Three-loop static octet potential

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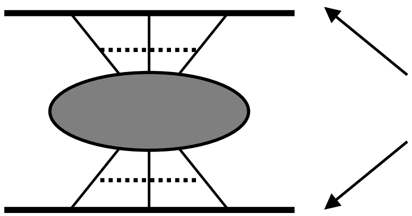
BASED ON "COLOR OCTET POTENTIAL TO THREE LOOPS", PHYS. REV. D 88, 054030 (2013)
IN COLLABORATION WITH M. PRAUSA, A. V. SMIRNOV, V. A. SMIRNOV, AND M. STEINHAUSER

WED. NOV. 12TH 2014

Calculation of static potential

- Static Potential: The energy of $q\bar{q}$ state
  - Essential to study heavy quarkonium : an important input parameter
  - Can be used to determine strong coupling constant accurately
  - Necessary for elucidating quark-antiquark interaction
- In perturbative QCD, static potential has been studies since late ‘70s
- In 2009, 3loop correction in singlet color configuration was calculated.
  - A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. **104** 112002 (2010)
  - C. Anzai, Y. Kiyo and Y. Sumino, Phys. Rev. Lett. **104** 112003 (2010)
- Color octet configuration case was calculated up to 2loop order.
  - B. A. Kniehl, A. A. Penin, Y. Schroder, V. A. Smirnov and M. Steinhauser, Phys. Lett. **B 607** 96 (2005)
  - T. Collet and M. Steinhauser, Phys. Lett. **B 704** 163 (2011)
- We have computed 3loop correction in color octet configuration.

Definition of static potential



$SU(N_c = 3)$ fundamental repl. $\mathbf{3} \times \overline{\mathbf{3}} = \mathbf{1} + \mathbf{8}$

The ellipse includes interaction among gluons and light quarks (n_l flavor)

We define static potential in representation c as

$$V^{[c]}(|\vec{q}|) = -4\pi C^{[c]} \frac{\alpha_s(|\vec{q}|)}{\vec{q}^2} \left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1^{[c]} + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2^{[c]} + \right.$$

Plan of talk

- Calculation of $a_3^{[8]}$
  - Pinch Singularity
  - Well-defined expression for diagrams with pinch singularities
  - Two different method for regularization
- Results for $a_3^{[8]}$
  - Result for octet case
  - Numerical Evaluation
  - Analytical structure of $\delta a_3^{[8]}$
- Conclusion

Calculation of $a_3^{[8]}$

Point of calculation

- We already computed 3loop potential in singlet case.
- Large parts of computations are not needed because we can use the result of previous work.
- Treatment of pinched diagrams are more complicated than singlet case.
- We discussed two different algorithm to resolve this problem.

Point of calculation

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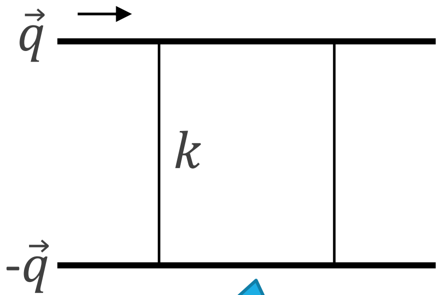
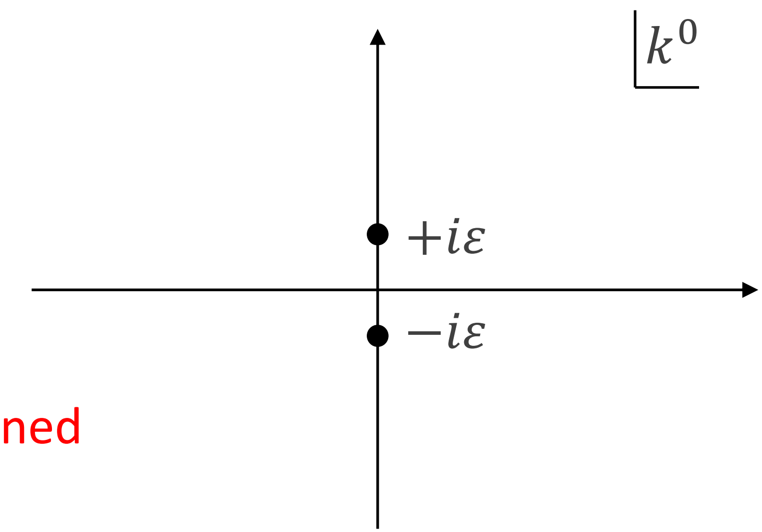
Pinch Singularities



$$= \int \frac{d^D k}{(4\pi)^D} f(k,\vec{q}) \frac{1}{k_0 + i\varepsilon} \frac{1}{k_0 - i\varepsilon}$$ (ill-defined)

HQ propagator

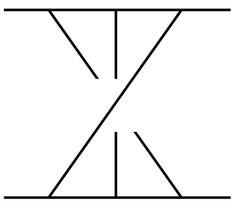
$$\frac{1}{(k-q)_0 + i\varepsilon} = \frac{1}{k_0 + i\varepsilon}$$



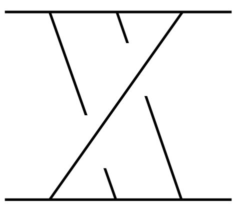
when $\varepsilon \to 0$, contour is ill-defined

Singlet Case

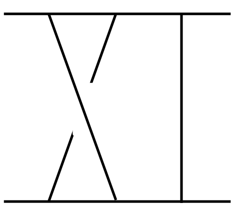
In singlet case, extraction of max non-abelian term is enough to eliminate pinch singularities and iteration terms from lower order potentials. e. g.



$$\propto \frac{1}{N_c C_F}\mathrm{Tr}_F\left(T_F^a T_F^b T_F^c T_F^a T_F^b T_F^c\right) = \frac{C_A^2}{2} - \frac{3}{2}C_A C_F + C_F^2$$



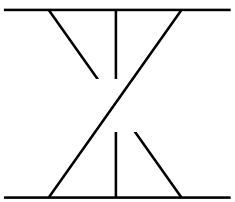
$$\propto \frac{1}{N_c C_F}\mathrm{Tr}_F\left(T_F^a T_F^b T_F^c T_F^b T_F^a T_F^c\right) = \frac{C_A^2}{4} - C_A C_F + C_F^2$$



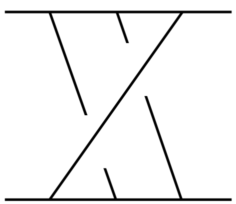
$$\propto \frac{1}{N_c C_F}\mathrm{Tr}_F\left(T_F^a T_F^b T_F^c T_F^c T_F^a T_F^b\right) = -\frac{1}{2}C_A C_F + C_F^2$$

Singlet Case

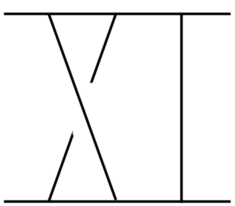
In singlet case, extraction of max non-abelian term is enough to eliminate pinch singularities and iteration terms from lower order potentials. e. g.



$$\propto \frac{C_A^2}{2} - \frac{3}{2} C_A C_F + C_F^2 \rightarrow \frac{C_A^2}{2}$$



$$\propto \frac{C_A^2}{4} - C_A C_F + C_F^2 \rightarrow \frac{C_A^2}{4}$$



$$\propto -\frac{1}{2} C_A C_F + C_F^2 \rightarrow 0$$

QED fact (not explained here)

↓

terms proportional to C_F vanish in total (singlet case)

= extract $C_A^{(loop\ order)}$ term

In octet case, this fact is not useful because $C_F \rightarrow C_F - \frac{1}{2} C_A$

A method for regularization of pinch singularities in octet case

- In singlet case, regularization of pinch singularities and exclusion of iterating terms can be performed with the Gatheral's method.
  - J. G. M. Gatheral, Phys. Lett. **B 133**, 90 (1983)
  - J. Frenkel and J. C. Taylor, Nucl. Phys. **B 246**, 231 (1984)
- With this method, color factors of non pinched diagrams are changed in such a way that the pinch contribution is taken into account.
  - For pinched diagrams, replaced color factors with method vanish automatically. Hence we can safely set pinch diagrams to zero.
- We can straightforwardly extend this method to octet case.
- new color factor is defined as follows:

  $E(x) = C(x) - \sum_{d\in \mathrm{Dec}'(x)} T^{-n(d)} E(d),$ Dec: decomposition to "web"

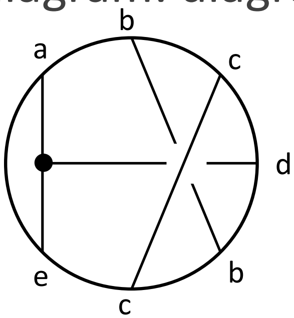
  where $T = N_c$ for singlet and $T = N_c C_F$ for octet

Color diagrams and web

- Color diagram: diagrammatical expression of color structures

  e. g.

$C =$



$= \mathrm{Tr}_F\left(T_F^a T_F^b T_F^c T_F^d T_F^b T_F^c T_F^e\right)\left(-i f^{ade}\right)$

Color diagrams and web

- Color diagram: diagrammatical expression of color structures

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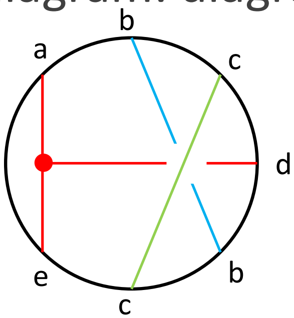
  $C =$  $= \mathrm{Tr}_F\left(T_F^a T_F^b T_F^c T_F^d T_F^b T_F^c T_F^e\right)\left(-if^{ade}\right)$

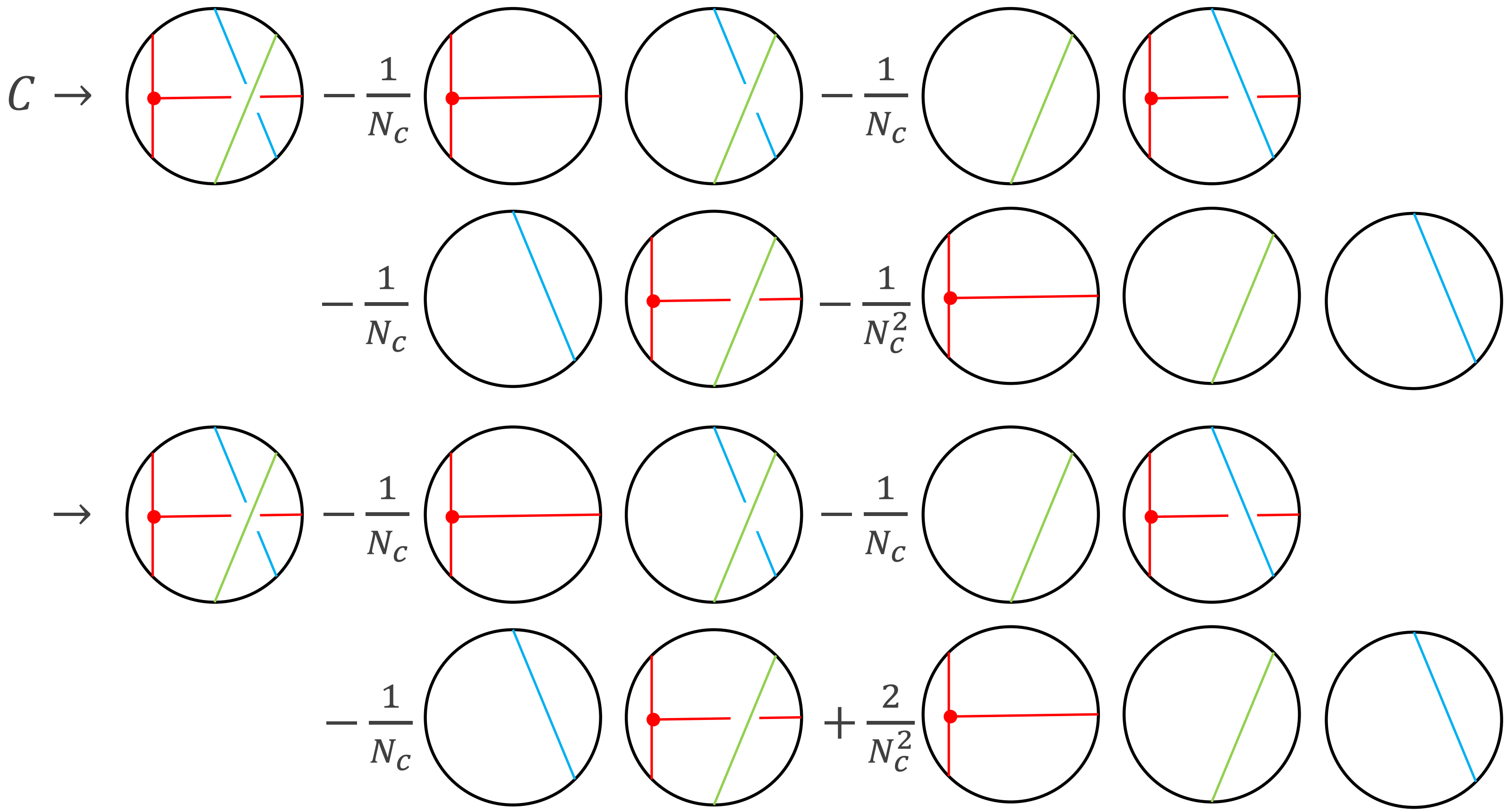
- Web: a set of gluons that cannot be partitioned without cutting one of its lines. (upper diagram contains 3 webs, R, G, and B)

- Changing color factor C of non-pinched diagrams to E, contributions from pinched diagrams are taken into account to them:

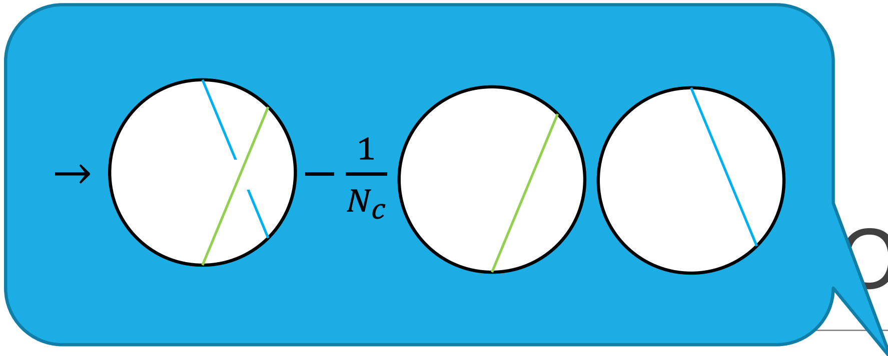
$$E(x) = C(x) - \sum_{d \in \mathrm{Dec}'(x)} T^{1-n(d)} E(d)\,,$$

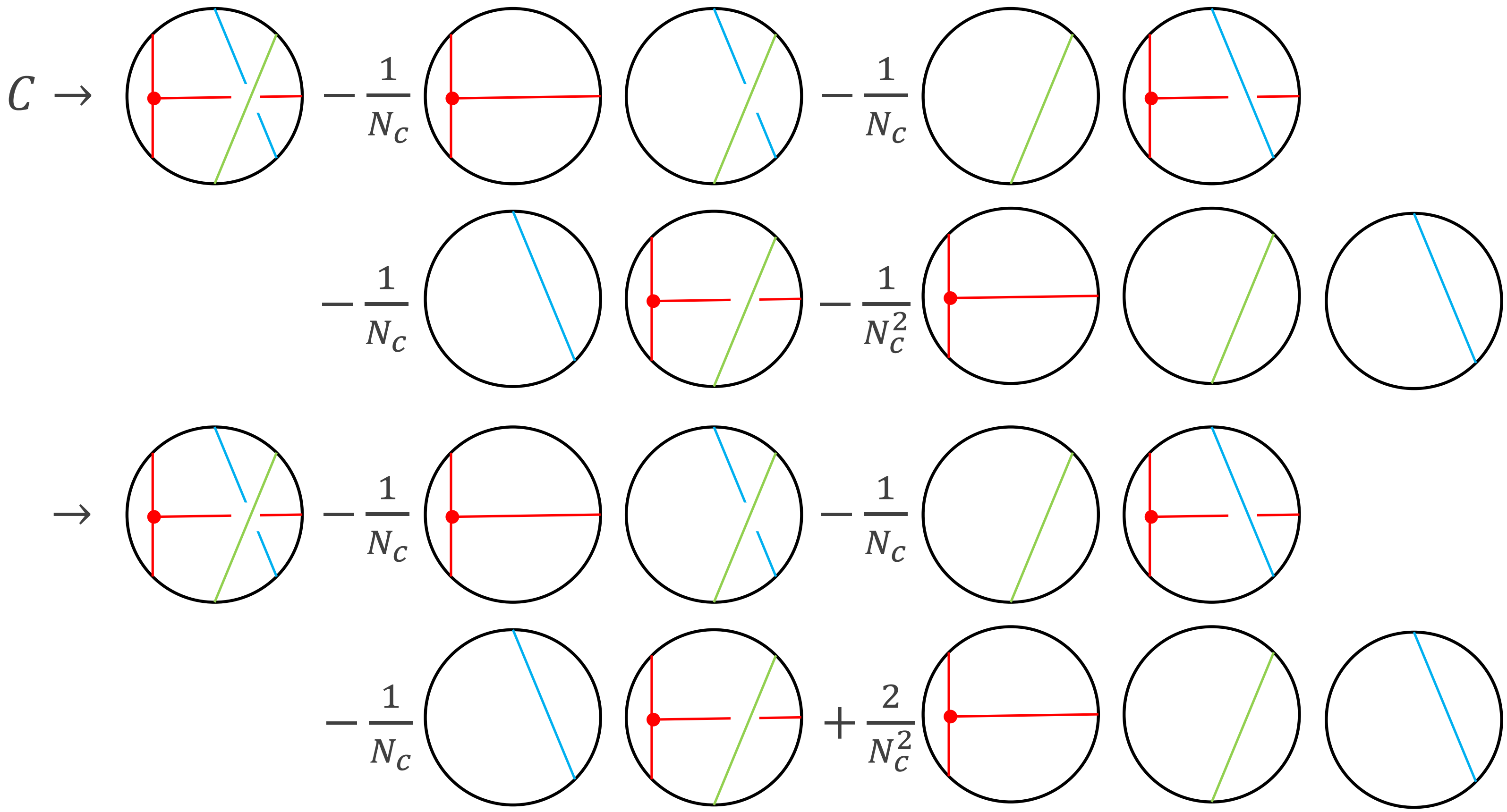
where $T = N_c$ for singlet and $T = N_c C_F$ for octet

Replacement of colorfactor



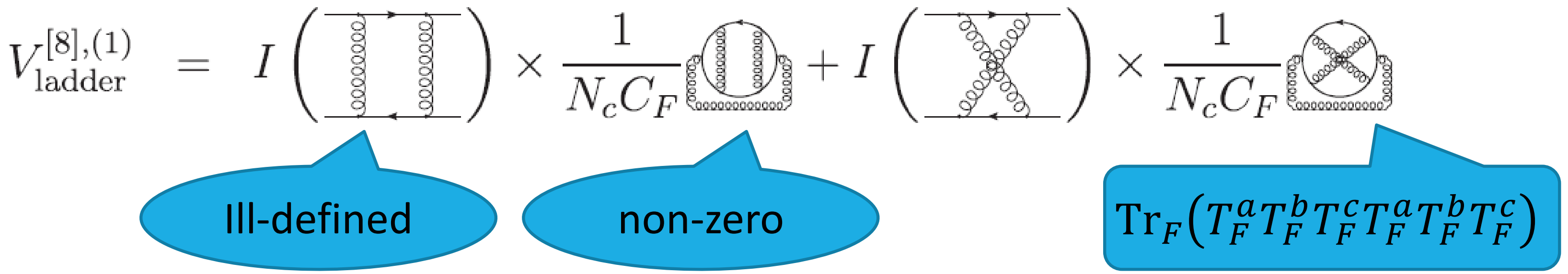
# of colorfactor



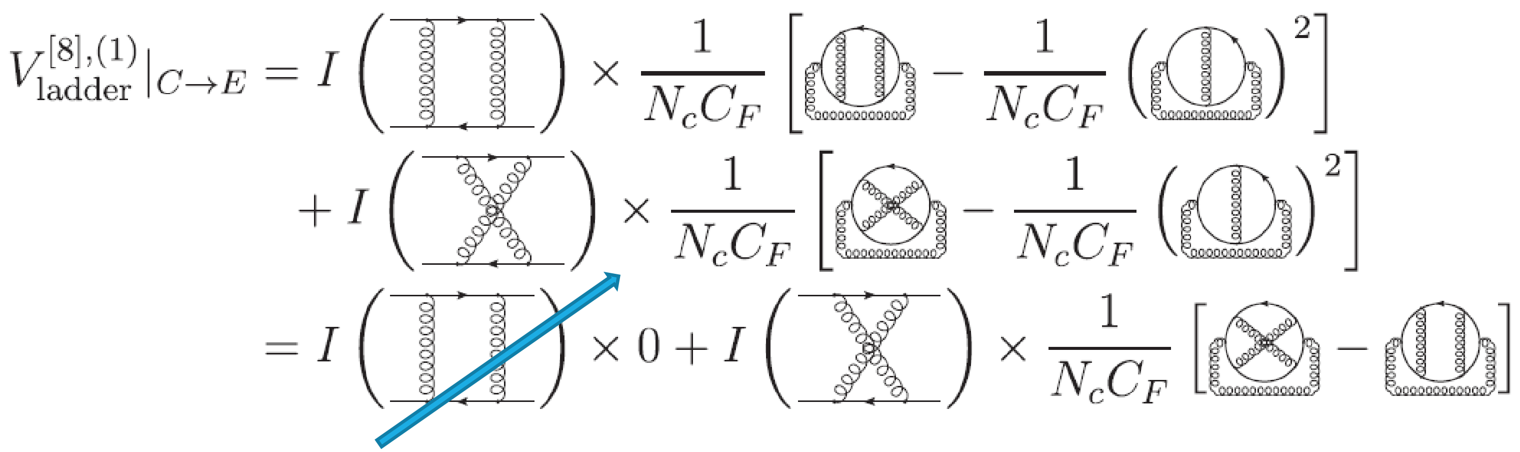


An example in octet case

- Diagrams with original colorfactor



- Diagrams with replaced colorfactor



Intermediate result

- With this algorithm, all pinched diagrams are expressed in terms of corresponding non-pinched ones and multiplication of lower order diagrams
- These multiplication terms are exactly identical to the predicted iteration terms from exponentiation of lower order potential (we checked it up to 3loop order in singlet case explicitly). Thus we can easily extract the pure 3loop contribution from the result by neglecting these terms.
- For some diagrams are non-zero only in octet case, the tables generated in computation of singlet case are not enough to reduce all diagrams appeared in octet potential. We use two individual code to reduce them. The result is expressed in terms of same 41 master integrals as singlet case.

We use these code for reduction

- FIRE: A. V. Smirnov, JHEP **10**, 107 (2008)
  A. V. Smirnov and V. A. Smirnov, arXiv:1302.5885[hep-ph]
- Our original code implemented in Mathematica 9 (not published)

Results for $a^{[8]}$

Results up to 2loop (known)

$$a_1^{[8]} = \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$

$$a_2^{[8]} = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta_3 \right) C_A^2 + N_c^2 \pi^2 (\pi^2 - 12)$$
$$- \left(\frac{1798}{81} + \frac{56}{3}\zeta_3 \right) C_A T_F n_l - \left(\frac{55}{3} - 16\zeta_3 \right) C_F T_F n_l$$
$$+ \left(\frac{20}{9} \right)^2 T_F^2 n_l^2$$

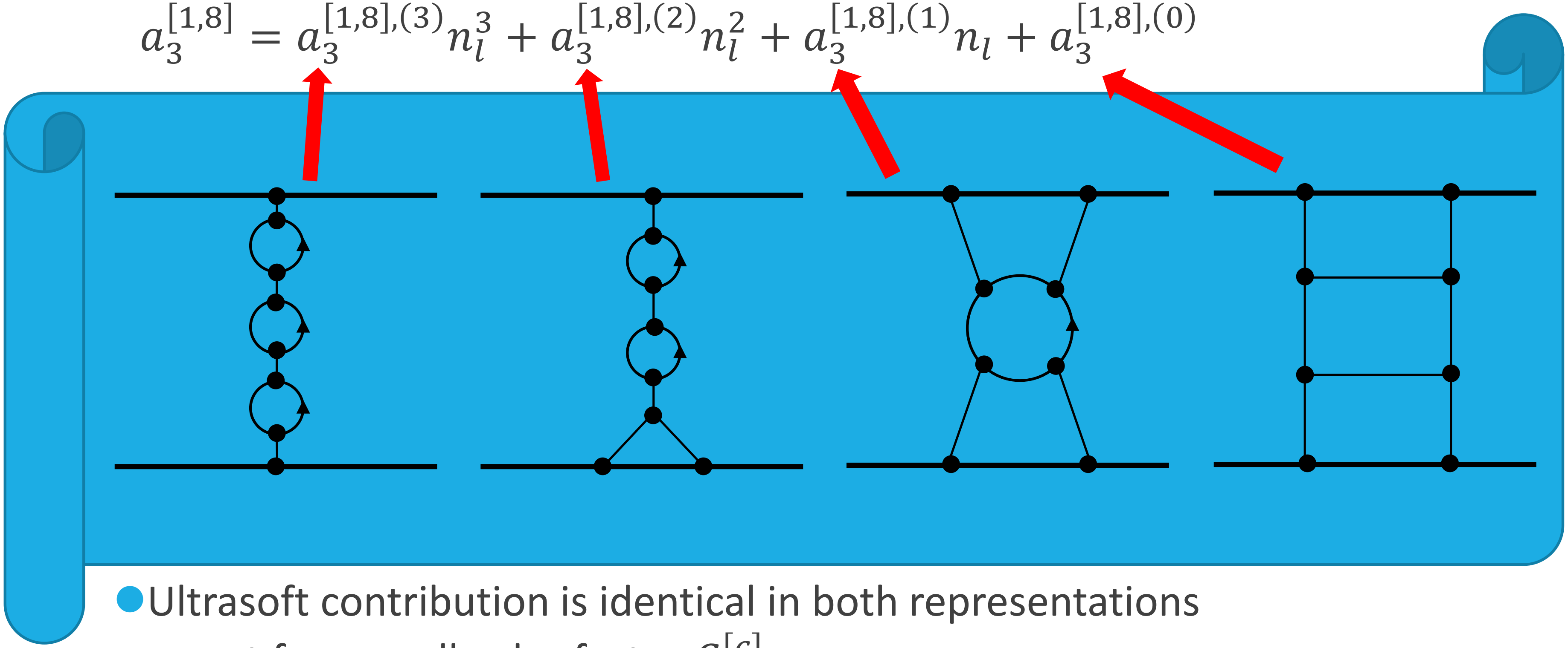
B. A. Kniehl, A. A. Penin, Y. Schroder, V. A. Smirnov, and M. Steinhauser, Phys. Lett. **B** 607, 96 (2005)

T. Collet and M. Steinhauser, Phys Lett. **B** 704, 163 (2011)

3loop result 1: Identical term

- If we decompose the result in this form:

$a_3^{[1,8]} = a_3^{[1,8],(3)} n_l^3 + a_3^{[1,8],(2)} n_l^2 + a_3^{[1,8],(1)} n_l + a_3^{[1,8],(0)}$



- Ultrasoft contribution is identical in both representations except for overall color factor $C^{[c]}$

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n_l^2 and n_l^3 terms are identical for both representations

$$a_3^{[1],(3)} = a_3^{[8],(3)} = -\left(\frac{20}{9}\right)^3 T_F^3$$

$$a_3^{[1],(2)} = a_3^{[8],(2)} = \left(\frac{12541}{243} + \frac{368\zeta_3}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} + \frac{416\zeta_3}{2}\right) C_F T_F^2$$

- Ultrasoft contribution is identical in both representations except for overall color factor $C^{[c]}$

$$V^{[c]}(|\vec{q}|) = -4\pi C^{[c]} \frac{\alpha_s(|\vec{q}|)}{\vec{q}^2}\left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1^{[c]} + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2^{[c]} + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^3 \left(a_3^{[c]} + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^2}\right) + \cdots\right]$$

- If we decompose the result in this form:

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- Ultrasoft contribution is identical in both representations except for overall color factor $C^{[c]}$

3loop result 2: Additional term

we present $SU(N_c)$ result in terms of the power of N_c:

$$a_3^{[1],(1)} = -367.319N_c^2 + 17.3611(7) - 12.597(2)\frac{1}{N_c^2}$$

$$a_3^{[8],(1)} = a_3^{[1],(1)} + 6.836(1) + 40.125N_c^2$$

$$= -327.193N_c^2 + \frac{66133}{648} - \frac{112\pi^2}{9} - \frac{272\zeta_3}{3} - \frac{32\pi^2\zeta_3}{3} + 20\zeta_5 - 12.597(2)\frac{1}{N_c^2}$$

$$a_3^{[1],(0)} = -499.396N_c^3 - 17.049(7)N_c$$

$$a_3^{[8],(0)} = a_3^{[1],(0)} - 97.579(16)N_c^3$$

we can obtain analytic result for N_c independent term in $a_3^{(1)}$, only for the octet case

Numerical values of $a^{[8]}$

Coefficient of $(\alpha_s/4)^i$ with renormalization scale $\mu = |\vec{q}|$ are as follows.

n_l is the number of light quarks; 3, 4, 5 correspond to the charm, bottom, top quark cases.

order	1loop ($a_1/4$)			2loop ($a_2/16$)			3loop ($a_3/64$)		
n_l	3	4	5	3	4	5	3	4	5
Singlet	1.750	1.472	1.194	16.80	13.19	9.740	81.25	49.39	22.83
Octet	1.750	1.472	1.194	4.973	1.366	-2.087	57.33	31.22	10.41

In these cases, all of total contributions of additional terms are negative.

- $\delta a_2/16 = N_c^2\pi^2(\pi^2 - 12)/16 = -11.8272$
- $\delta a_3/64 = \{[6.836 + 40.125N_c^2]n_l - 97.579N_c^3\}/64 = 5.7494n_l - 41.166$

In 2loop or higher order, these terms compensate the large part of the corrections appeared in the singlet case.

Analytic structures of $\delta a^{[8]}$

- In 2 loop case, the difference between singlet and octet case is proportional to π^2: $\delta a_2^{[8]} = N_c^2 \pi^2 (\pi^2 - 12)$
- It is true for N=4 super Yang-Mills theory (M. Prausa and M. Steinhauser, Phys. Rev. **D 88**, 020529):

$$\delta a_{2,\mathrm{SYM}}^{[8]} = N_c^2 \pi^2 (\pi^2 - 12)$$

- For 3loop case, we have only numerical result. However, because of the feature of master integrals appeared in the result, we can conclude that $\delta a_3^{[8]}$ contains the factor π^2 (next slide).

Master Integrals in $\delta a_3^{[8]}$

There are two MIs in $\delta a^{[8]}$, that can be evaluated only numerically. Both of them are expressed in this form:

$$I = \iiint \frac{d^D k}{(4\pi)^D} \frac{d^D p}{(4\pi)^D} \frac{d^D l}{(4\pi)^D} \frac{1}{+k_0 + i\varepsilon} \frac{1}{+p_0 + i\varepsilon} f(k, p, l, q),$$

where q is the external momentum. With appropriate variable transformation, I is represented in

$$I = \iiint \frac{d^D k}{(4\pi)^D} \frac{d^D p}{(4\pi)^D} \frac{d^D l}{(4\pi)^D} \frac{1}{-k_0 + i\varepsilon} \frac{1}{+p_0 + i\varepsilon} f(k, p, l, q)$$

$$= \iiint \frac{d^D k}{(4\pi)^D} \frac{d^D p}{(4\pi)^D} \frac{d^D l}{(4\pi)^D} \frac{1}{+k_0 + i\varepsilon} \frac{1}{-p_0 + i\varepsilon} f(k, p, l, q)$$

$$= \iiint \frac{d^D k}{(4\pi)^D} \frac{d^D p}{(4\pi)^D} \frac{d^D l}{(4\pi)^D} \frac{1}{-k_0 + i\varepsilon} \frac{1}{-p_0 + i\varepsilon} f(k, p, l, q)$$

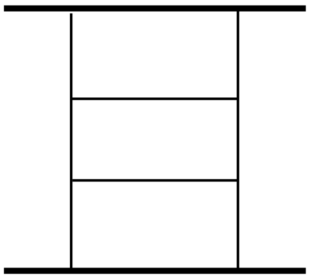
Master Integrals in $\delta a_3^{[8]}$

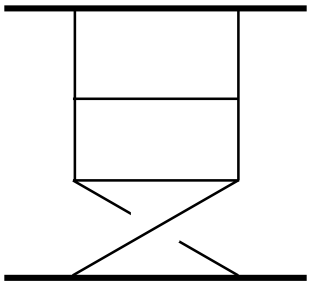
Then,

$$I = \frac{1}{4}\iiint \frac{d^Dk}{(4\pi)^D}\frac{d^Dp}{(4\pi)^D}\frac{d^Dl}{(4\pi)^D}\left(\frac{1}{k_0+i\varepsilon}+\frac{1}{-k_0+i\varepsilon}\right)\left(\frac{1}{p_0+i\varepsilon}+\frac{1}{-p_0+i\varepsilon}\right)f(k,p,l,q)$$

$$= -\pi^2 \iiint \frac{d^Dk}{(4\pi)^D}\frac{d^Dp}{(4\pi)^D}\frac{d^Dl}{(4\pi)^D}\delta(k_0)\delta(p_0)f(k,p,l,q)$$

Diagrammatically,





$\propto \pi^2$

This might be related to the symmetry of the color space.

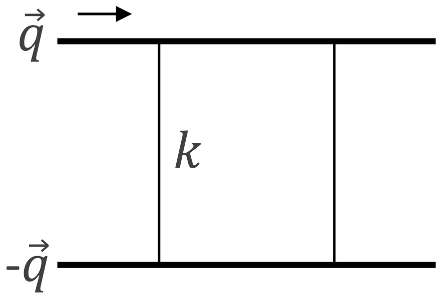
Conclusion

Conclusion

- We calculated the three loop correction for static potential between quarks in color octet configuration.
- In octet case, because of the contributions from pinched diagrams, calculation is more complicated than singlet case.
- We reduce these diagrams to non-pinched ones with two different algorithms, and results coincide.
- The results are identical to singlet case, except for additional terms proportional to π^2. If the coefficients of a_3 are represented in terms of N_c, two of five are same for both color configuration.
- Additional terms are negative for $SU(3)$, $n_l = 3,4, \text{and } 5$, corresponding charm, bottom, and top quark case.

Backup

Principal Value prescription



$$= \int \frac{d^D k}{(4\pi)^D} f(k,\vec{q}) \frac{1}{k_0+i\varepsilon} \frac{1}{k_0-i\varepsilon}$$ (ill-defined)

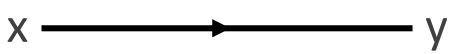
One treatment for this diagram is principal-value prescription:

$$\frac{1}{k_0+i\varepsilon}\frac{1}{k_0-i\varepsilon} \to \frac{1}{2}\left[\frac{1}{(k_0+i\varepsilon)^2}+\frac{1}{(k_0-i\varepsilon)^2}\right],$$

but for 2loop or beyond, naïve application of this does not work (because of diagrams with two or more pinches)

Diagrams in coordianate space

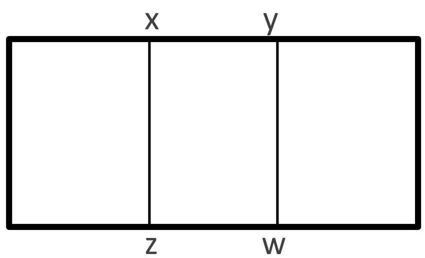
- Feynman rule of heavy quark in coordinate space



$$= \theta(y_0 - x_0)$$

- Without color structure, one loop ladder diagram is expressed by this form (corresponding "QED" diagram):

$$\alpha_s^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dx_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dy_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dz_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dw_0 \ \theta(y_0 - x_0)\theta(w_0 - z_0) D_{00}(x_0 - z_0, \vec{r}) D_{00}(y_0 - w_0, \vec{r})$$



- In general: $\left[\theta(y_2^0 - y_1^0)\cdots\theta(y_i^0 - y_{i-1}^0)\theta(x_1^0 - y_i^0)\theta(x_2^0 - x_1^0)\cdots\theta(x_j^0 - x_{j-1}^0)\right]\left[(x,y) \leftrightarrow (z,w)\right]$

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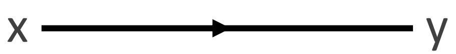

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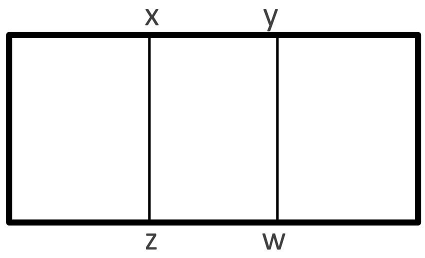
- Without color structure, one loop ladder diagram is expressed by this form (corresponding “QED” diagram):

$$\alpha_s^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dx_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dy_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dz_0 \int_{-\frac{T}{2}}^{\frac{T}{2}} dw_0 \;\; \theta(y_0 - x_0)\theta(w_0 - z_0) D_{00}(x_0 - z_0, \vec{r}) D_{00}(y_0 - w_0, \vec{r})$$



- In general: $[\theta(y_2^0 - y_1^0) \cdots \theta(y_i^0 - y_{i-1}^0)\theta(x_1^0 - y_i^0)\theta(x_2^0 - x_1^0) \cdots \theta(x_j^0 - x_{j-1}^0)][(x, y) \leftrightarrow (z, w)]$

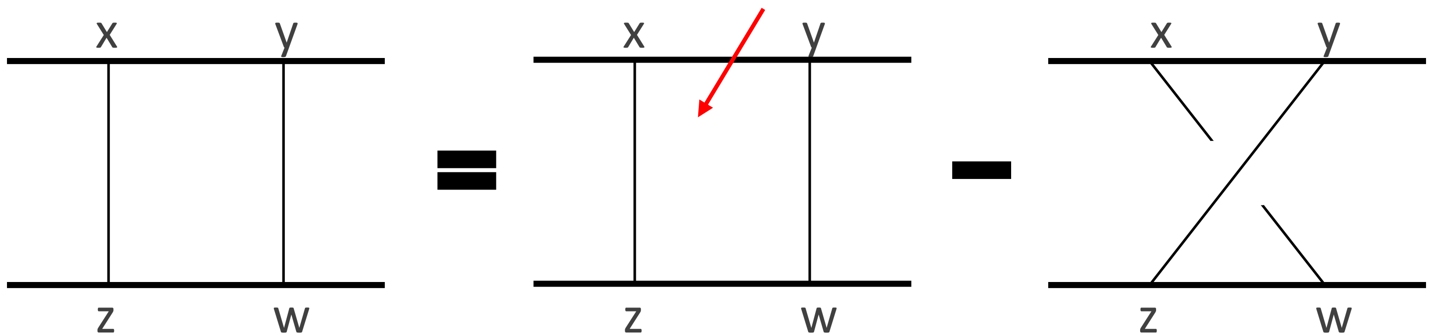
Replace by 1

Treatment of pinched diagram

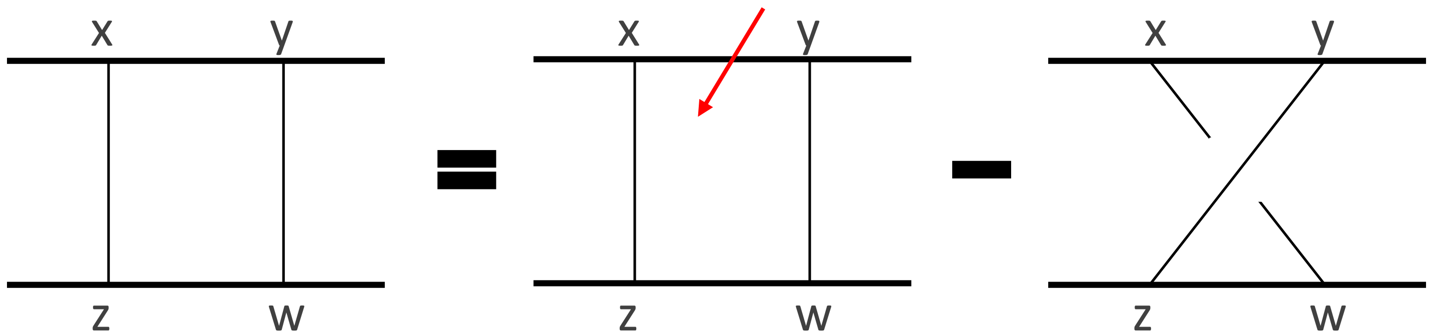
- Step function satisfies $\theta(y_0 - x_0) + \theta(x_0 - y_0) = 1$

  ⟹ planer ladder + crossed ladder = omission of one θ function

- Cut of heavy quark line



$\theta(y_0 - x_0) \times \theta(w_0 - z_0)$ = $1 \times \theta(w_0 - z_0)$ - $\theta(x_0 - y_0) \times \theta(w_0 - z_0)$

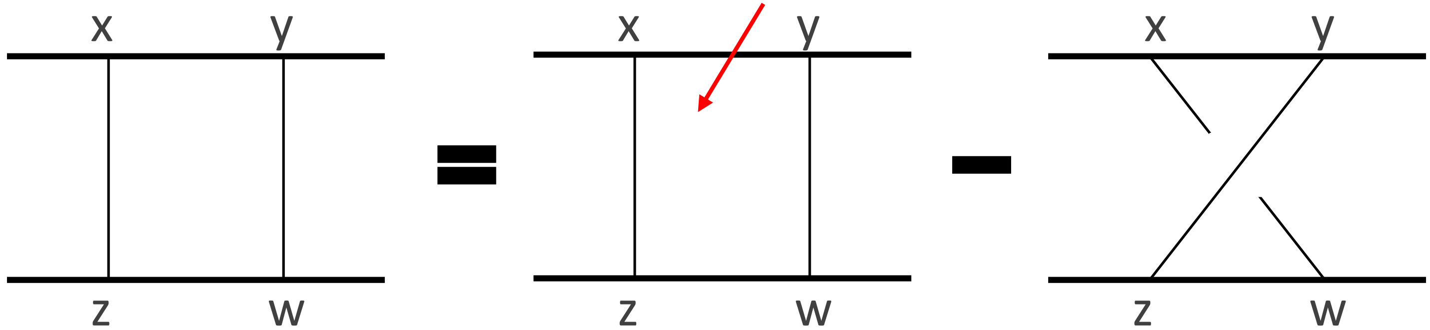


$\theta(y_0 - x_0) \times \theta(w_0 - z_0)$

$1 \times \theta(w_0 - z_0)$

$\theta(x_0 - y_0) \times \theta(w_0 - z_0)$

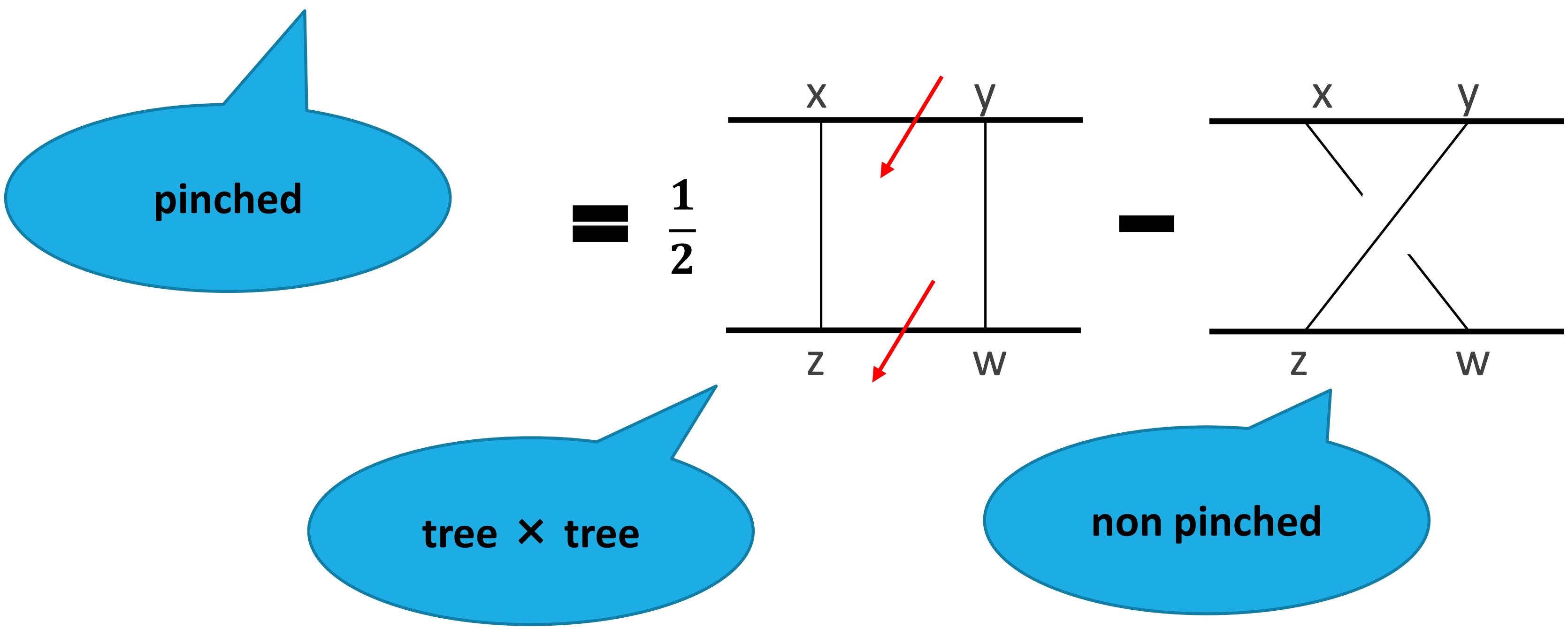
$$
\begin{aligned}
&\int dx_0 dy_0 dz_0 dw_0 D_{00}(x-z) D_{00}(y-w) 1 \\
&= \int dx_0 dy_0 dz_0 dw_0 D_{00}(x-z) D_{00}(y-w) \theta(w_0 - z_0) \\
&\quad + \int dx_0 dy_0 dz_0 dw_0 D_{00}(x-z) D_{00}(y-w) \theta(z_0 - w_0) \\
&= 2 \int dx_0 dy_0 dz_0 dw_0 D_{00}(x-z) D_{00}(y-w) \theta(w_0 - z_0)
\end{aligned}
$$

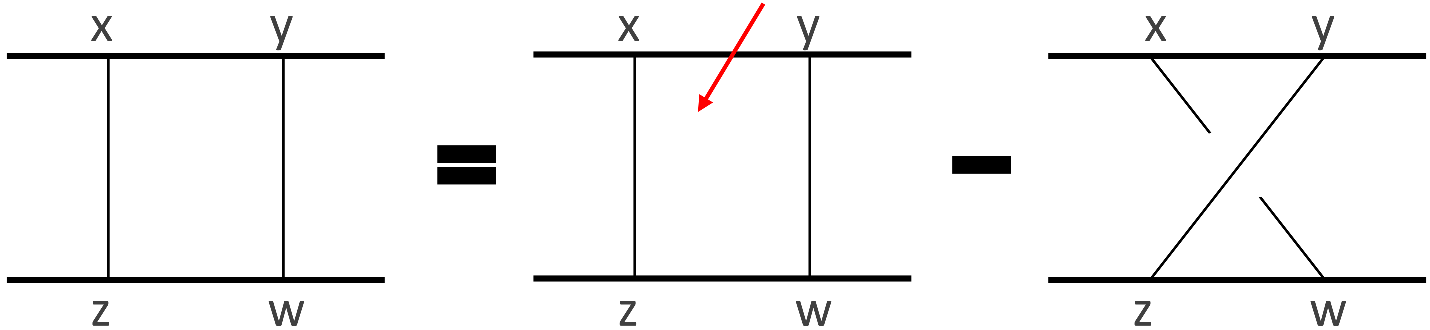


$\theta(y_0 - x_0)$ $\times \theta(w_0 - z_0)$

1 $\times \theta(w_0 - z_0)$

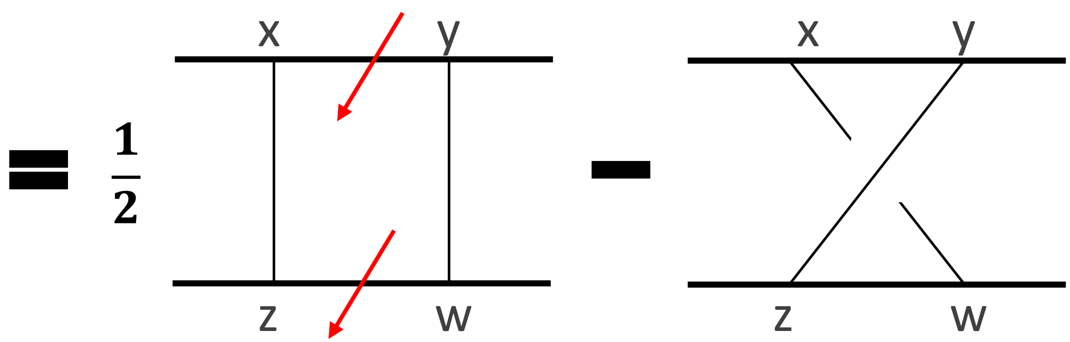
$\theta(x_0 - y_0)$ $\times \theta(w_0 - z_0)$





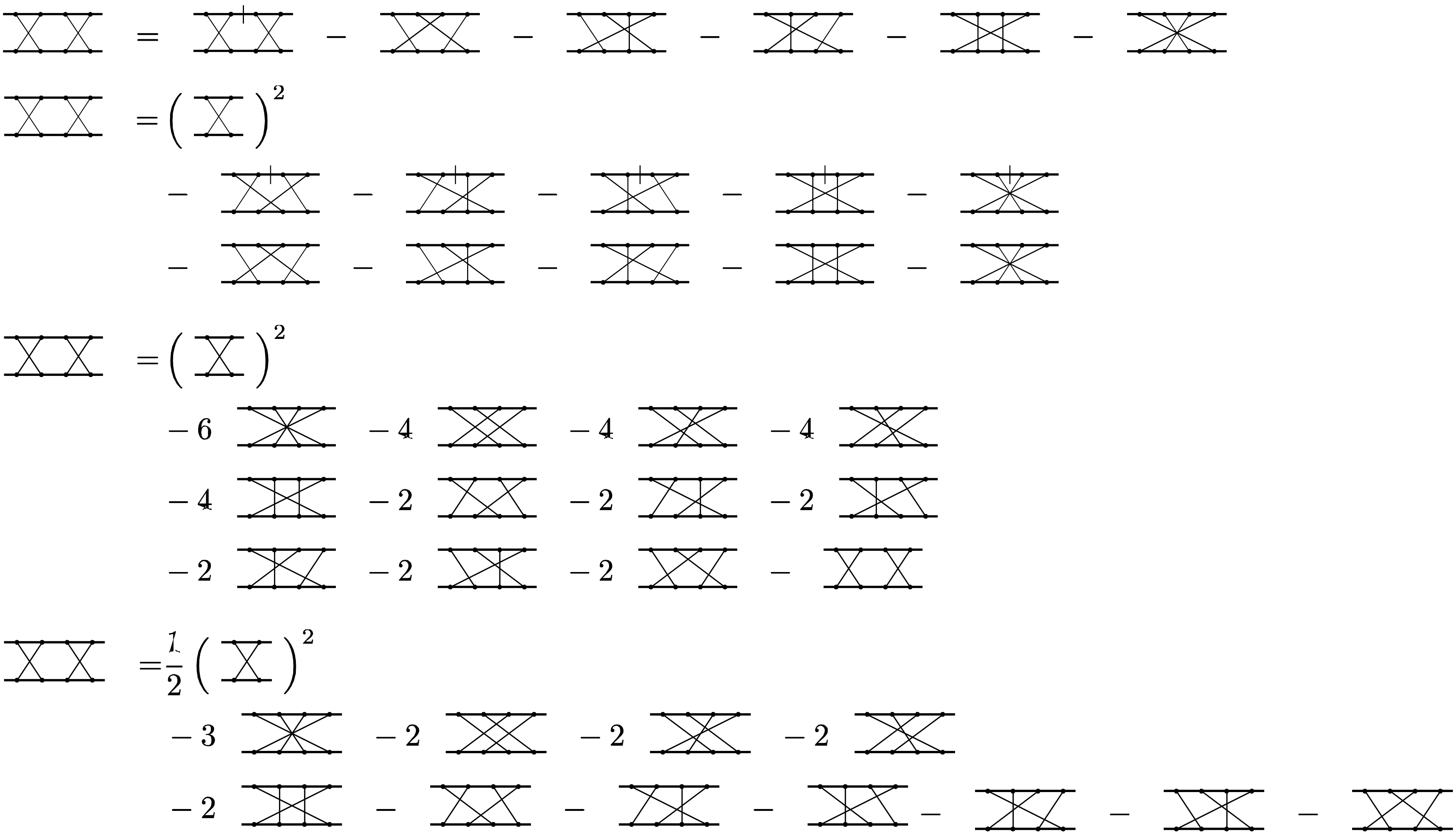
$\theta(y_0 - x_0) \times \theta(w_0 - z_0)$ $1 \times \theta(w_0 - z_0)$ $\theta(x_0 - y_0) \times \theta(w_0 - z_0)$

pinched

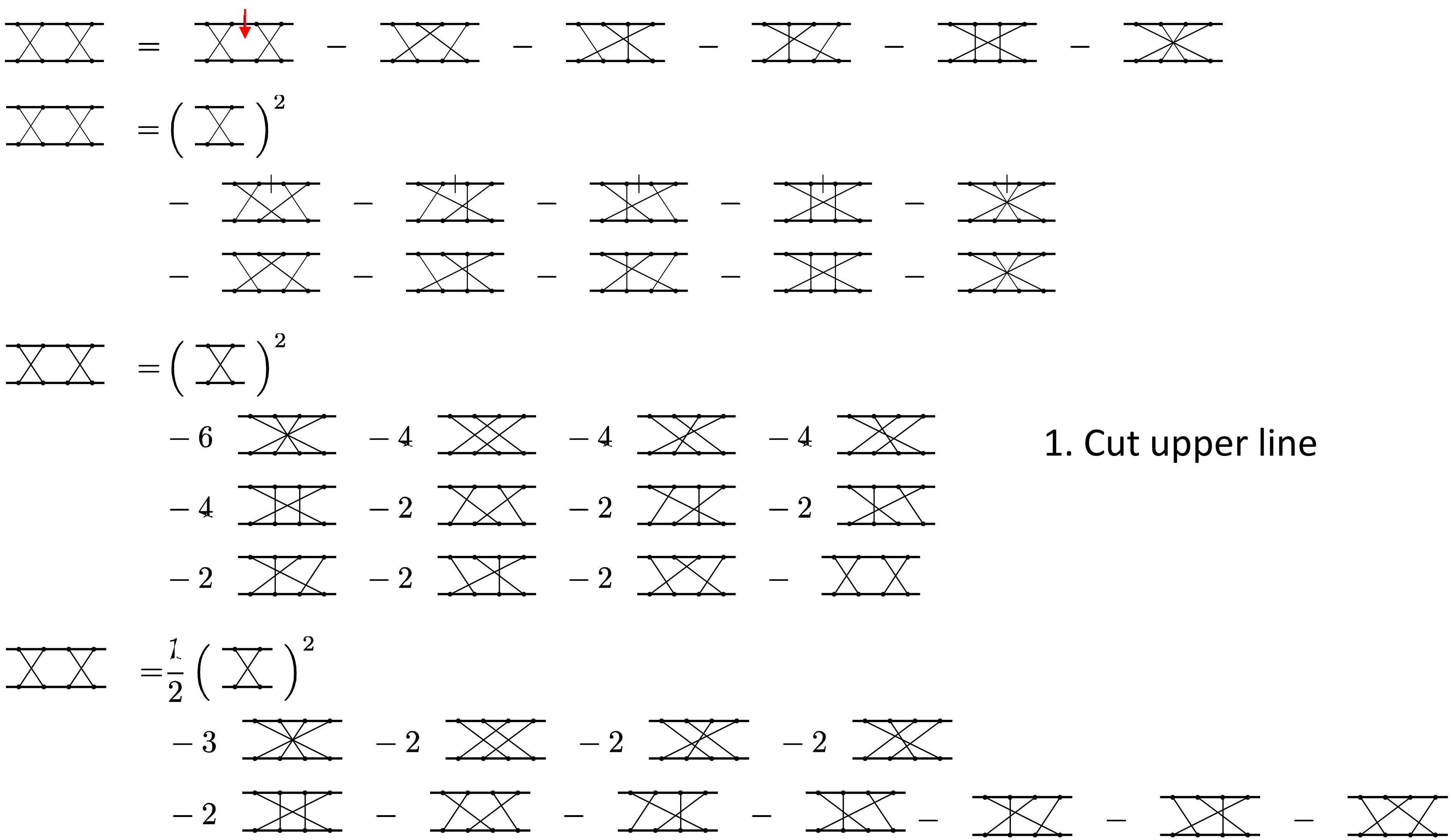


well-defined in momentum space

3-loop example

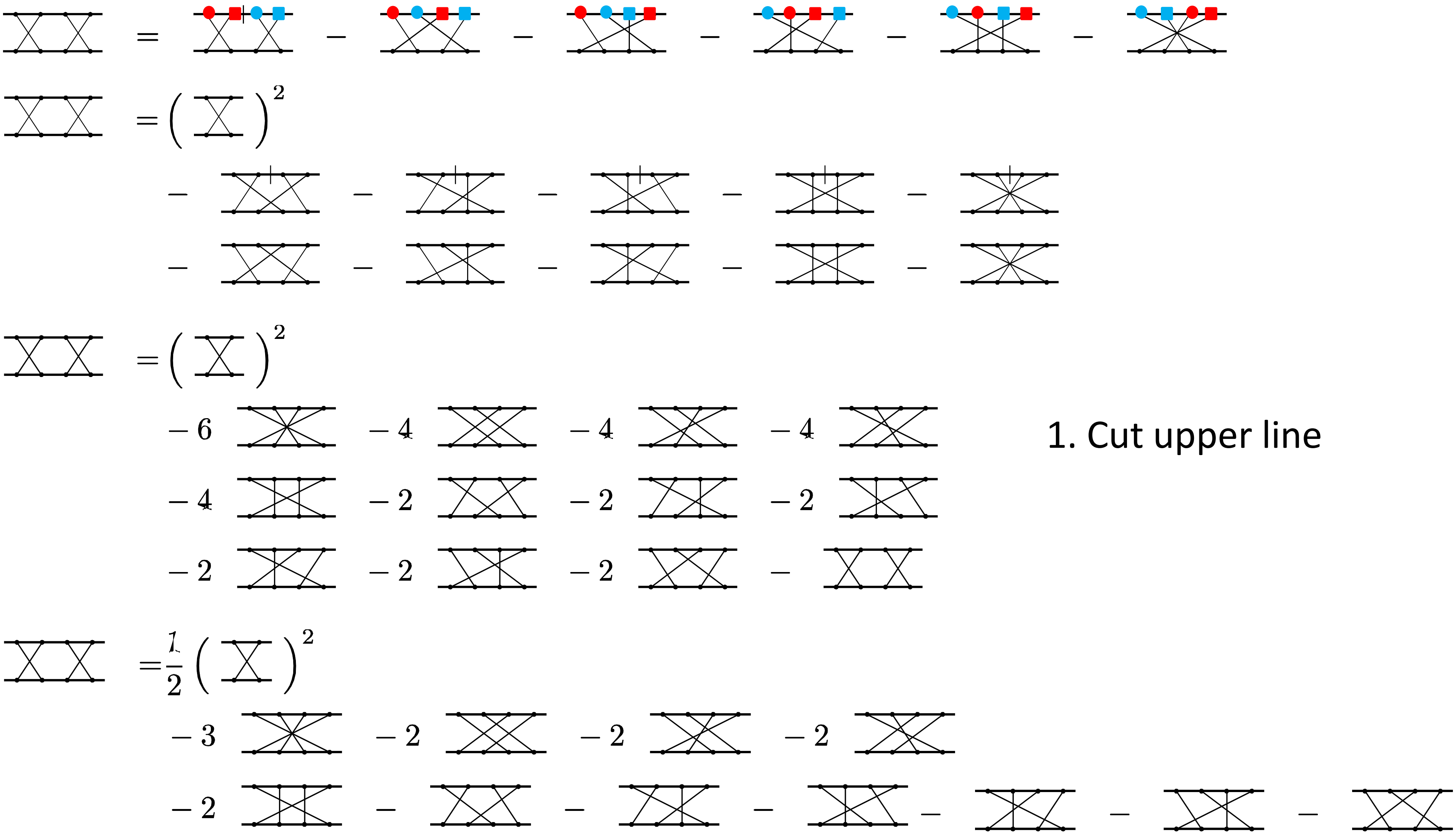


3-loop example



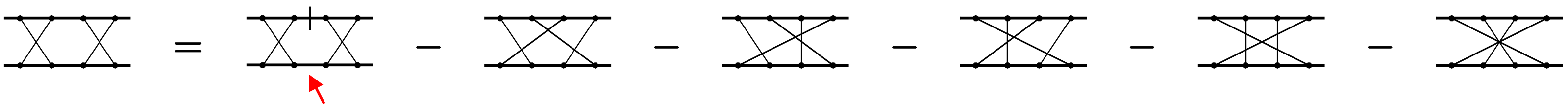
1. Cut upper line

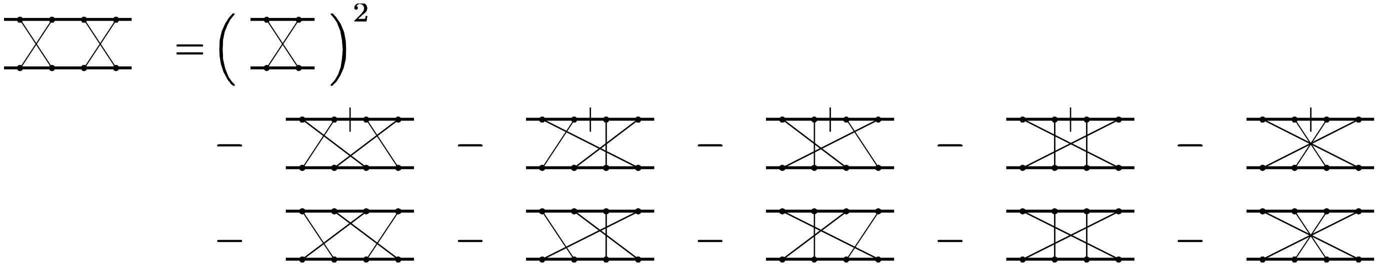
3-loop example

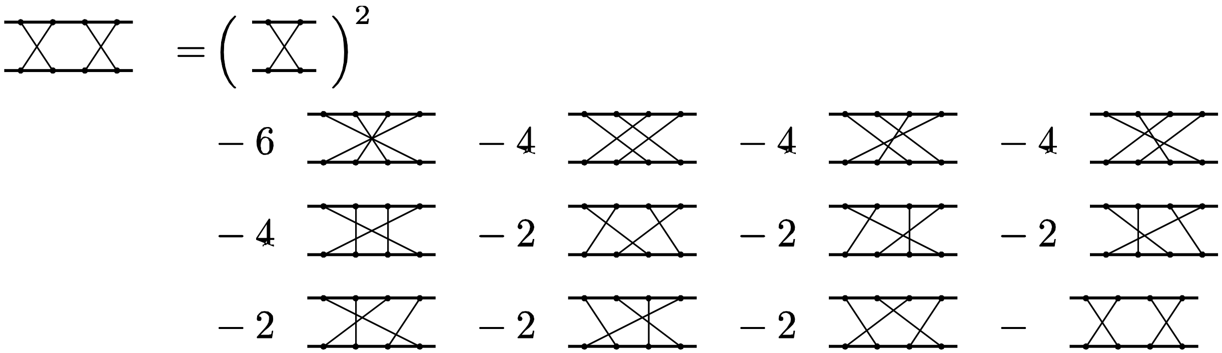


1. Cut upper line

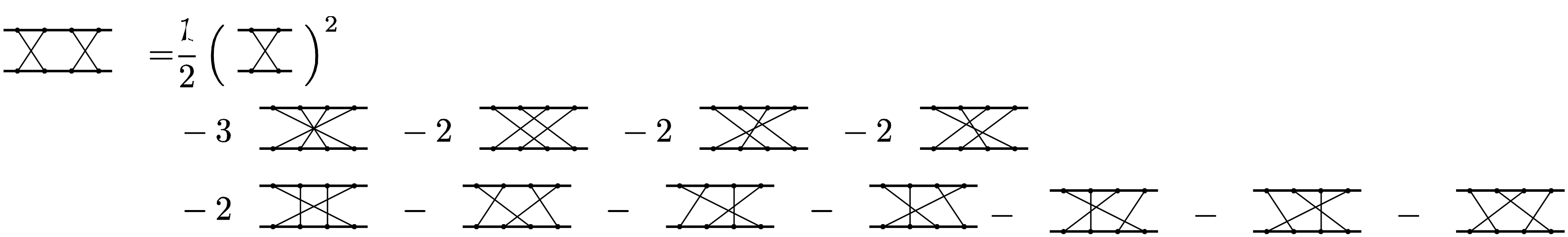
3-loop example



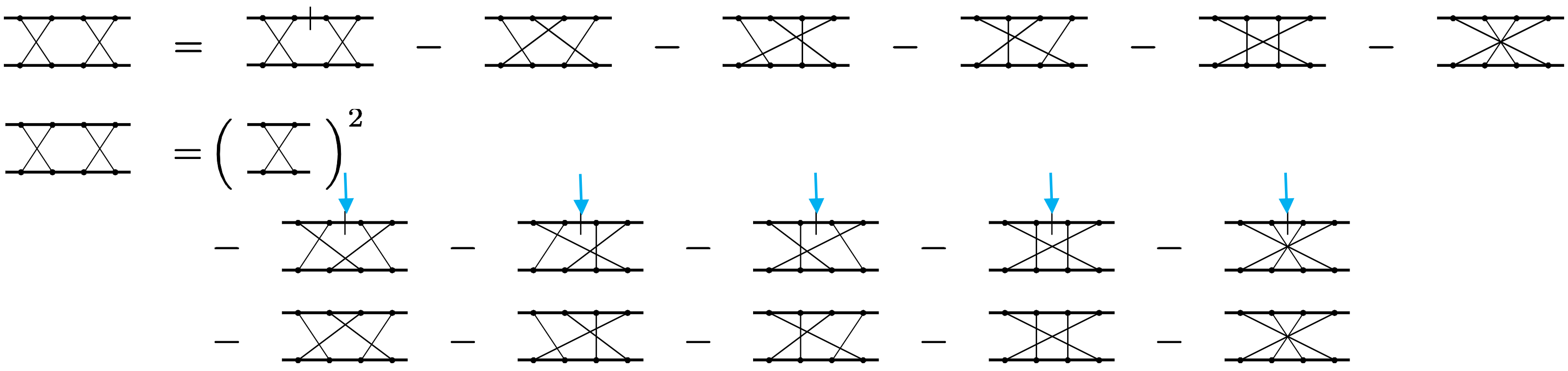


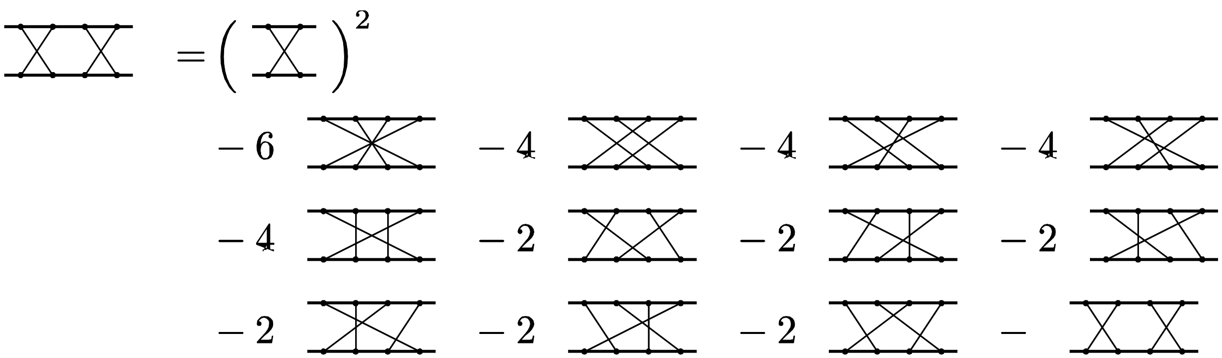


1. Cut upper line
2. Cut lower line

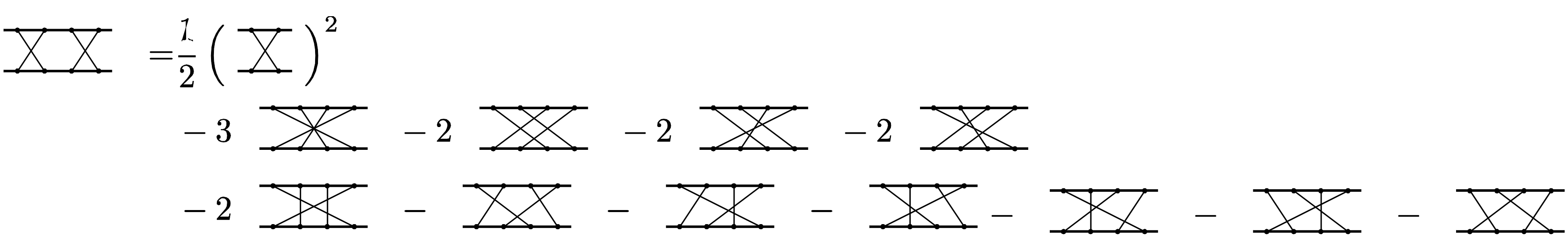


3-loop example

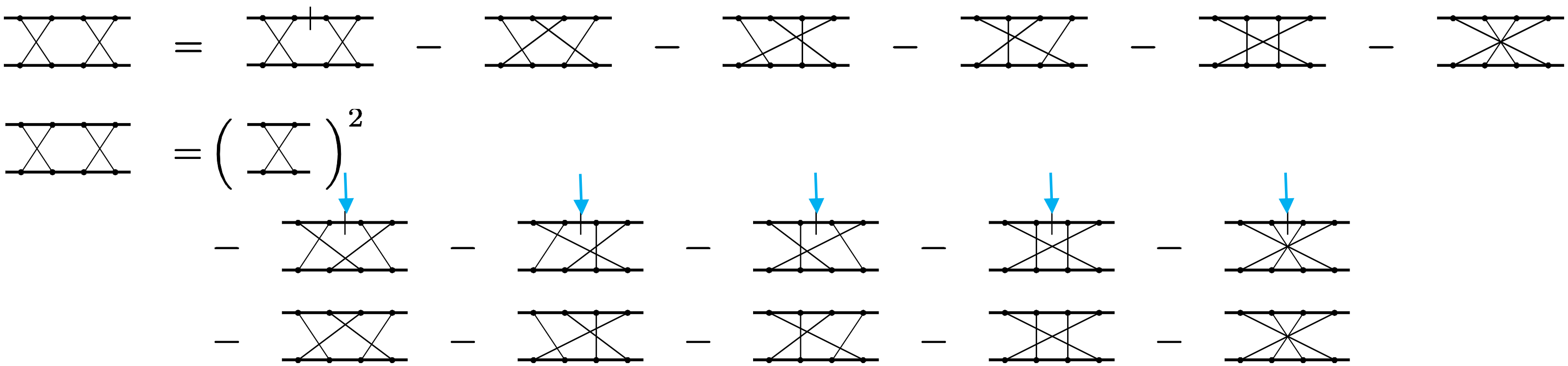


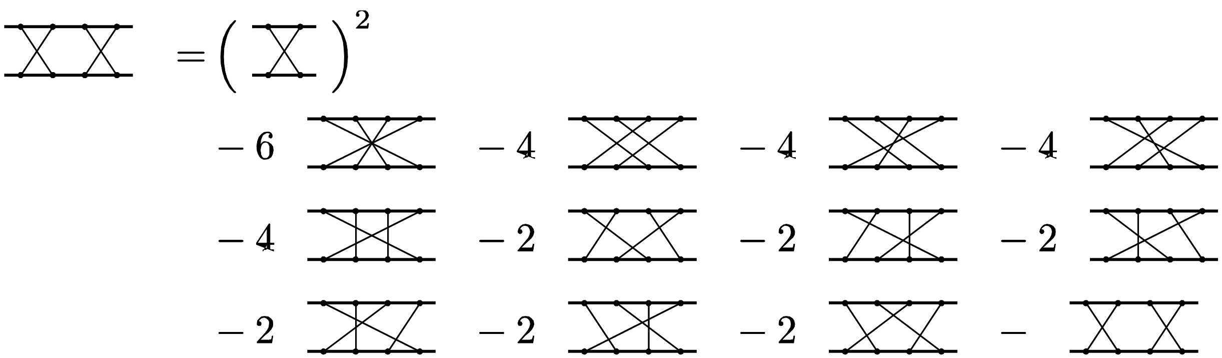


1. Cut upper line
2. Cut lower line
3. Insert propagator

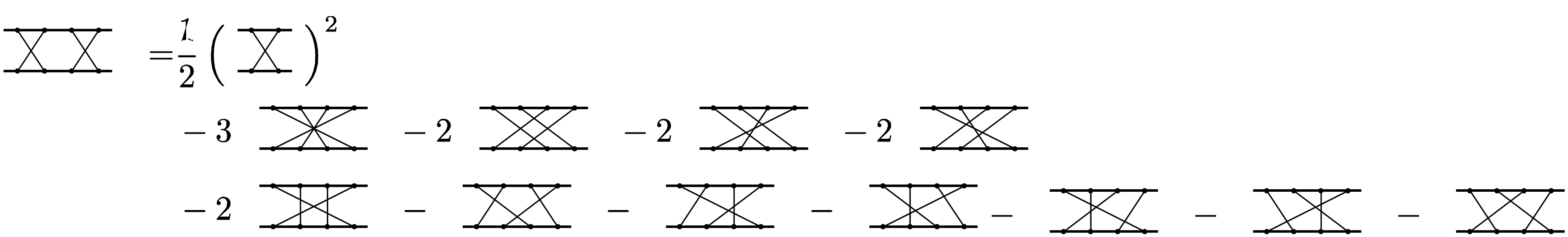


3-loop example





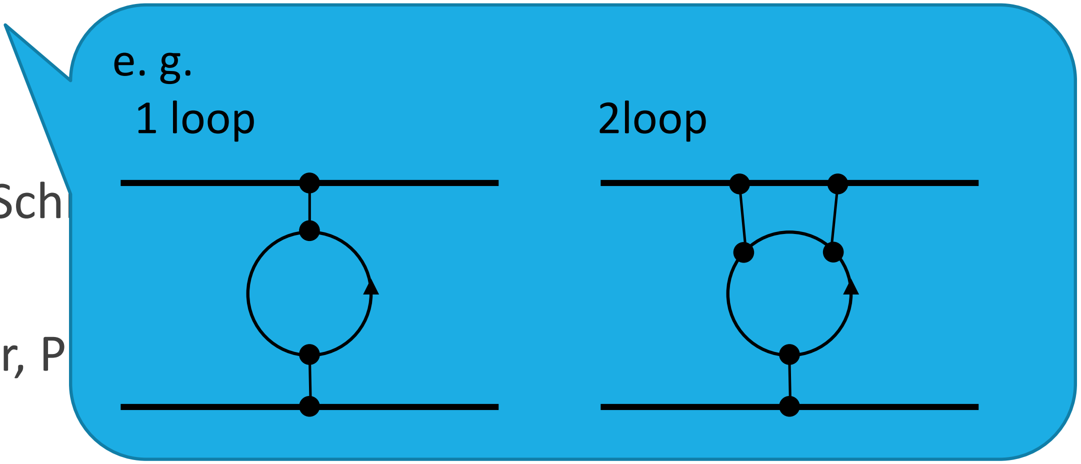
1. Cut upper line
2. Cut lower line
3. Insert propagator
4. Solve relation(s)



Results up to 2loop (known)

$$a_1^{[8]} = \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$

$$a_2^{[8]} = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta_3\right) C_A^2 + N_c^2 \pi^2 (\pi^2 - 12)$$
$$- \left(\frac{1798}{81} + \frac{56}{3}\zeta_3\right) C_A T_F n_l - \left(\frac{55}{3} - 16\zeta_3\right) C_F T_F n_l$$
$$+ \left(\frac{20}{9}\right)^2 T_F^2 n_l^2$$



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