# Boosted Top Quark Pair Production in Soft Collinear Effective Theory

**Andrea Ferroglia**

**New York City College of Technology**
**NYCCT Center for Theoretical Physics**

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# Outline

1. Soft Gluon Emission Resummation
2. Boosted Tops: Soft Gluons + Small Mass

# NNLL Resummation and Approximate NNLO

*in collaboration with*

V. Ahrens, M. Neubert, B. Pecjak, and L. L. Yang

Phys.Rev. D84 (2011) 074004 (arXiv:1106.6051 [hep-ph])
Phys.Lett. B703 (2011) 135-141 (arXiv:1105.5824 [hep-ph])
JHEP 1109 (2011) 070 (arXiv:1103.0550 [hep-ph])
JHEP 1009 (2010) 097 (arXiv:1003.5827 [hep-ph])
Phys.Lett. B687 (2010) 331-337 (arXiv:0912.3375 [hep-ph])

# Soft Limits

The partonic top quark pair production process:

$$p_i(p_1) + p_j(p_2) \longrightarrow t(p_3) + \bar{t}(p_4) + X(k) \qquad (i,j \in \{q, \bar{q}, g\})$$

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- Top quark $p_T$ dist. $\frac{d\sigma}{dp_T dy}$ ("1PI kin.")

$$s_4 = (p_4 + k)^2 - m_t^2 \to 0$$

# Factorization

In the soft limit one finds a clear hierarchy among physical scales

PIM kinematics $s, M^2, m_t^2 \gg s(1-z)^2 \gg \Lambda^2_{\rm QCD}$

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$$d\hat{\sigma} \sim \mathrm{Tr}\left[\mathbf{H}(M, m_t, \cos\theta, \mu)\mathbf{S}(\sqrt{s}(1-z), m_t, \cos\theta, \mu)\right] + \mathcal{O}(1-z)$$

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- Only tree-level channels ($q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$) contribute
- **H** $\to$ virtual corrections, same for PIM and 1PI
- **S** $\to$ real corrections, different in PIM and 1PI
- Separation of hard and soft scales

# What to do with factorization?

- $\mathbf{H}$ and $\tilde{\mathbf{s}}$ (Laplace transf. of $\mathbf{S}$ ) obey RGE of the form

$$\frac{d}{d\ln\mu}\mathbf{H} = \mathbf{\Gamma}_H\mathbf{H} + \mathbf{H}\mathbf{\Gamma}_H^\dagger \qquad \frac{d}{d\ln\mu}\tilde{\mathbf{s}} = -\mathbf{\Gamma}_s\tilde{\mathbf{s}} - \tilde{\mathbf{s}}\mathbf{\Gamma}_s^\dagger$$

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One can use this to obtain

I) All order resummation of large soft logs up to NNLL accuracy

II) Approximate NNLO formulas for the partonic cross section

$$d\hat{\sigma}_{\text{NNLO}} = D_3 P_3(\lambda) + D_2 P_2(\lambda) + D_1 P_1(\lambda) + D_0 P_0(\lambda) + C_0\delta(\lambda) + R(\lambda)$$

with $\lambda \in \{z, s_4\}$

$$P_n(z) = \left[\frac{\ln^n(1-z)}{(1-z)}\right]_+ \qquad P_n(s_4) = \left[\frac{\ln^n(s_4/m_t^2)}{s_4}\right]_+$$

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$D_0, \cdots, D_3$ are calculated exactly (generic $s, t, m_t$), only $\ln\mu$ in $C_0$ can be determined

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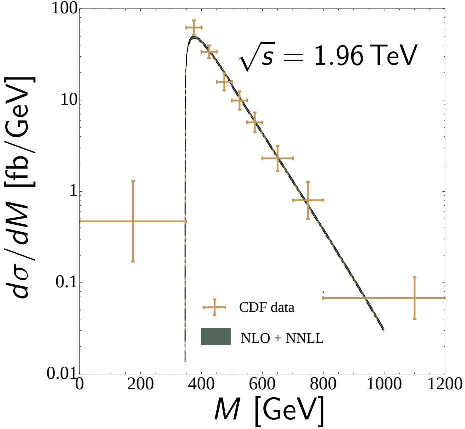
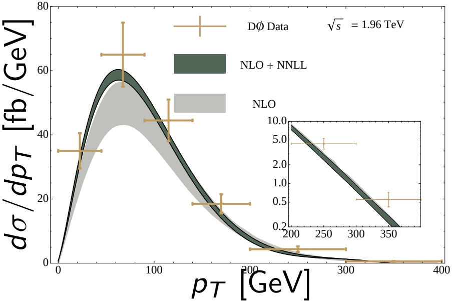
**Dynamical threshold enhancement**

Due to the steep fall-off of the PDFs away from the soft region, we can obtain good predictions for hadronic cross sections

# Invariant Mass Distribution and $p_T$ Distribution Versus Tevatron Data

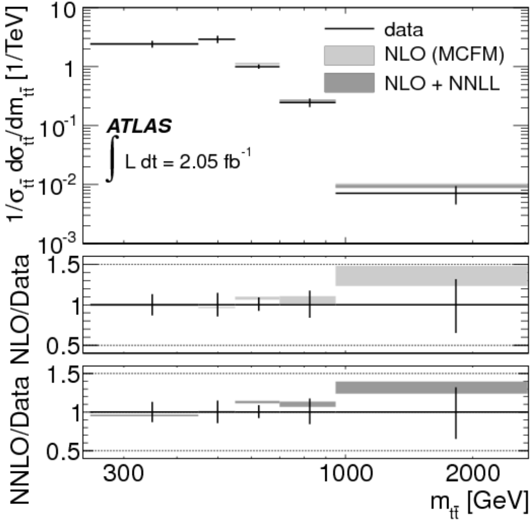



Normalization and shape of the distributions are consistent with data

# Invariant Mass Distribution Versus LHC Data

ATLAS Measurement of the invariant mass distribution at 7 TeV (1207.5644)



# Boosted Tops

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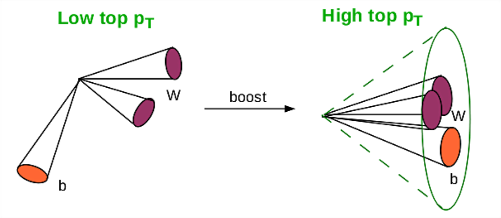


JHEP 1401 (2014) 028 (arXiv:1310.3836 [hep-ph])
JHEP 1309 (2013) 032 (arXiv:1306.1537 [hep-ph])
JHEP 1210 (2012) 180 (arXiv:1207.4798 [hep-ph])
Phys.Rev. D86 (2012) 034010 (arXiv:1205.3662 [hep-ph])

# Top Pairs at Large Invariant Mass

Many models of physics beyond the Standard Model predict the existence of new particles which decay into energetic top quarks and whose characteristic signal would be either resonant bumps or more subtle distortions in the high invariant mass region of the differential distribution

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In order analyze the tail of the invariant mass distribution one needs to consider the following scale hierarchy

$$s, M^2 \gg m_t^2 \gg s(1-z)^2 \gg \Lambda_{\mathrm{QCD}}^2$$

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**Goal**: Framework for simultaneous resummation of

$$\left[\frac{\ln^n(1-z)}{1-z}\right]_+ \text{ and } \ln^n\left(\frac{m_t}{M}\right)$$

# Factorization in PIM Kinematics

**Small Mass Limit**

$M \gg m_t$

Massless quark production cross section $\otimes$ Perturbative fragmentation functions for each of the heavy quarks

$$C_{ij}(z, M, m_t, t_1, \mu) = C_{ij}^{ab}(z, M, t_1, \mu) \otimes D_{a/t}(z, m_t, \mu) \otimes D_{b/t}(z, m_t, \mu)$$

**Soft Limit**

$z \longrightarrow 1$

Hard function $\otimes$ Soft function

$$C_{ij}^{t\bar{t}} = \mathrm{Tr}\left[\mathbf{H}(M)\mathbf{S}(\sqrt{s}(1-z))\right]$$

**Soft Limit**

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Collinear function $\otimes$ Soft-collinear function

$$D_{t/t} = C_D(m_t) S_D(m_t(1-z))$$

(schematically)

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- but we know more
  - Fragmentation function and its soft component known to NNLO [Melnikov and Mitov ('06), Neubert ('07)]
  - Hard functions for $m_t = 0$ known to NNLO [Anastasiou, Glover, Tejeda-Yeomans *et al.* ('00-'04), Bern, de Freitas, Dixon ('02-'04), Broggio, AF, Pecjak, Zhang (soon)]
  - NNLO soft function for $m_t = 0$ calculated [AF, Pecjak and Yang ('12)]

We can calculate a full soft+virtual NNLO approximation to the partonic cross section

# Soft+Virtual NNLO Approximation (in PIM)

Remember the structure of the NNLO partonic CS

$$d\hat{\sigma}_{\rm NNLO} = D_3 P_3(z) + D_2 P_2(z) + D_1 P_1(z) + D_0 P_0(z) + C_0 \delta(1-z) + R(z)$$

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$$C_0 = A \ln^2 \frac{m_t^2}{M^2} + B \ln \frac{m_t^2}{M^2} + C + \mathcal{O}\left(\frac{m_t^2}{M^2}\right)$$

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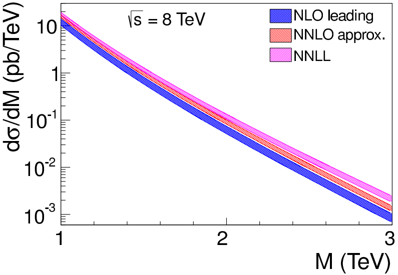
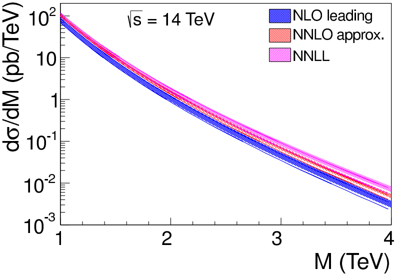
Combine $D_i(M, t_1, m_t, \mu)$ with $\lim_{m_t \to 0} C_0(M, t_1, m_t, \mu)$ to obtain an almost complete soft + virtual NNLO cross section (only positive powers of $m_t$ in $C_0$ missing)

# Soft+Virtual NNLO Impact




- Soft+virtual NNLO approximation produces enhancements of the differential CS compared to approx. NNLO from soft limit alone. The relative enhancement is larger at lower values of $M$ and at higher collider energy.
- At large values of $M$ NNLL soft gluon resummation effects are large and cannot be neglected

# Boosted Tops in 1PI Kinematics

Assume a hierarchy: $s \gg m_t^2 \gg s_4 = (p_4 + k)^2 - m_t^2 \gg \Lambda_{\rm QCD}$

**Soft Limit**

$s_4 \to 0$

Massive hard function $\otimes$ Massive soft function

$$C_{ij} = \mathrm{Tr}\left[\mathbf{H}^m(s,m_t)\mathbf{S}^m(s_4,m_t)\right]$$

**Small Mass Limit**

$m_t \to 0$

Collinear factor $\otimes$ Massless hard function

$$\mathbf{H}^m(s,m_t) = C_D\left(\ln\frac{m_t}{\mu}\right)\mathbf{H}(s)$$

**Small Mass Limit**

$m_t \to 0$

Soft wide angle $\otimes$ Collinear to top $\otimes$ Collinear to antitop

$$\mathbf{S}^m(s_4,m_t) \sim \mathbf{S}\left(\frac{s_4}{\sqrt{s}}\right) S_D\left(\frac{s_4 m_t}{s}\right) S_B\left(\frac{s_4}{m_t}\right)$$

(schematically)

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  $S_D$ scale $\to \mu_d \sim m_t s_4/s$ same as in PIM

  $S_B$ scale $\to \mu_b \sim s_4/m_t$ [Jain, Scimemi, Stewart ('08)]

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- We can built the soft + virtual NNLO approximation

$$
\begin{aligned}
d\hat{\sigma}_{\text{NNLO}} = {} & D_3 P_3(s_4) + D_2 P_2(s_4) + D_1 P_1(s_4) \\
& + D_0 P_0(s_4) + C_0 \delta(s_4) + R(s_4)
\end{aligned}
$$

I) We obtain $\lim_{m_t \to 0} D_i \Longrightarrow$ Check factorization

II) $\lim_{m_t \to 0} C_0 \Longrightarrow$ New information

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- Work is in progress to improve predictions for the boosted top production. A resummation scheme is available for the double resummation of soft and small mass effects in the pair invariant mass distribution as well as for the $p_T$ distribution.
- Future goals:
  - Implement double small-mass and soft-gluon resummation numerically
  - Include electroweak corrections (important at high M)

# Backup Slides

# Dynamical Threshold Enhancement

Does the soft limit provide a good approximation of the exact result?
ex. invariant mass distribution

$$z = \frac{M^2}{s} \qquad \tau = \frac{M^2}{s_{\text{had}}}$$

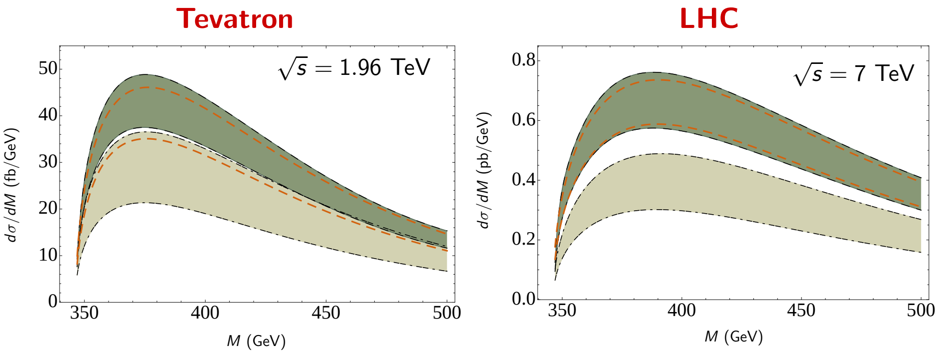
$$\frac{d\sigma}{dM} = \frac{8\pi\beta}{3M}\int_\tau^1 \frac{dz}{z} \sum_{ij=(q\bar{q},gg,\bar{q}q)} L_{ij}\left(\frac{\tau}{z},\mu\right) \underbrace{C_{ij}\left(z,\ldots,\mu\right)}_{\text{partonic cross section}}$$

The limit $z \to 1$ provides a good approximation of the complete result if

a) $\tau \sim 1$; ...but the interesting region is $\tau < 0.3$

b) $L_{ij} \to 0$ for $z \to \tau$; Dynamical Threshold Enhancement

# Dynamical Threshold Enhancement




- Exact NLO result (dark gray band) obtained with MCFM (`Campbell, Ellis`)
- The NLO threshold expansion $\longrightarrow$ band between the dashed lines ($200\,\text{GeV} \leq \mu \leq 800\,\text{GeV}$; close to $M/2 \leq \mu \leq 2M$)
- The threshold expansion agrees quite well with the exact result, even in the low invariant mass region

# Definition of the Soft Function - PIM

In momentum space

$$\mathbf{S}\left(\omega, \frac{t_1}{M^2}, \mu\right) = \frac{1}{d_R} \sum_{X_s} \langle 0|\mathbf{O}_s^\dagger(0)|X_s\rangle\langle X_s|\mathbf{O}_s(0)|0\rangle \delta\left(\omega - (n_1 + n_2)\cdot p_{X_s}\right)$$

with $d_R = N$ for $qbarq$ and $d_R = N^2 - 1$ for $gg$ and

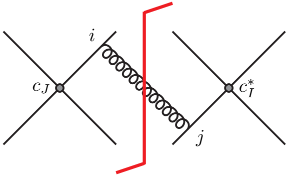
$$\mathbf{O}(x) = [\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3\mathbf{S}_4](x)$$

$\mathbf{S}_i$ are Wilson lines defined as

$$\mathbf{S}_i = \mathcal{P}\exp\left(ig_s \int_{-\infty}^{0} ds\, n_i \cdot A^a(x + sn_i)\mathbf{T}_i^a\right)$$

with $n_i^2 = 0$. For massive legs, $n_i \to v_i$ with $v_i^2 = 1$
The bare functions contain poles $\longrightarrow$ need renormalization

# Soft Function at NLO




Feynman diagram proportional to (using dim. reg. in $d = 4 - 2\epsilon$ dimensions)

$$I_1(\omega, a_{ij}) = \int d^d k \delta(k^2)\theta(k_0) \frac{n_i \cdot n_j}{n_i \cdot k \; n_j \cdot k} \delta(\omega - n_0 \cdot k) = \pi^{1-\epsilon} e^{-\epsilon\gamma_E} \omega^{-1-2\epsilon} \bar{I}_1(a_{ij})$$

$$a_{ij} \equiv 1 - \frac{n_0^2 n_i \cdot n_j}{2 n_0 \cdot n_i n_0 \cdot n_j} \qquad \bar{I}_1(a) = \frac{2 e^{\epsilon\gamma_E} \Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} (1-a)^{-\epsilon} {}_2F_1(-\epsilon, -\epsilon, 1-\epsilon, a)$$

where $n_0 = n_1 + n_2$

Bare soft function matrix obtained by summing over legs and evaluating color factors

$$\left[ S_{\text{bare}}^{(1)} \right]_{IJ} = \frac{2}{\omega} \left( \frac{\mu}{\omega} \right)^{2\epsilon} \sum_{\text{legs}} \langle c_I | \mathbf{T}_i \cdot \mathbf{T}_j | c_J \rangle \bar{I}_1(a_{ij})$$

# Order of the limits

In deriving the factorization formula for boosted top in PIM we

1. took the small mass limit $m_t/M \to 0$
2. took the soft emission limit $z \to 1$

It must be possible invert the order of the limits and to obtain the same result by taking the small mass limit of

$$d\hat{\sigma} = \mathrm{Tr}\left[\mathbf{H}^m \mathbf{S}^m\right]$$

# Order of the limits

Massive/massless amplitudes relation + IR renormalization (argument $t_1$ suppressed)

$$\left|\mathcal{M}(\epsilon, M, m_t, \mu)\right\rangle = Z_{[q]}\left(\epsilon, m_t, mu\right)\left|\mathcal{M}(\epsilon, M, \mu)\right\rangle$$ [Mitov, Moch]

$$\lim_{\epsilon\to 0}\mathbf{Z}_m^{-1}(\epsilon, M, m_t, \mu)\left|\mathcal{M}(\epsilon, M, m_t, \mu)\right\rangle = \left|\mathcal{M}(M, m_t, \mu)\right\rangle$$ [Becher, Neubert]

$$\lim_{\epsilon\to 0}\mathbf{Z}^{-1}(\epsilon, M, \mu)\left|\mathcal{M}(\epsilon, M, m_t, \mu)\right\rangle = \left|\mathcal{M}(M, \mu)\right\rangle$$ [Catani Becher, Neubert]

Combine to get a relation between $\mathbf{Z}$ factors and finite matching function $f$

$$Z_{[q]}\left(\epsilon, m_t, mu\right)\mathbf{Z}_m^{-1}(\epsilon, M, \mu) = f(m_t, \mu)\mathbf{Z}^{-1}(\epsilon, M, m_t, \mu)$$

**Soft factorization of the hard function**

$$\mathbf{H}_{ij}^{m}(M, m_t, \mu) = f^2(m_t, \mu)\mathbf{H}_{ij}(M, m_t, \mu)$$

We checked that it works to NLO, and to NNLO for $\mu$ dependent terms.

# Order of the limits

- In order to match the factorization formula we obtained starting from the small mass limit we should find

$$\mathbf{S}_{ij}^{m}\left(\sqrt{s}(1-z), m_t, \mu\right) = \mathbf{S}_{ij}\left(\sqrt{s}(1-z), \mu\right) \otimes \frac{C_D(m_t,\mu) S_D(m_t(1-z),\mu)}{f(m_t,\mu)} \otimes \frac{C_D(m_t,\mu) S_D(m_t(1-z),\mu)}{f(m_t,\mu)}$$

- The soft function is related to real emission. All real emission in the fragmentation function is associated to $S_D$: One would expect

$$f(m_t,\mu) = C_D(m_t,\mu)$$

- Instead we found difference at $\mathrm{N}^3\mathrm{LL}$

$$f(m_t,\mu) = C_D(m_t,\mu) - \left(\frac{\alpha_s}{4\pi}\right)^2 4\pi^2 C_A C_F$$

We have an (annoying) mismatch between $f$ and $C_D$ (or between $S_D$ and the shape function)!