin, JK, Sturm; Boughezal, Czakon, Schutzmeier)

$\bar{C}_2$ and $\bar{C}_3$ (2008) Program "Crusher", Marquard & Seidel

(Maier, Maierhöfer, Marquard, A. Smirnov)

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser, Laporta, Broadhurst, Kniehl et al.)

$\bar{C}_4 - \bar{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard, 2009)

(Also: $q^2$-dependence of scalar, vector,... correlator)

## Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\Big|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$$

Perturbation theory: $\bar{C}_n$ is function of $\alpha_s$ and $\ln\frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s\, \frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

➪ $\mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

➪ $m_c$

# 3.a) Results from Experiment

## Ingredients (charm)

**experiment:**

- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & BABAR (PDG)
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

**theory:**

- N$^3$LO for $n = 1, 2, 3, 4$
- include condensates

$$\delta\mathcal{M}_n^{\rm np} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi}\bar{b}_n\right)$$

- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of $R_{uds}$
- careful definition of $R_c$

## **Results** ($m_c$)  (2009)

Error budget

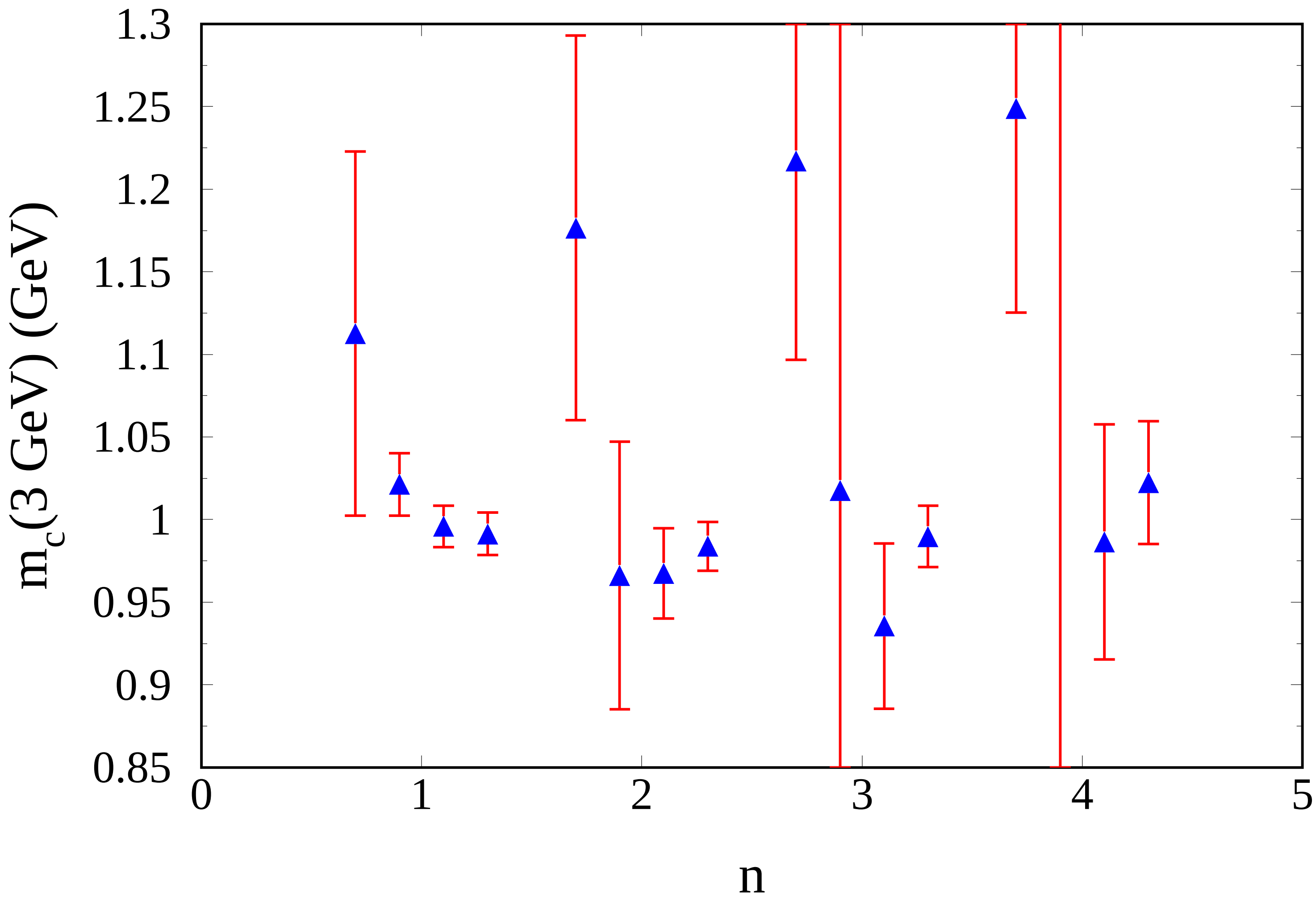
| $n$ | $m_c(3\,\mathrm{GeV})$ | exp | $\alpha_s$ | $\mu$ | np | total |
|---|---|---|---|---|---|---|
| 1 | 986 | 9 | 9 | 2 | 1 | 13 |
| 2 | 976 | 6 | 14 | 5 | 0 | 16 |
| 3 | 978 | 5 | 15 | 7 | 2 | 17 |
| 4 | 1004 | 3 | 9 | 31 | 7 | 33 |

Remarkable consistency between $n = 1, 2, 3, 4$
and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);
prefered scale: $\mu = 3\,\mathrm{GeV}$,

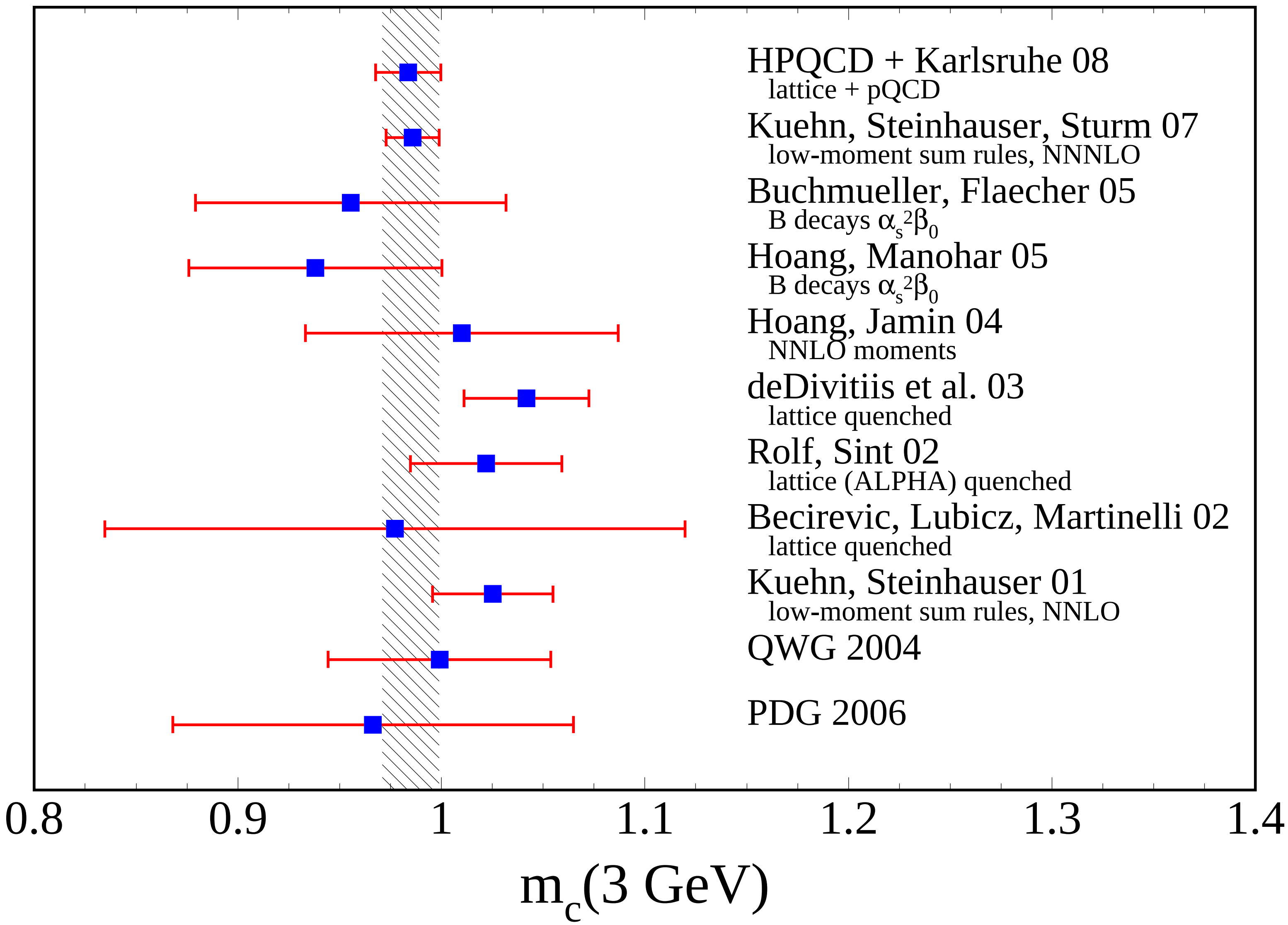
- $m_c(3\,\mathrm{GeV}) = 986 \pm 13\,\mathrm{MeV}$

conversion to $m_c(m_c)$:

- $m_c(m_c) = 1279 \pm 13\,\mathrm{MeV}$




dependence of $m_c$ on number of moment $n$ and on $\mathcal{O}(\alpha_s^i)$ for $i = 0, \ldots, 3$

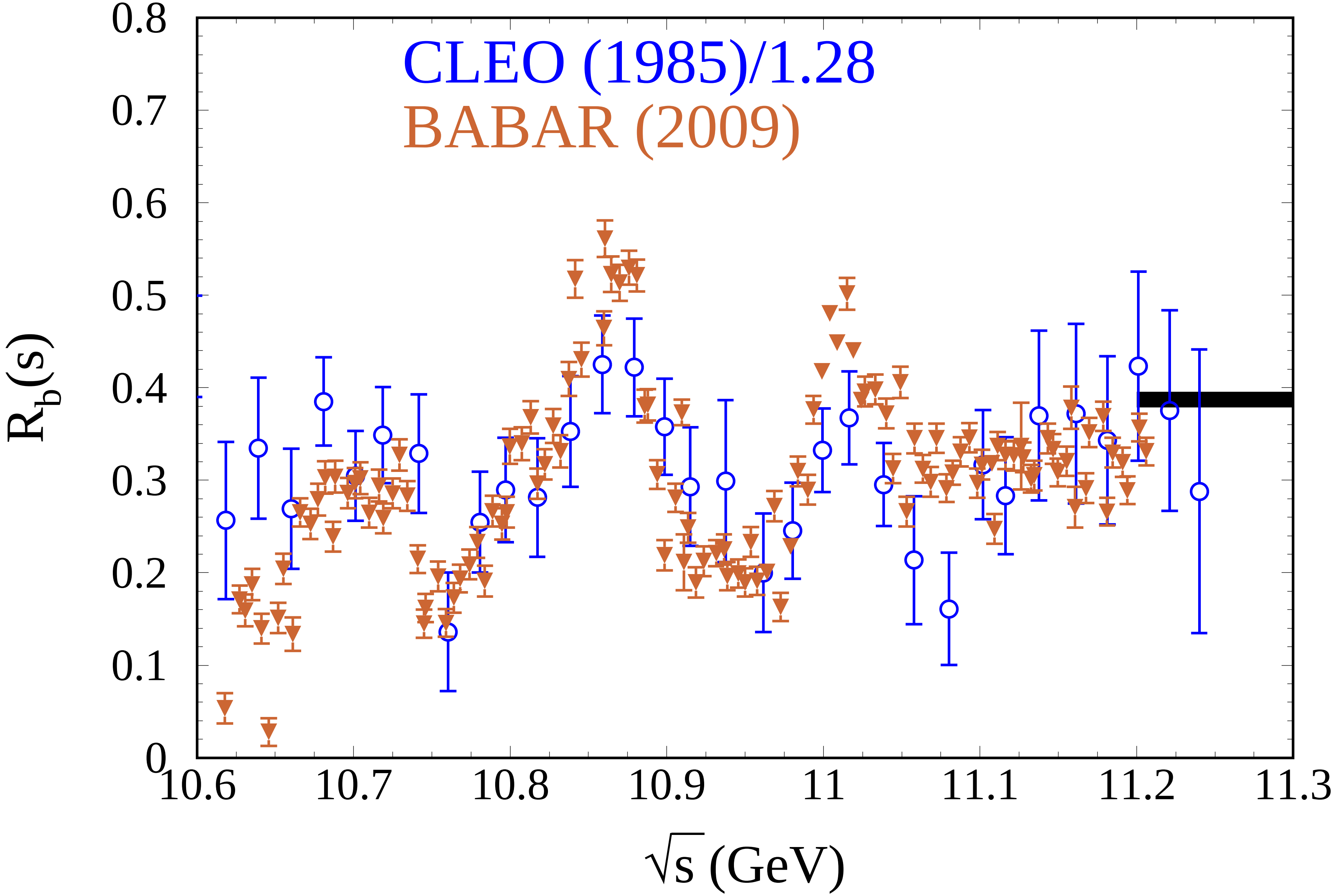

HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2006
0.8
0.9
1
1.1
1.2
1.3
1.4
$m_c$(3 GeV)

## Experimental Ingredients for $m_b$

Contributions from

- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$) (PDG)
- threshold region (10.618 GeV − 11.2 GeV) (BABAR 2009)
- perturbative continuum ($E \geq 11.2$ GeV) (Theory)
- different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$

| $n$ | $\mathcal{M}_n^{\text{res,(1S−4S)}}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$ | $\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$ |
|---|---|---|---|---|
| 1 | 1.394(23) | 0.287(12) | 2.911(18) | 4.592(31) |
| 2 | 1.459(23) | 0.240(10) | 1.173(11) | 2.872(28) |
| 3 | 1.538(24) | 0.200(8) | 0.624(7) | 2.362(26) |
| 4 | 1.630(25) | 0.168(7) | 0.372(5) | 2.170(26) |

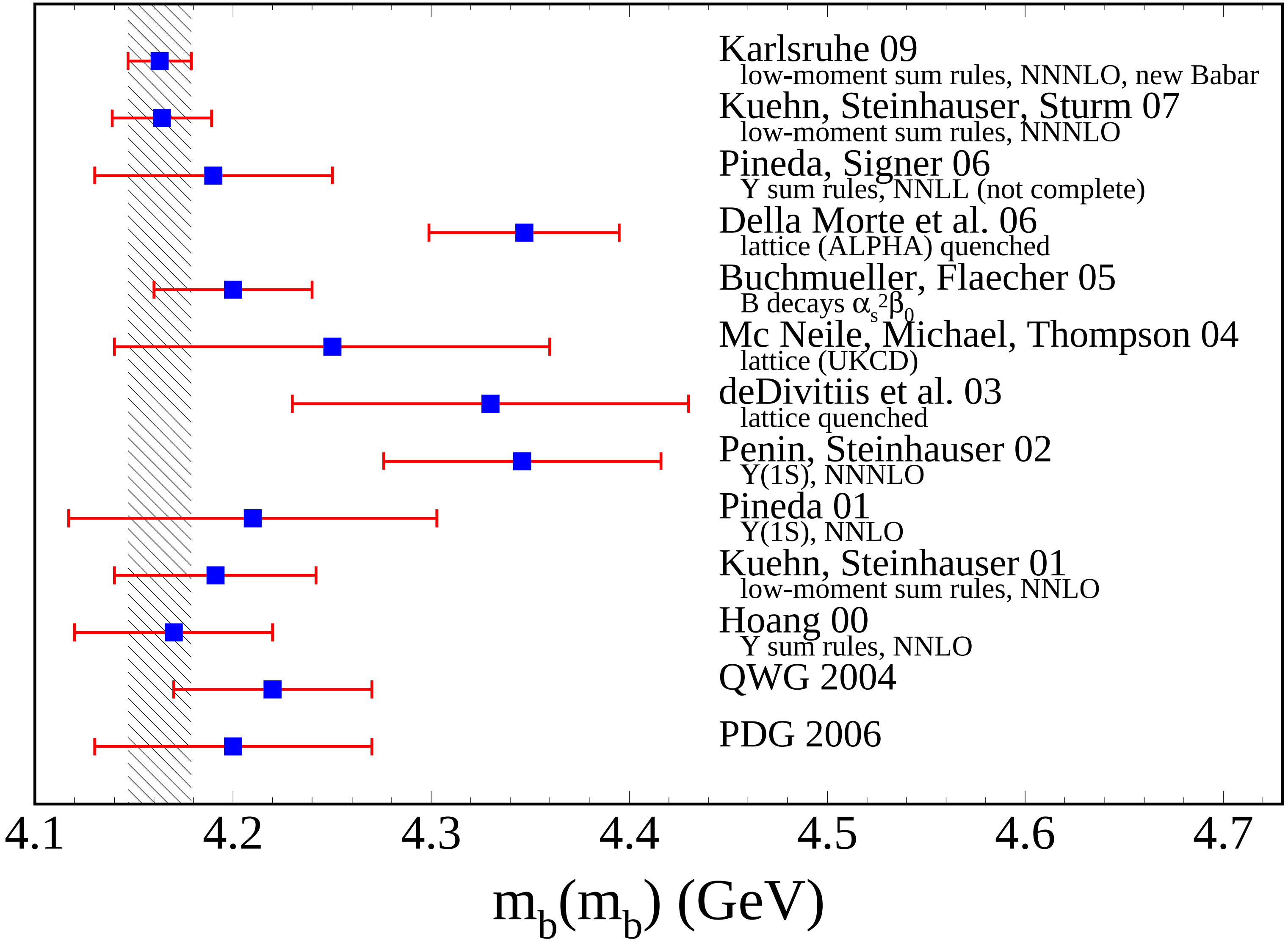

CLEO (1985)/1.28
BABAR (2009)
$R_b(s)$
$\sqrt{s}$ (GeV)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3

| $n$ | $m_b(10\,\text{GeV})$ | exp | $\alpha_s$ | $\mu$ | total | $m_b(m_b)$ |
|---|---|---|---|---|---|---|
| 1 | 3597 | 14 | 7 | 2 | 16 | 4151 |
| 2 | 3610 | 10 | 12 | 3 | 16 | 4163 |
| 3 | 3619 | 8 | 14 | 6 | 18 | 4172 |
| 4 | 3631 | 6 | 15 | 20 | 26 | 4183 |

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

- $m_\text{b}(10\,\text{GeV}) = 3610 \pm 16\,\text{MeV}$
- $m_\text{b}(m_\text{b}) = 4163 \pm 16\,\text{MeV}$

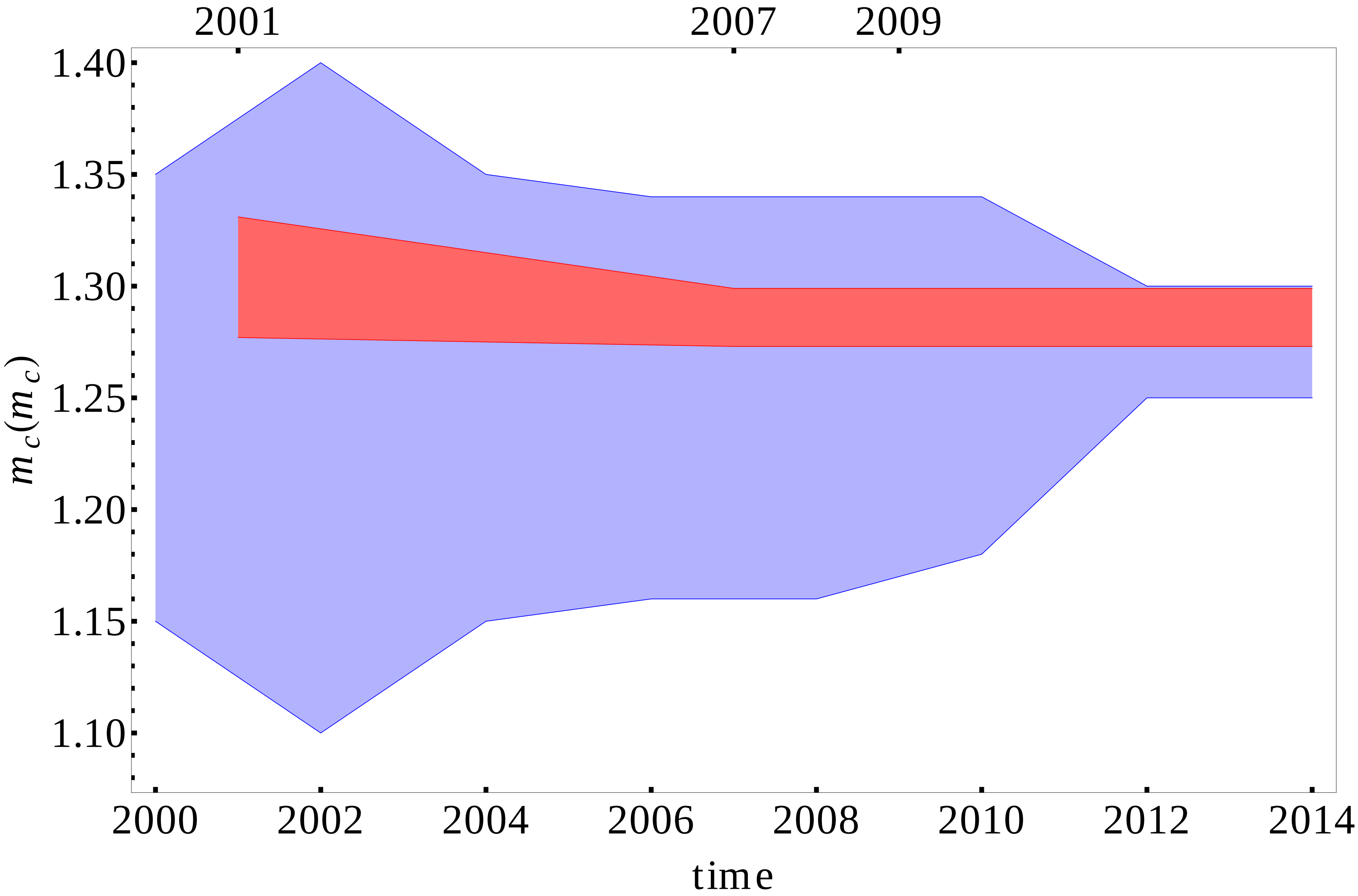
well consistent with KSS 2007


Karlsruhe 09
low-moment sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Pineda, Signer 06
Υ sum rules, NNLL (not complete)
Della Morte et al. 06
lattice (ALPHA) quenched
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Mc Neile, Michael, Thompson 04
lattice (UKCD)
deDivitiis et al. 03
lattice quenched
Penin, Steinhauser 02
Υ(1S), NNNLO
Pineda 01
Υ(1S), NNLO
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
Hoang 00
Υ sum rules, NNLO
QWG 2004
PDG 2006
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ (GeV)

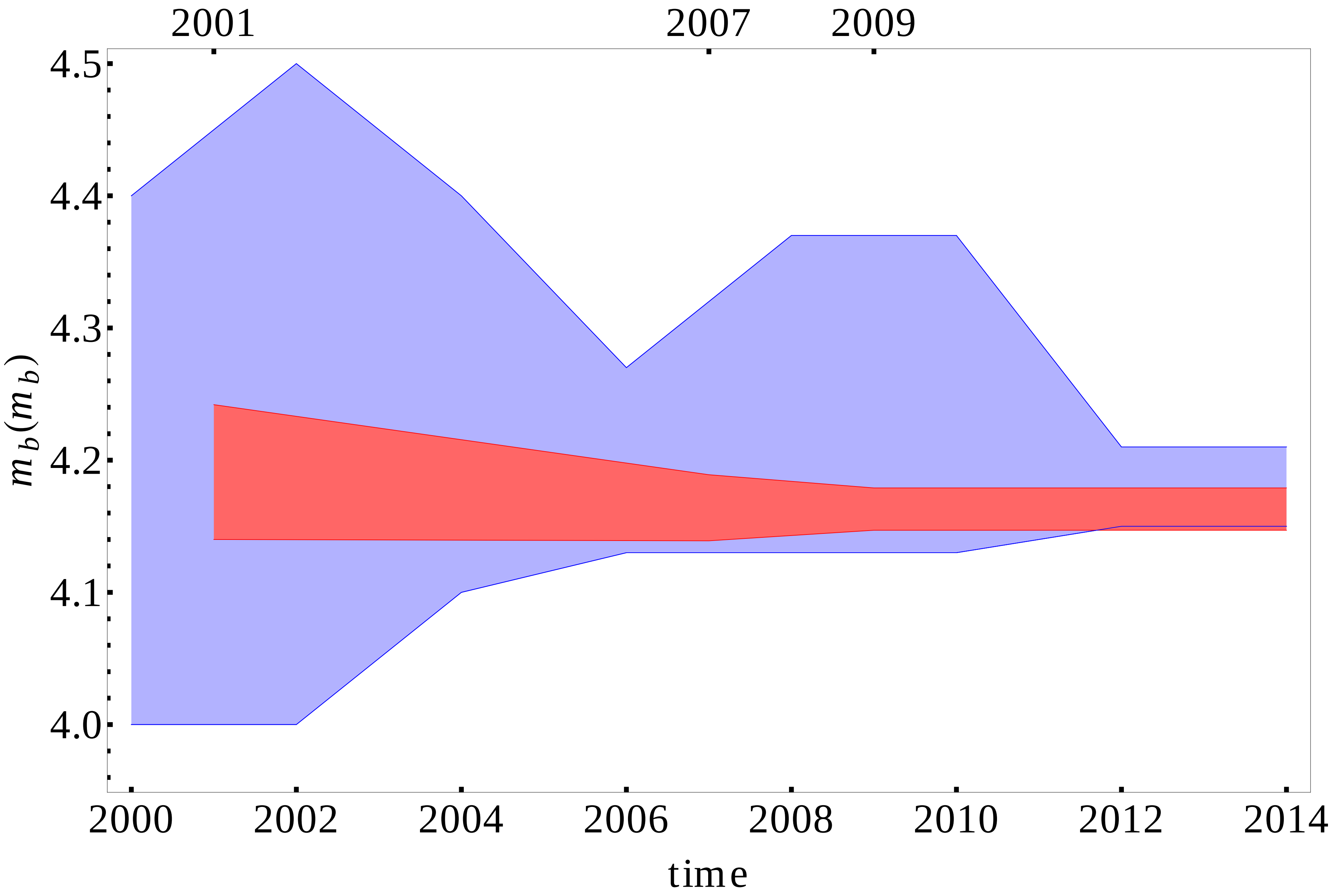
## Time evolution:

results from 2001, 2007 and 2009 internally consistent, driving the PDG result

| | $m_c(m_c)$ | $m_b(m_b)$ |
|---|---|---|
| PDG 2000 | $1.15 - 1.35$ GeV | $4.0 - 4.4$ GeV |
| sum rules 2001 | $1304 \pm 27$ MeV | $4191 \pm 51$ MeV |
| sum rules 2007 | $1286 \pm 13$ MeV | $4164 \pm 25$ MeV |
| sum rules 2009 | $1286 \pm 13$ MeV | $4163 \pm 16$ MeV |
| PDG 2014 | $1275 \pm 25$ MeV | $4180 \pm 30$ MeV |


2001
2007
2009
1.40
1.35
1.30
1.25
1.20
1.15
1.10
$m_c(m_c)$
2000
2002
2004
2006
2008
2010
2012
2014
time

oder


2001
2007
2009
4.5
4.4
4.3
4.2
4.1
4.0
$m_b(m_b)$
2000
2002
2004
2006
2008
2010
2012
2014
time

oder

# 3.b) Results from Lattice

lattice & pQCD (HPQCD + SFB/A1)

lattice evaluation of pseudoscalar correlator

$\Rightarrow$ replace experimental moments by lattice simulation

input: $M(\eta_c) \hat{=} m_c\,,\quad M(\Upsilon(1S)) - M(\Upsilon(2S)) \hat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

"all" moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

the lowest moment is dimensionless

$\Rightarrow$ $\alpha_{\mathsf{s}}(3\mathrm{GeV}) \Rightarrow \alpha_{\mathsf{s}}(M_Z) = 0.1174(12)$

higher moments: $\sim m_{\mathsf{c}}^2 \times \left(1 + ...\frac{\alpha_{\mathsf{s}}}{\pi}...\right)$,

$\Rightarrow$ $m_{\mathsf{c}}(3\mathrm{GeV}) = 986(10)$ MeV
to be compared with 986(13) MeV from $e^+e^-$ !

update: HPQCD 2010

$\alpha_{\mathsf{s}}(3\mathrm{GeV}) \Rightarrow \alpha_{\mathsf{s}}(M_Z) = 0.1183(7)$

$m_{\mathsf{c}}(3\mathrm{GeV}) = 986(6)$ MeV

$m_{\mathsf{b}}(10\mathrm{GeV}) = 3617(25)$ MeV

# SUMMARY

best determinations of $m_c(3\text{ GeV})$

986(13) MeV low moments; $e^+e^-$ data SFB/A1

986(6) MeV low moments; lattice SFB/A1+HPQCD

combined 986(5) MeV

best determinations of $m_b(10\text{ GeV})$

3610(16) MeV low moments; $e^+e^-$ data

3617(25) MeV HPQCD

$m_b(10\text{ GeV}) = 3621(9)$ MeV non relativistic sum rules, Penin, 2014

combined 3618(7) MeV

further improvements needed and possible