Solutions for the Optics Tutorial

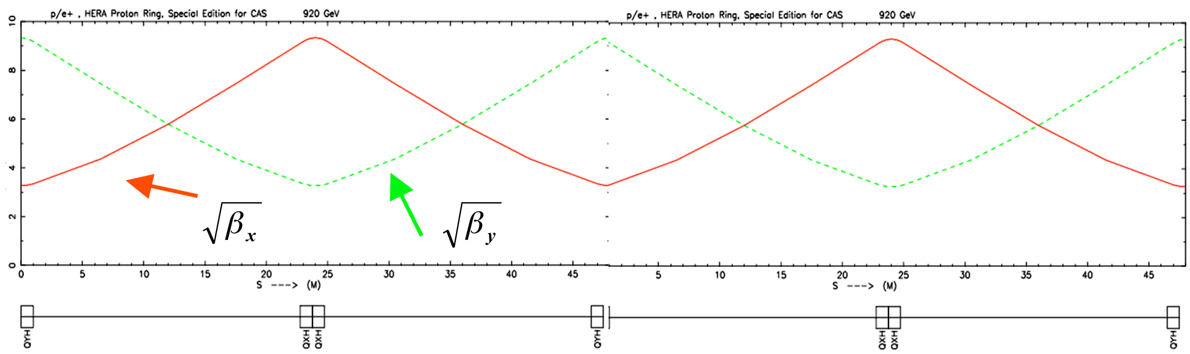
CAS_Prag Lectures 2014,
B. Holzer, CERN

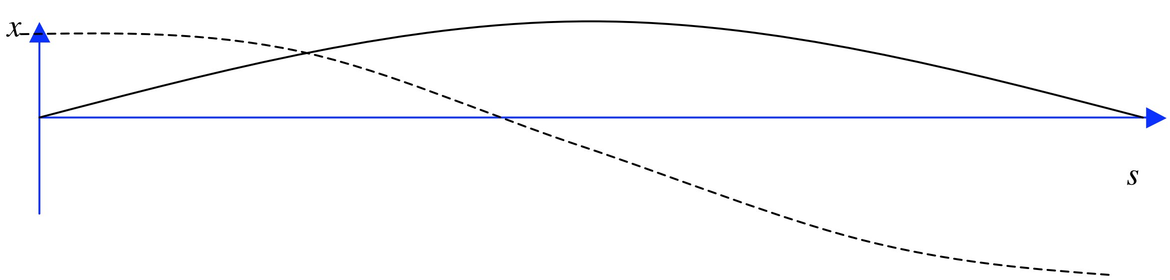
1.) Can you explain in your own words the meaning of …

phase advance
beam emittance
β-function

Concerning the two parameters β-function and beam emittance: they both determine the beam envelope. Can you explain the difference ?

Consider the FODO structure in the figure: The phase advance per cell is 90 degrees and the red curve is the horizontal, the green line the vertical beta function. . Can you draw in the plot below two particle trajectories, that propagate through the two cells, one starting with x=0, x´>0 and the other one starting with x>0, x´=0 ?





Assume somewhere in the storage ring there is a position where α = 0.
Where would such a situation occur typically?
α=0 means that the beta function has an extreme value: maximum or minimum.
Such a situation occurs in general in the center of quadrupoles (in a pure FoDo or similar structure this is always the case) or in the middle of a beam waist.

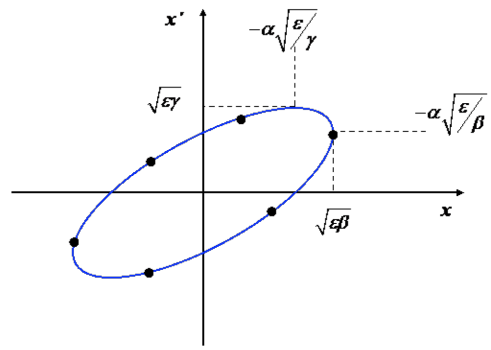
How will the phase space ellipse look like?

Depending on the beta (maximum or minimum) the ellipse will be flat in the horizontal plane or upright.

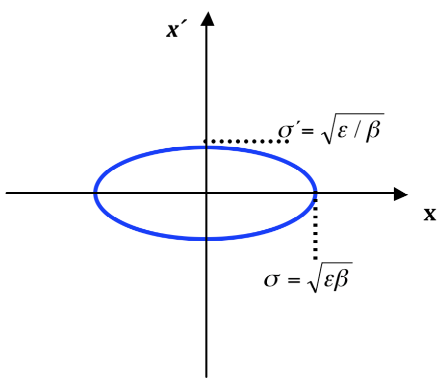
General case:

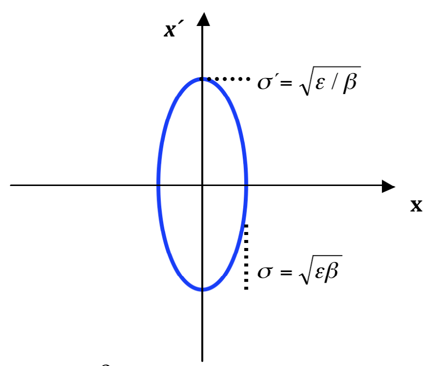
remember that

$$\gamma = \frac{1+\alpha^2}{\beta}$$



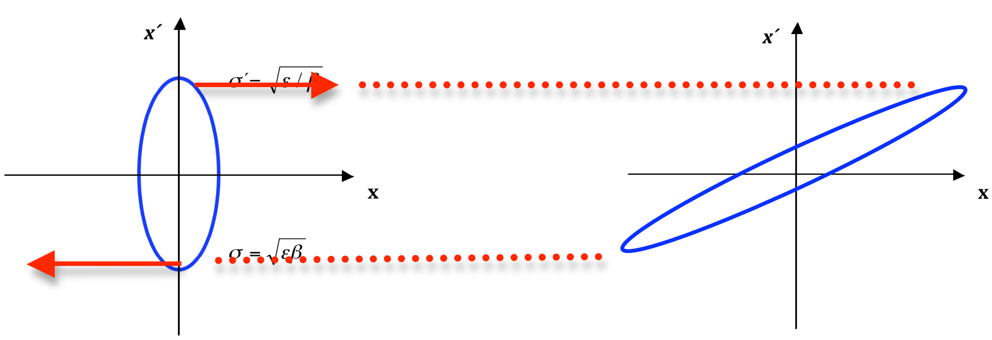
If α = 0 we get the following picture: The beam will need the largest aperture if the ellipse is flat, or it will have the largest divergence.





Can you give a physical interpretation of the β function at such a place ?

Starting from a mini-beta insertion with α=0, how does the phase space picture develop in the long drift space ?



2.) Beam rigidity & particle momentum

… or the stupid problem: after all we have to deal with a relativistic beam!

A synchrotron of 25m radius accelerates protons from a kinetic energy of 50 MeV to 1000 MeV. What is the maximum energy of a deuteron beam (Z=1, A=2) that could be accelerated in the machine ?

The beam rigidity relates the magnetic field to the particle momentum:

$$B*\rho = \frac{p}{e} = 3.333*p(GeV/c)$$

The momentum is given just by the magnetic dipole field and the bending radius – independent of the particle that is stored in the machine !
Calculation of the momentum of a 1000 MeV proton beam:

$$p^2c^2 = E^2 - E_0^2$$

kinetic energy: $T = 1000MeV$
rest energy of a proton: $E_0 = m_0c^2 = 938MeV$
overall energy = rest energy + kin. energy: $E = m_0c^2 + T$

$$p^2c^2 = (m_0c^2 + T)^2 - (m_0c^2)^2$$

$$p^2c^2 = (0.938GeV + 1.0GeV)^2 - (0.938GeV)^2$$
$$pc = \sqrt{2.876}\ GeV$$

This is the maximum momentum that can be carried by the machine.
To calculate the kinetic energy for the deuteron we set:

$$p^2c^2 = (T_{deut} + 0.938GeV + 0.939GeV)^2 - (0.938GeV + 0.939GeV)^2 = 2.876GeV^2$$

and solve for T_{deut}

$$T_{deut} = 0.653\ GeV$$

Nota bene: *We do not need the bending radius to obtain this result. But we could use it to calculate the magnetic field B that we need in this machine:*

$$B*\rho = 3.33*p\left[GeV/c\right]$$
$$= 3.33*\sqrt{2.876}$$

And as $\rho = 25m$ we get $B = 0.22T$.

3.) LHC: particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN will collide proton beams with a maximum momentum of $p = 7\ TeV/c$ *per beam.*
The main parameters of this machine are:

Circumference	$C_0 = 26658.9m$	
particle momentum	$p = 7\ TeV/c$	
main dipoles	$B = 8.392\ T$	$l_B = 14.2m$
main quadrupoles	$G = 235\ T/m$	$l_q = 5.5m$

Calculate the magnetic rigidity of the design beam and the bending radius of the main dipole magnets in the arc…

The beam rigidity is obtained in the usual way by the golden rule:

$$B * \rho = \frac{p}{e} = \frac{1}{0.299792} * p[GeV/c]$$
$$= \frac{1}{0.299792} * 7000\ Tm = 23350\ Tm$$

and knowing the magnetic dipole field we get

$$\rho = \frac{7000\ Tm}{8.392T * 0.299792} = 2780m$$

… and determine the number of dipoles that is needed in the machine.

The bending angle for one LHC dipole magnet:

$$\alpha = \frac{l}{\rho} = \frac{14.2m}{2780m} = 5.108\ mrad$$

and as we want to have a closed storage ring we require an overall bending angle of 2π:

$$N = \frac{2\pi}{\alpha} = \frac{6.28rad}{5.108mrad} = 1230\ Magnets$$

Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens ?

We can use the beam rigidity (or the particle momentum) to calculate the normalized quadrupole strength:

$$k = \frac{g}{B*\rho} = \frac{g}{p/e} = 0.299792 * \frac{g}{p[GeV/c]}$$

$$k = 0.299792 * \frac{235\, T/m}{7000\, [GeV/c]} = 0.01 \frac{1}{m^2}$$

$$f = \frac{1}{k*l_q} = \frac{1}{0.01/m^2 * 5.5m} = 18.2m \quad > \quad l_q$$

The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

How does the matrix for such a (foc.) magnet look like?
How would you establish a description of this magnet in thin lens approximation?
Compare the matrix elements.
Nota bene: in our notation a foc. magnet has a negative k-value.

The matrix of a focusing quadrupole is given by

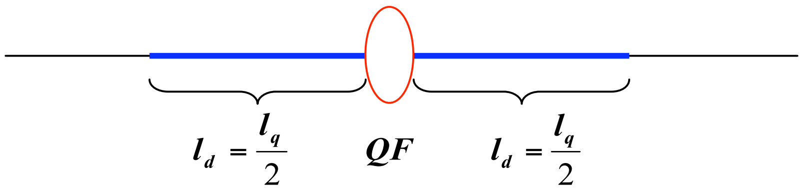
$$M_{QF} = \begin{pmatrix} \cos(\sqrt{|k|}*l) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}*l) \\ -\sqrt{|k|}*\sin(\sqrt{|k|}*l) & \cos(\sqrt{|k|}*l) \end{pmatrix}$$

$$M_{QF} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \dfrac{-1}{|f|} & 1 \end{pmatrix} \quad \text{with the focal length } f = \frac{1}{|kl_q|} = \frac{1}{0.01/m^2 * 5.5m} = 18.2m$$

But we should not forget the overall length of the beast: The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



So we write:

$$M_{thinlens} = \begin{pmatrix} 1 & l_q/2 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_q/2 \\ 0 & 1 \end{pmatrix}$$

Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2} * k * l_q & \frac{l_q}{2} * (2 + k * l_q * \frac{l_q}{2}) \\ k * l_q & 1 + k * l_q \end{pmatrix}$$

With the parameters in the example we get finally

$$\left.\begin{aligned} &\frac{l_q}{2} = 2.75\, m, \\ &k * l_q = \frac{-1}{|f|} = \frac{-1}{18.2m} = -0.0549 \frac{1}{m} \end{aligned}\right\} \qquad M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

which is still quite close to the result of the exact calculation above.

4.) *Questions to be discussed in the evening, having a good beer*

a.) Can you explain in your own words the meaning of …
dispersion
chromaticity

b.) The largest contribution to the chromaticity ξ in a storage ring is – due to the high β -values and strong quadrupole strengths the interaction region.
Would it be a good idea to install sextupole magnets there to compensate ξ locally ?

In principle yes. It turns out that a local compensation is always the best that we can do. But unfortunately we have to get rid of the dispersion at the interaction regions. Dispersive orbits would spoil the luminosity and lead to large background contributions. Therefore (nearly) always the dispersion will be zero close to the interaction point. And a sextupole magnet located there would be useless.

5.) *Apertures and Beam Envelopes:*

The LHC magnet structure in the arcs consists of a symmetric FoDo with 90° phase advance per cell and an aperture radius of r_0 =20mm.
a.) Given the value of β_{max} = 500m in a QF quadrupole lens, what beam emittance would just touch the vacuum chamber ? (We call this value the "acceptance" of the machine).

$$\sigma_{\max} = \sqrt{\varepsilon_{\max}\beta} = r_0 = 20\,mm$$

$$\varepsilon_{\max} = \frac{\boldsymbol{r}_0^2}{\beta} = \frac{400\boldsymbol{mm}^2}{500\boldsymbol{m}} = 8*10^{-7}\,\boldsymbol{rad\ m}$$

b.) If the typical emittance of a stored beam at 450 GeV injection energy is $\varepsilon \approx 7*10^{-9}$ rad m, how many σ of beam envelope fit into the vacuum chamber for β=500m ?

$$\sigma_{\boldsymbol{beam}} = \sqrt{\varepsilon_{\boldsymbol{typical}} * \beta} = \sqrt{7*10^{-9}\,\boldsymbol{radm} * 500\boldsymbol{m}}$$

$$\sigma_{\boldsymbol{beam}} = 1.9\boldsymbol{mm} \qquad \rightarrow \qquad \boldsymbol{r}_0 \approx 10\sigma$$

the vacuum aperture corresponds to 10 sigma of the stored beam ... indeed there is not much space for errors and distortions.

c) what will happen if – keeping the beam optics constant – you accelerate the beam to an energy of E = 7000 GeV ?

The beam emittance will shrink as a function of the relativistic parameters β*γ. In the given energy range $\beta \approx 1$ and so we can simplify the scaling and set

$$\varepsilon_{7000} = \varepsilon_{450} * \frac{\gamma_{450}}{\gamma_{7000}} = 4.5*10^{-10}$$

The beam dimension will shrink (in both planes) and the beam lifetime and background rates will improve.

During luminosity operation at this energy we require at least 14 sigma aperture due to background and quench safety reasons. What is the maximum beta function that can be accepted if the aperture of our mini beta quadrupoles is 20mm?

$$\boldsymbol{r}_0 = 14*\sqrt{\varepsilon\beta} \quad \rightarrow \frac{\boldsymbol{r}_0^{\,2}}{196} = 4.5*10^{-10} * \beta_{\max}$$

$$\beta_{\max} = \frac{400*10^{-6}\,\boldsymbol{m}^2}{196*4.5*10^{-10}\,\boldsymbol{m}} = 4.5\boldsymbol{km}$$

6.) Question just for the fun of it:
Beam rigidity and – well a little bit of special relativity

Let´s build a real cheap storage ring. Just put it to the north pole and use the magnetic field of the earth whose field lines are perpendicular to the surface at that nice place.
Forget about focusing … that's for nitpickers.
What will be the size of the ring for a 10 keV electron beam if the earth magnetic field is about 0.5 Gauß ?

As we all know it is the momentum that defines the magnetic field:

$$\boldsymbol{B} * \rho = \frac{\boldsymbol{p}}{\boldsymbol{e}}$$

So we have to calculate the momentum of the electron beam first and – as it is neither ultra relativistic nor in the classical energy regime – we have to apply the full relativistic stuff.

Overall energy of a particle: $\boldsymbol{E} = \sqrt{\boldsymbol{p}^2\boldsymbol{c}^2 + \boldsymbol{m}^2\boldsymbol{c}^4}$

$$\left.\begin{array}{ll} \text{rest energy of an electron:} & \boldsymbol{E}_0 = \boldsymbol{mc}^2 = 511\ \boldsymbol{keV} \\[2ex] \text{kinetic energy of the beam:} & \boldsymbol{E}_{kin} = 10\ \boldsymbol{keV} \end{array}\right\} \quad \boldsymbol{E}_{total} = 5.21*10^4\,\boldsymbol{eV}$$

calculation of the momentum:

$$p^2c^2 = E_{total}^2 - m^2c^4$$

$$\rightarrow \qquad \frac{p}{e} = \frac{1.02*10^5 \, eV}{2.99792*10^8 \, m/s * e} = 3.4*10^{-4} \, \frac{Vs}{m}$$

$$p = \frac{\sqrt{E_{total}^2 - m^2c^4}}{c} = \frac{\sqrt{(5.21*10^5 \, eV)^2 - (5.11*10^5 \, eV)^2}}{2.99792*10^8 \, m/s}$$

$$Tm$$

bending radius:

$$\rho = \frac{p/e}{B} = \frac{3.4*10^{-4} \, Tm}{5*10^{-5} \, T}$$

$$\rightarrow \rho = 6.8 \, m$$

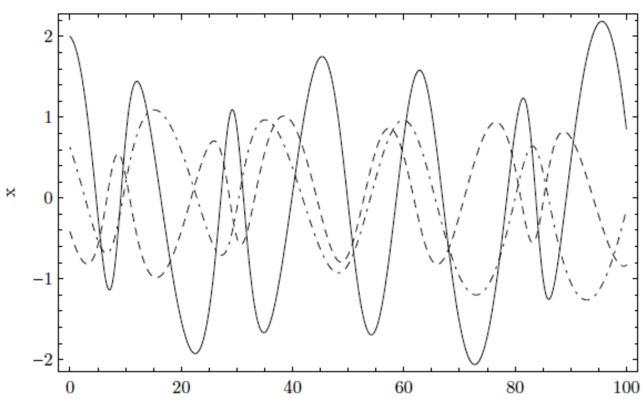
It is astonishing: The storage ring is very small, or in other words: the magnetic field of the earth is quite strong. Indeed, even in HERA we had to compensate for it.

And now for the beer:

10 keV is similar to the energy in our conventional TV screens. So - if all these considerations are true - given a length of 30cm (for the distance between the TV gun and the screen) the displacement at the screen due to the earth magnetic field is a few millimetres.
So … turning the TV screen around the colours should change !
Do they ?? Or is that all nonsense ???

7.) Particle Trajectories:

The following plot represents the trajectories of three particles traveling in the the same non-dispersive transfer line.



Among the three particles only one is significantly off-momentum. Which one (full, dashed or dot-dashed line)? Is its rigidity higher or lower than the on-momentum particles? Justify the answer.