The CERN Accelerator School

Introduction to Free Electron Lasers

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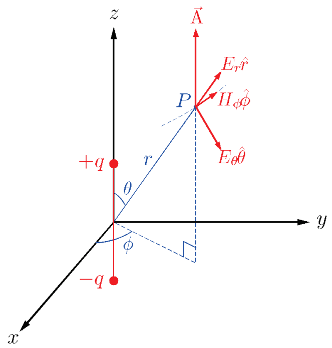
Free electron lasers (FELs) are sources of synchrotron radiation that can produce radiation with peak power and brightness orders of magnitude larger than the radiation produced by conventional sources such as dipoles and wigglers.

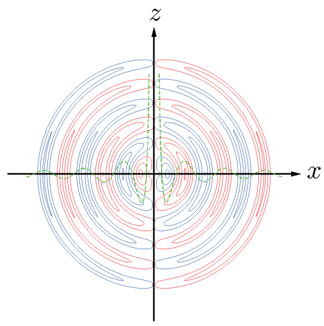
In this lecture, we shall:

- consider ways to enhance the intensity of radiation from undulators (developing the principles behind FELs);
- discuss some of the different types of free electron laser.

Radiation from an Oscillating Electric Charge

Consider a single oscillating electron. The amplitude of the electric and magnetic fields in the radiation produced by the electron are proportional to the charge on the electron.





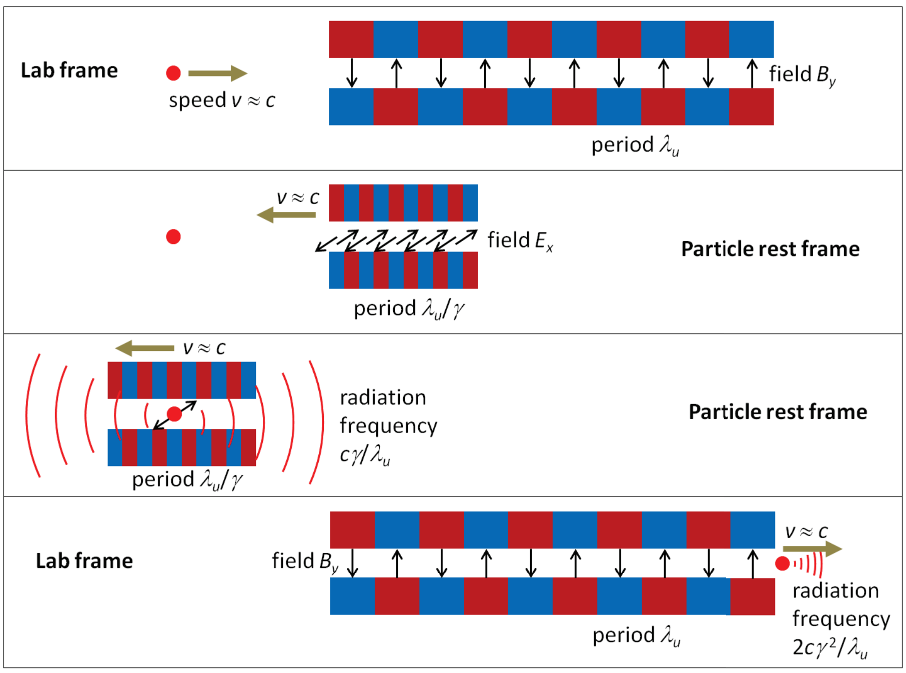
The total radiation power from a Hertzian dipole is:

$$P = \frac{ck^4 l^2 q^2}{12\pi\epsilon_0}. \tag{1}$$

We might expect the radiation power from an undulator to increase with the square of the bunch charge: but in practice, the power increases linearly with the charge. Why?

Synchrotron Radiation from an Undulator

Recall that synchrotron radiation is produced from an undulator by the oscillations induced on electrons passing through the undulator.



Consider a bunch of ultra-relativistic electrons with relativistic factor γ, passing through an undulator. In the lab frame, the bunch length is σ_z, and the undulator period is λ_u.

In the rest frame of the electrons, the bunch length is $\gamma\sigma_z$, and the undulator period is λ_u/γ.

Typically, σ_z is a few mm, and λ_u is a few cm; but if γ is large:

$$\sigma_z \gg \frac{\lambda_u}{\gamma^2}. \qquad (2)$$

In other words, in the rest frame of the electrons, the bunch length is much larger than the undulator period.

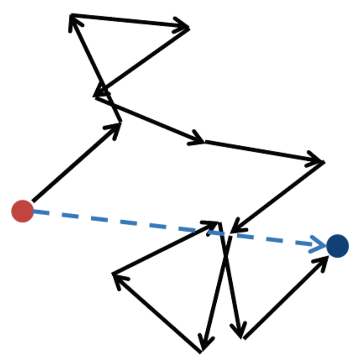
Therefore, in the rest frame of the electrons, the electrons are oscillating at different (random) phases.

The total electric field is the sum of the fields from all the electrons:

$$E_{\text{total}} = \sum_{n=1}^{N} E_0 e^{i\phi_n}, \qquad (3)$$

where E_0 is the field due to a single electron, and ϕ_n is the phase of the electric field from the n^{th} electron.

Since the electrons are oscillating at random phases, summing the fields is equivalent to a random walk in the complex plane.



Incoherent Radiation

Since the radiation is produced from particles oscillating at random phases, the generation is said to be *incoherent*.

In that case:

$$|E_{\text{total}}| \approx \sqrt{N} E_0. \quad (4)$$

For a large number of electrons N, the total *field strength* is proportional to the *square root* of the number of electrons.

Since the energy carried by an electromagnetic wave is proportional to the square of the field amplitude, the *power* of "incoherent" synchrotron radiation from an undulator is proportional to the current in the undulator.

Coherent Radiation

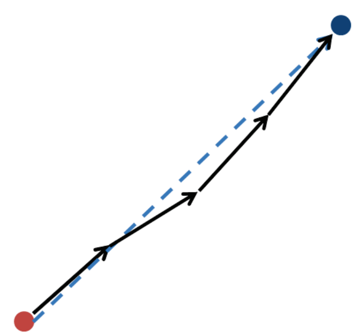
If all the electrons are oscillating in phase (or with phase difference strictly related to the distance between the electrons), then the fields in the synchrotron radiation add coherently. This situation can be represented by:

$$\phi_n \approx \phi_0, \quad (5)$$

where ϕ_0 is a constant. Then:

$$|E_{\text{total}}| \approx N E_0. \quad (6)$$

The total field is proportional to the number of electrons, and the radiation power will be proportional to the square of the number of electrons. The radiation is emitted coherently. This can occur if the bunch length is less than the radiation wavelength.



Coherent Radiation from an Undulator

Since the number of electrons in a bunch can be very large, the enhancement of radiation power from coherent emission compared to incoherent emission is potentially dramatic.

An undulator produces radiation at a wavelength:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \qquad \text{where} \qquad K = \frac{eB_u\lambda_u}{2\pi m_e c}. \tag{7}$$

The condition for coherent radiation can be written:

$$\sigma_z \ll \lambda_u/\gamma^2 \approx \lambda. \tag{8}$$

This condition can be difficult to achieve; however, one effect of the interaction between electrons and radiation with an undulator can be the development of microbunches within a beam.

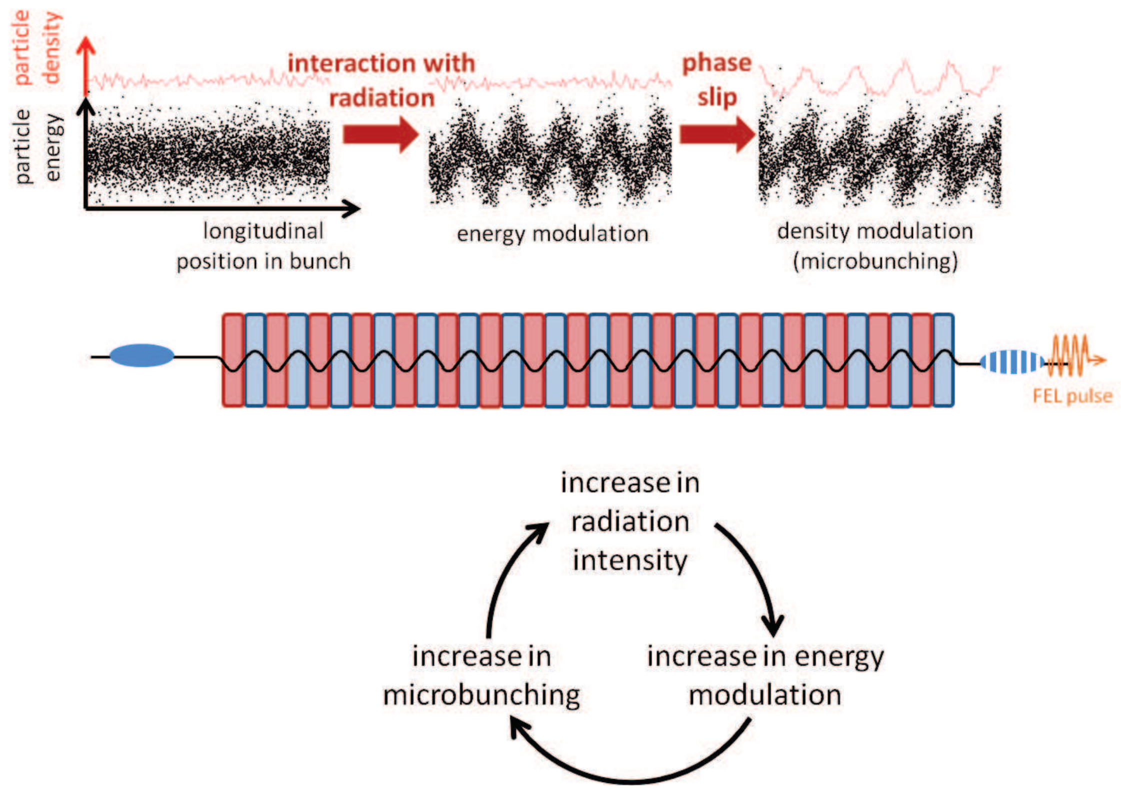
If the size of each microbunch is less than a radiation wavelength, then each microbunch can radiate coherently.

Coherent Radiation from an Undulator

Electrons in a bunch passing through an undulator can interact with the radiation produced by other electrons within the bunch. The forces on the electrons from the radiation field leads to a transfer of energy between the electrons and the radiation.

- In the *low-gain regime*, the radiation intensity can be treated as approximately constant. There is some transfer of energy between the electrons and the radiation, but the main effect is a change in the motion of the electrons so that "microbunches" start to develop, with length of order of the radiation wavelength.
- In the *high-gain regime*, the microbunching becomes strong enough that coherent radiation gives a rapid increase in radiation intensity. This in turn enhances the microbunching: the result is a rapid exponential increase in the radiation intensity with distance along the undulator.

Coherent Radiation from an Undulator



Types of FEL

FELs can be categorised according to how the microbunching is achieved. For example:

- In a *resonator* FEL, the incoherent radiation produced in an undulator is trapped within an optical cavity. Each electron bunch passing through the undulator adds to the radiation intensity, which leads to an increase in the rate at which microbunching takes place.
- In a *seeded amplifier* FEL, a radiation pulse (e.g. from a laser) is co-propagated with an electron bunch in an undulator. This initiates microbunching, which then develops rapidly along the undulator as the electrons within each microbunch radiate coherently.
- A *SASE* (Self-Amplified Spontaneous Emission) FEL works in a similar way to a seeded amplifier, except that the initial microbunching is initiated by the spontaneous radiation within the undulator, rather than from an external "seed".

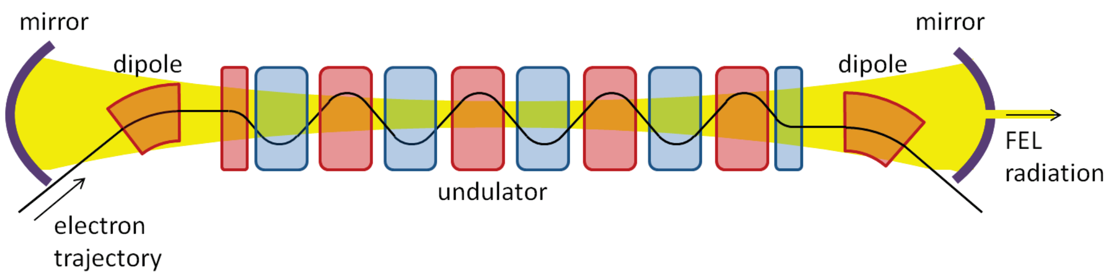
Types of FEL

Each type of FEL has certain advantages and disadvantages, and can be suitable for meeting different user requirements.

For example, the lack of suitable materials to build x-ray mirrors, and the lack of suitable sources for seeding an x-ray FEL means that x-ray FELs have to work on the SASE principle.

But since the FEL pulse effectively grows from "noise" (i.e. small random perturbations in the bunch density), it can be difficult to control the output from a SASE FEL. One feature of this is that SASE FELs lack temporal coherence, i.e. the power (and wavelength) within a single FEL pulse can fluctuate significantly over the duration of the pulse.

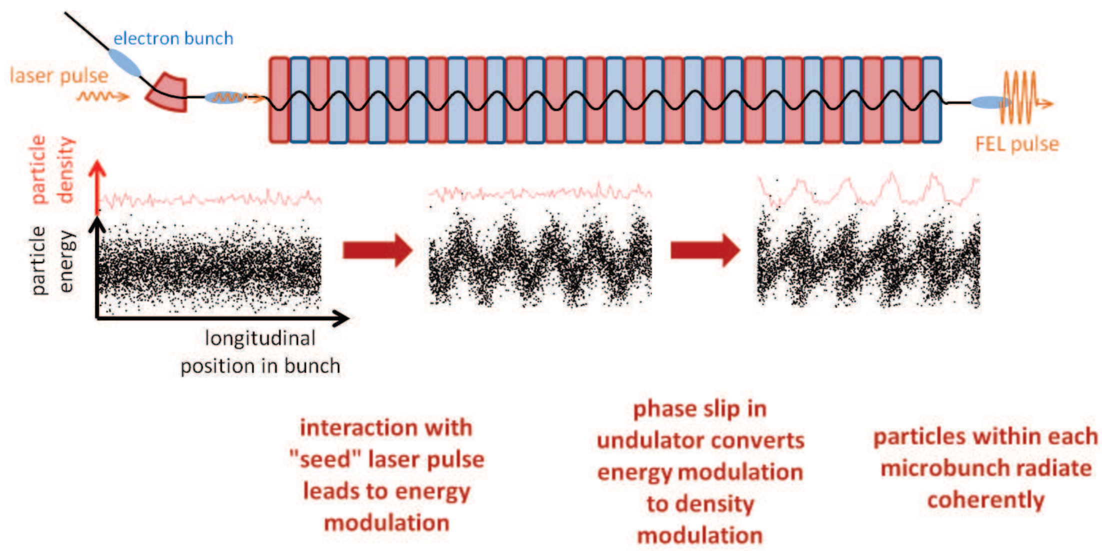
A Resonator FEL



In a resonator FEL, an undulator is placed between two mirrors that reflect and focus the synchrotron radiation produced by the undulator. Electrons are steered onto a trajectory along the undulator using dipoles at either end.

Successive electron bunches interact with the radiation stored in the undulator, leading to an increase in the radiation power, and to microbunching in the electron bunches. Some of the radiation can be extracted using an out-coupling hole in one of the mirrors.

A Seeded Amplifier FEL



A SASE FEL: Euro XFEL

A SASE FEL typically consists of a long linac, feeding one or more long undulators...

The European XFEL, presently under construction near Hamburg, is scheduled to start user operation in 2017.

Total length	3.4 km
Electron beam energy	17.5 GeV
Radiation wavelength	0.05 - 6 nm
X-ray pulse length	<100 fs
Peak brilliance	5×10^{33} γ/s/mm^2/mrad^2/0.1% bw

The Physics of Free Electron Lasers

To understand the properties of FELs, it is necessary to solve the equations of motion for particles in the combined fields of the undulator and the radiation, and the equations describing the radiation.

Fundamentally, the equations describing the motion of the electrons and the radiation are just Maxwell's equations, together with the Lorentz force equation.

But since the equations need to be solved self-consistently (i.e. taking into account simultaneously the changes in the particle distribution and the radiation) the full solutions become mathematically very complicated.

Low-Gain Regime: the Pendulum Equations

To illustrate the mathematical analysis of an FEL, we shall consider the low-gain regime. This is defined by the assumption that the intensity of the radiation is approximately constant.

The trajectory of an electron in the undulator is given by:

$$\frac{d^2x}{dz^2} = -\frac{B_u}{B\rho}\sin(k_u z) \quad \therefore \quad x = \frac{B_u}{k_u^2 B\rho}\sin(k_u z) = \frac{K}{\gamma k_u}\sin(k_u z), \tag{9}$$

where x is the horizontal transverse position with respect to the axis of the undulator, and z is the longitudinal distance along the axis of the undulator.

Now suppose that there is an electromagnetic wave also travelling through the undulator, with electric field given by:

$$E_x = E_0 \cos(kz - \omega t + \phi), \tag{10}$$

where $\omega/k = c$.

The force on the electron from the electric field leads to a change in energy of the electron.

The rate of change of the energy of the electron is given by:

$$mc^2\frac{d\gamma}{dt} = eE_x\frac{dx}{dt} = \frac{ecE_0K}{\gamma}\cos(k_uz)\cos(kz-\omega t+\phi), \quad (11)$$

$$= \frac{ecE_0K}{2\gamma}\left[\cos(\theta+\phi)+\cos(\tilde{\theta}+\phi)\right], \quad (12)$$

where the *ponderomotive* phase θ is given by:

$$\theta = (k+k_u)z-\omega t, \quad (13)$$

and:

$$\tilde{\theta} = (k-k_u)z-\omega t. \quad (14)$$

We shall manipulate equation (12), to derive equations describing how the energy γ and ponderomotive phase θ vary as a function of distance z along the undulator.

Our immediate goal is to eliminate the time t from equation (12). This will give us an equation describing how the energy of a particle varies with position along the undulator.

The change of energy is a result of the interaction with the radiation field in the undulator. Hence, we will be able to calculate the change in energy of the radiation that results from the interaction with the particles in the electron beam.

To begin, we write an expression for the position z of a particle along the undulator at time t:

$$z = v_zt+z_0, \quad (15)$$

where z_0 is a constant. The longitudinal velocity v_z varies along the undulator because of the oscillation in the magnetic field of the undulator. For an ultra-relativistic particle:

$$v_z \approx \left(1-\frac{1}{2\gamma^2}\right)c - \frac{cK^2}{2\gamma^2}\Big(1+\cos(2k_uz)\Big). \quad (16)$$

The average longitudinal velocity is:

$$\bar{v}_z = \left(1 - \frac{1 + K^2/2}{2\gamma^2}\right) c. \tag{17}$$

Using:

$$\frac{dz}{dt} \approx \bar{v}_z, \tag{18}$$

we find from equations (13) and (14):

$$\frac{d\theta}{dz} = k_u - \left(\frac{1 + K^2/2}{2\gamma^2}\right) k, \tag{19}$$

and:

$$\frac{d\tilde{\theta}}{dz} = -k_u - \left(\frac{1 + K^2/2}{2\gamma^2}\right) k. \tag{20}$$

Let us suppose that the radiation in the undulator has wavelength $\lambda = 2\pi/k$, where:

$$\lambda = \left(\frac{1 + K^2/2}{2\gamma_r^2}\right) \lambda_u, \tag{21}$$

for some particular value of γ_r. If the electrons have energy such that $\gamma \approx \gamma_r$, then λ will be equal to the wavelength of the synchrotron radiation produced by the electrons in the undulator.

In that case:

$$\frac{d\theta}{dz} \ll k_u, \quad \text{and} \quad \frac{d\tilde{\theta}}{dz} \approx -2k_u. \tag{22}$$

The ponderomotive phase θ varies slowly as an electron moves along the undulator, whereas the phase $\tilde{\theta}$ varies rapidly. As a result, over several periods of the undulator, the term in $\tilde{\theta}$ on the right hand side of equation (12) averages to zero, while the term in θ can lead to some significant change in γ.

The final step is to introduce the variable η to describe the energy of the electron:

$$\eta = \frac{\gamma - \gamma_r}{\gamma_r}. \tag{23}$$

Assuming that η is small (i.e. $\gamma \approx \gamma_r$), equations (19) and (12) become respectively:

$$\frac{d\theta}{dz} = 2k_u\eta, \tag{24}$$

and:

$$\frac{d\eta}{dz} = -\frac{eE_0\tilde{K}}{2\gamma_r^2 mc^2}\sin(\theta). \tag{25}$$

Equations (24) and (25) take the same form as the equations of motion for a pendulum: they are therefore known as the *pendulum equations*.

Note that equation (25) is written in terms of the modified undulator parameter $\tilde{K}$, defined by:

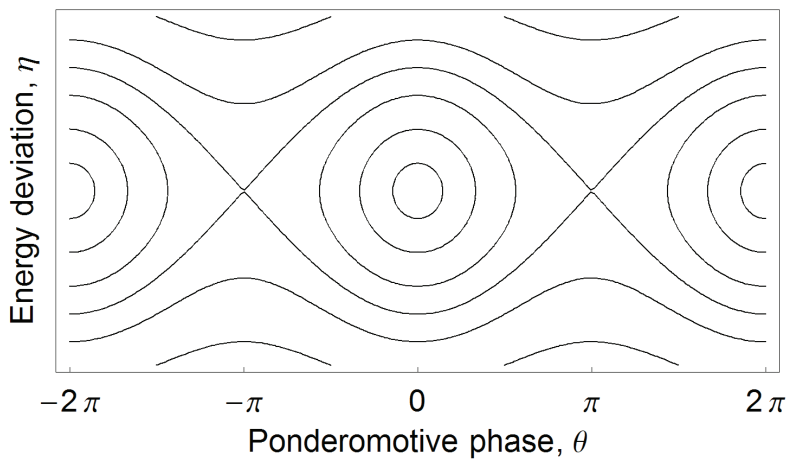
$$\tilde{K} = K\left[J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)\right], \tag{26}$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions.

This takes into account the modulation of the longitudinal component of the velocity of the electrons, which affects the coupling between the electrons and the radiation.

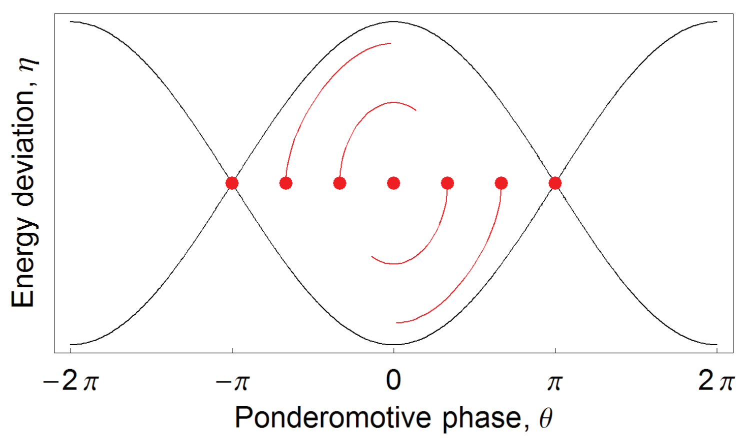
For $K = 1$, the modified undulator parameter is $\tilde{K} \approx 0.91$.

The dynamics within an FEL in the low-gain regime can be represented on a plot of η versus θ. Note that θ represents the phase of the energy transfer between a particle and the radiation. As an electron moves along the undulator, it traces a line in the plot of η versus θ.



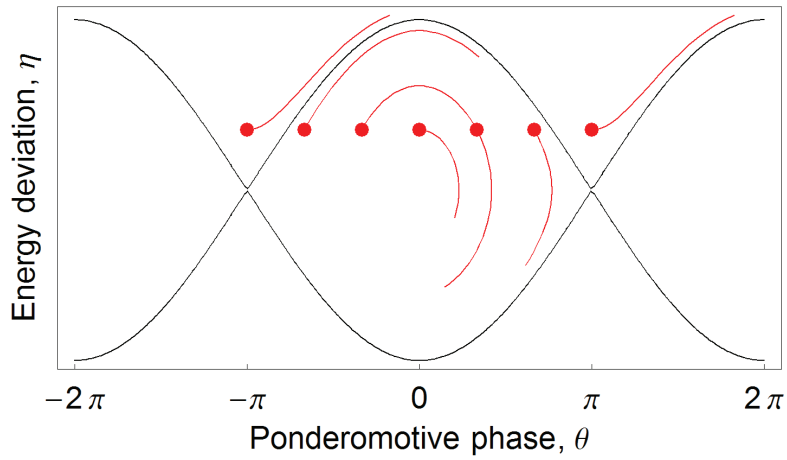
Consider a bunch of particles with initial energy $\eta = 0$ (i.e. with $\gamma = \gamma_r$). Since a typical bunch length is much larger than the radiation wavelength, we can assume that the particles cover the whole range of θ from $-\pi$ to π.

As the particles move along the undulator, some of the particles gain energy from the radiation; others lose energy to the radiation. Overall, the net energy transfer between the bunch and the radiation is zero.



Low-Gain Regime: the Pendulum Equations

Now consider what happens if $\eta > 0$, i.e. $\gamma > \gamma_r$. There are still some particles that gain energy, while others lose energy. However, the symmetry is broken: there is now a net energy transfer from the bunch to the radiation.



Low-Gain Regime: the Gain Equation

Consider a pulse of radiation with some frequency ω that propagates along the undulator with a bunch of electrons that have energy such that $\gamma = \gamma_r$.

Using the description (developed in the previous slides) of the interaction between the radiation and the electrons, it is possible to show that the intensity of the radiation changes by a factor $(1 + G(\xi))$, where:

$$G(\xi) = -\frac{\pi^2 r_e \tilde{K}^2 L_u^3 n_e}{\gamma_r^3 \lambda_u} g(\xi), \qquad (27)$$

where $r_e = e^2/4\pi\epsilon_0 m_e c^2$ is the classical radius of the electron, n_e is the number of electrons per unit volume in the bunch, and L_u is the total length of the undulator.

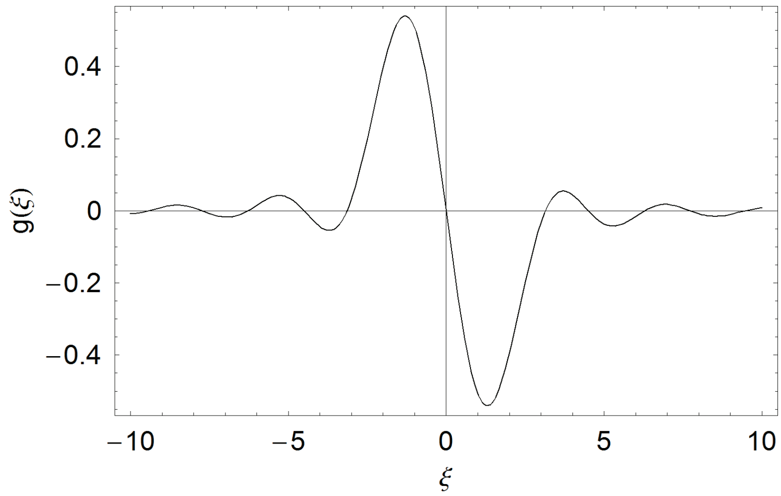
Note that the gain depends on the degree of overlap between the electron beam and the radiation beam: equation (27) assumes an "optimal" overlap.

Low-Gain Regime: Madey's Theorem

The gain function $g(\xi)$ and the dimensionless parameter ξ are defined by:

$$g(\xi) = \frac{d}{d\xi}\left(\frac{\sin^2(\xi)}{\xi^2}\right), \qquad \xi = \pi N_u \frac{\omega_r - \omega}{\omega_r}. \tag{28}$$

ω_r is the frequency of the "spontaneous" synchrotron radiation produced in the undulator by electrons with $\gamma = \gamma_r$.



Low-Gain Regime: Madey's Theorem

Equation (27) allows us to work out the gain (in the low-gain regime) of an oscillator FEL, given the electron beam and undulator parameters.

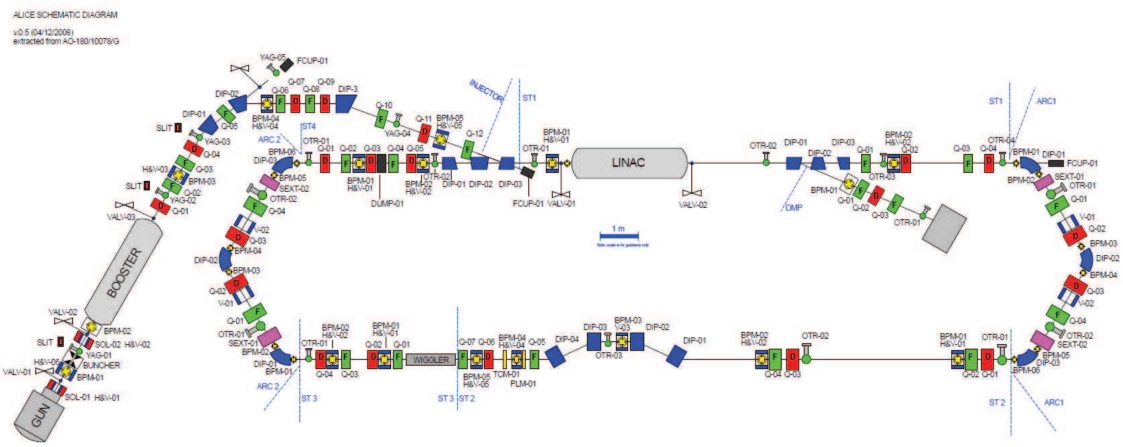
The dependence of the gain on the wavelength is contained within the function $g(\xi)$: note that this is the derivative of the function $\text{sinc}^2(\xi)$ that describes the line width of the spontaneous radiation. This relationship between the FEL gain (in the low-gain regime) and the line width of the spontaneous undulator radiation is known as *Madey's theorem*.

Note that for given electron energy, the gain is a function (through the parameter ξ) of the radiation wavelength. For some wavelengths, the gain is positive; for others, it is negative.

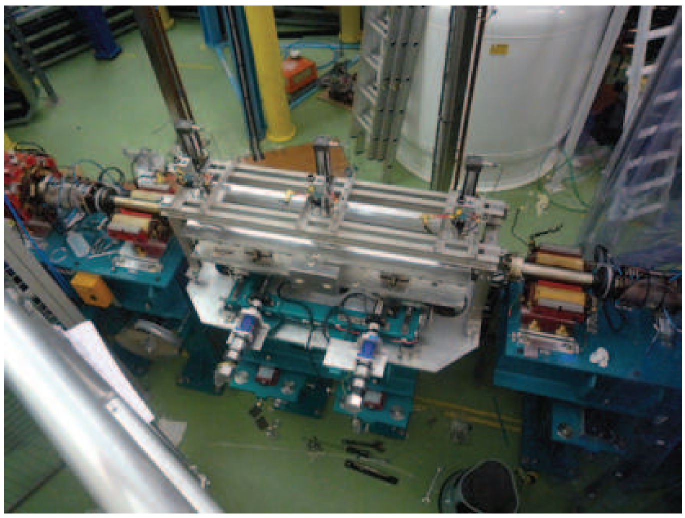
In practice, the gain function determines the frequency and bandwidth of light from a low-gain oscillator FEL.

Example of a Resonator FEL: the ALICE IR FEL

As an example of a resonator FEL, consider the IR FEL on the ALICE accelerator test facility at Daresbury Laboratory, UK.



Example of a Resonator FEL: the ALICE IR FEL



Beam energy	26.5 MeV
Bunch charge	80 pC
Bunch length	1 mm
Transverse beam size	700 μm
Undulator parameter, K	1.0
Undulator period	27 mm
Undulator length	1.08 m

Example of a Resonator FEL: the ALICE IR FEL

The radiation produced from the undulator has wavelength:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \approx 7.2\,\mu\text{m}. \qquad (29)$$

Since $\sigma_z \gg \lambda/\gamma^2$, without microbunching the FEL operates in the low-gain regime.

Substituting values into the gain equation (27), and assuming an optimal overlap between the electron beam and the radiation beam within the undulator, we find:

$$\text{maximum } G \approx 0.27, \qquad (30)$$

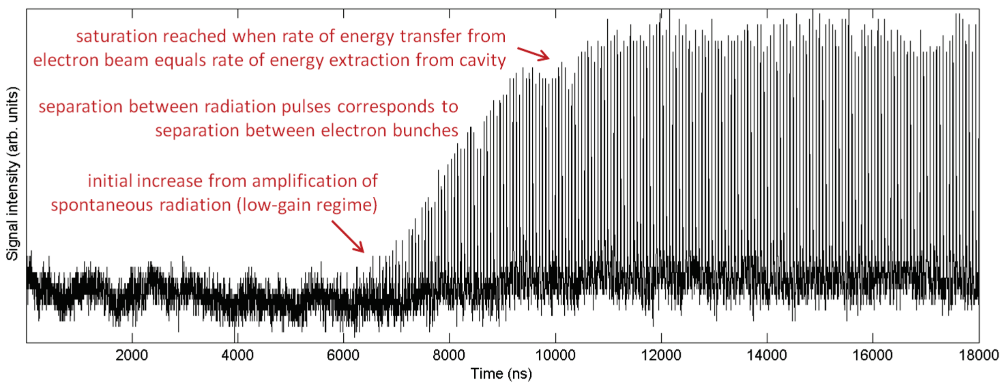
i.e. the maximum gain (in the low-gain regime) is about 27%.

Once the radiation starts to build up in the optical cavity, it can cause microbunching to develop. There is a limit in the amount of energy that can be extracted from each bunch. The radiation intensity saturates when the rate of energy extracted from the electron beam equals the rate at which energy is lost through the out-coupling hole.

Example of a Resonator FEL: the ALICE IR FEL

The time structure of the FEL output can be observed using a fast detector.

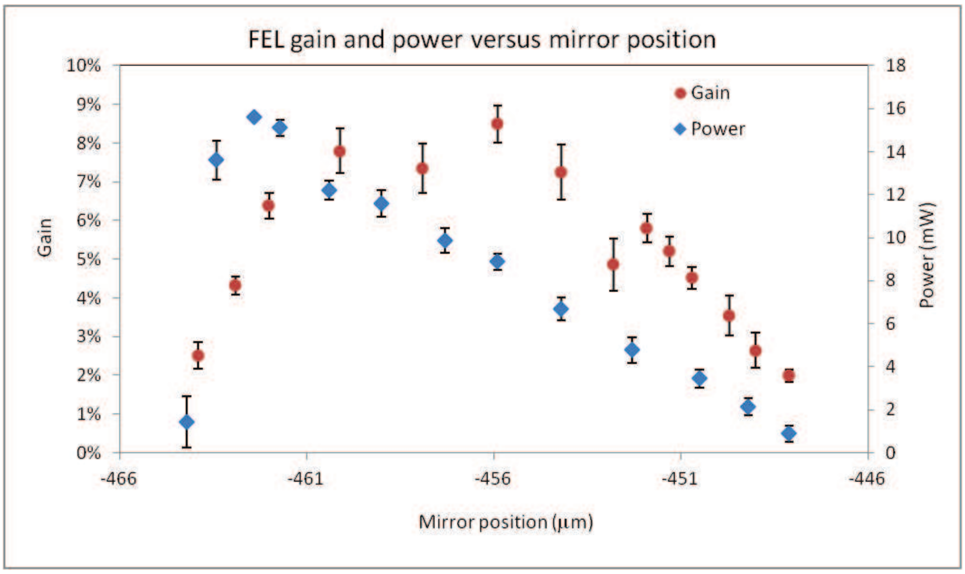
Each machine pulse of $100\,\mu$s contains 1625 bunches.



By fitting the exponential rise in intensity, it is possible to estimate the gain.

Example of a Resonator FEL: the ALICE IR FEL

The gain and intensity both depend on the length of the optical cavity, as well as on things like the overlap between the radiation and the electron beam. The power output can be high even when the gain is low, if the power level at which saturation is reached is high.



SASE: Self-Amplified Spontaneous Emission

Resonator FELs have limits in the wavelengths they can achieve, because of the need for high-quality reflectors.

Seeded amplifier FELs also become more difficult at shorter wavelengths, because of the lack of convenient short-wavelength high power lasers to provide the seed.

To construct an x-ray FEL, we need to exploit the fact that the spontaneous radiation from an undulator can act as the seed required to generate microbunches which then radiate coherently: this process is known as *Self-Amplified Spontaneous Emission* (SASE).

SASE FEL Power Curve

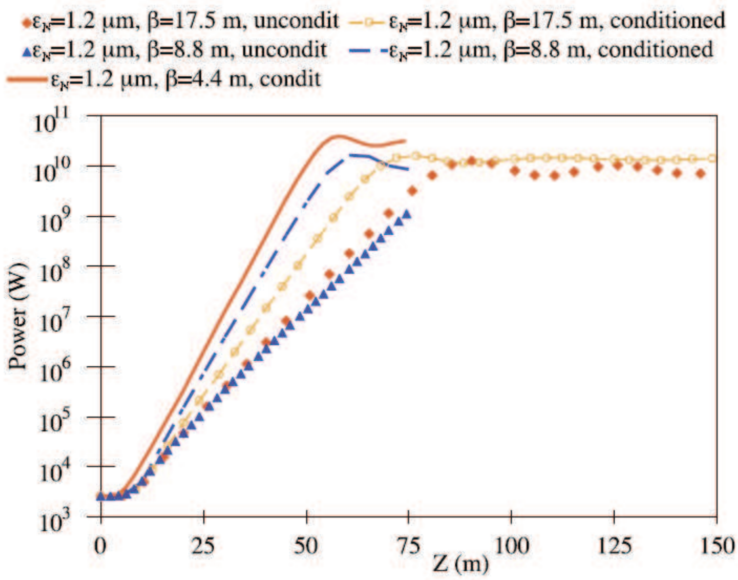


FIG. 3. (Color) Radiation power as a function of undulator length for LCLS, with different beta functions, and with conditioned and unconditioned beams.

A. Wolski, G. Penn, A. Sessler, J. Wurtele, PRST-AB, 7, 080701 (2004).

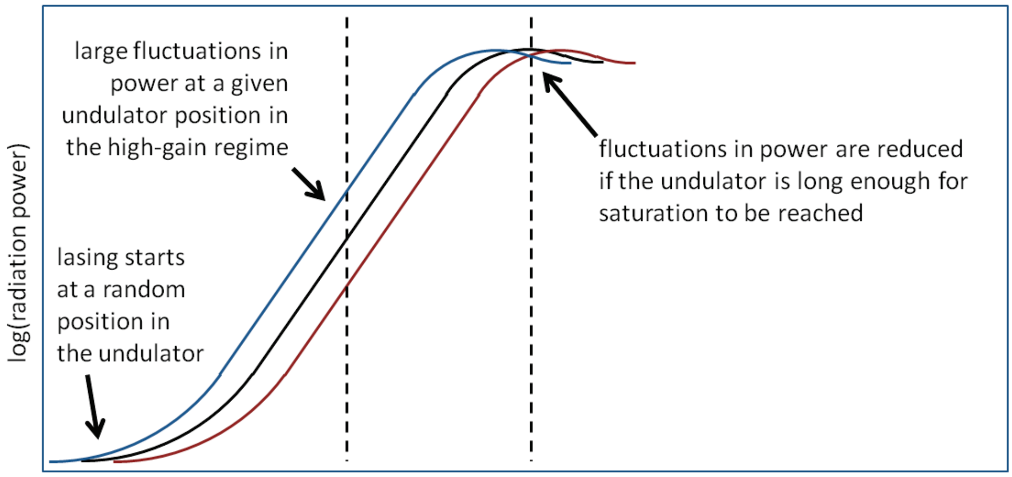
SASE FEL Power Curve

The plot on the previous slide illustrates some characteristic features of the power of a SASE FEL as a function of undulator length:

- There is an initial slow increase in radiation power as spontaneous radiation is generated, and microbunching starts to happen (the low-gain regime).
- This is followed by a rapid (exponential) increase in radiation power, as microbunching develops in the electron beam (driven by the radiation): this is the high-gain regime.
- Eventually, the power saturates once maximum microbunching is achieved. There is even some drop in power as the electron beam re-absorbs some of the energy from the radiation.

SASE FEL Power Curve

Since the radiation in a SASE FEL grows from random fluctuations in the beam density, it is difficult to control the point in the undulator at which the beam enters the high-gain regime. This means that to minimise fluctuations in output power, it necessary to build the undulator long enough for the power to reach saturation – and no shorter!



Properties of SASE FELs: Gain Length

The detailed analysis of a high-gain FEL is complicated, and we do not investigate it further here. However, some of the main results can be stated fairly simply.

The first important result is for the gain length. In the high-gain regime, the radiation power in an FEL increases exponentially with distance z, so that:

$$P(z) = P(0)\exp(z/L_{g0}). \tag{31}$$

The gain length L_{g0} is given by:

$$L_{g0} = \frac{1}{\sqrt{3}}\left(\frac{\gamma_r^3\lambda_u}{2\pi^2 r_e \tilde{K}^2 n_e}\right)^{\frac{1}{3}}. \tag{32}$$

This expression neglects the effects of energy spread and transverse emittance of the electron beam (hence the subscript '0' on the gain length L_{g0}).

Properties of SASE FELs: Bandwidth

The second important result is for the bandwidth of the amplification. We already saw in Madey's theorem that the FEL gain in the low-gain regime depended on the wavelength of the radiation: the same is true in the high-gain regime.

A useful measure of the bandwidth is given by the *FEL parameter* ρ_{FEL} (sometimes known as the Pierce parameter):

$$\rho_{\text{FEL}} = \frac{1}{4\pi\sqrt{3}} \frac{\lambda_u}{L_{g0}}. \tag{33}$$

Typically, ρ_{FEL} is of order 10^{-3} (so $L_{g0} \approx 50\lambda_u$).

At $z = 4L_{g0}$, the full width at half maximum of the gain curve is approximately $2\rho_{\text{FEL}}$. This drops to about ρ_{FEL} at $z = 16L_{g0}$.

The bandwidth of a SASE FEL is given by:

$$\frac{\sigma_\omega}{\omega} = 3\sqrt{2}\,\rho_{\text{FEL}} \sqrt{\frac{L_{g0}}{z}}. \tag{34}$$

Properties of SASE FELs: Saturation

Finally, it is possible to derive an expression for the level at which the radiation power from an FEL saturates. Saturation occurs when the amplitude of the longitudinal density modulation reaches a maximum.

At saturation, the radiation power is given by:

$$P_{\text{rad}} \approx \rho_{\text{FEL}}\, P_{\text{e−beam}}. \tag{35}$$

That is, the maximum radiation power as a fraction of the electron beam power is given by the FEL parameter ρ_{FEL}.

Final Remarks

FEL physics is a fascinating and challenging branch of accelerator science, and a highly active area of research.

In this lecture, we have outlined some of the basic principles, and given just a few of the key results.

There are many variations on the principles and designs that have been outlined here. The aim is not just to achieve the maximum possible brightness (at the shortest possible wavelengths), but to be able to manipulate the properties of the radiation, e.g. to provide extremely short (<fs) pulses, temporal coherence, precise synchronisation with external laser pulses...

New ideas and techniques are regularly being proposed and actively developed, and new applications being found.

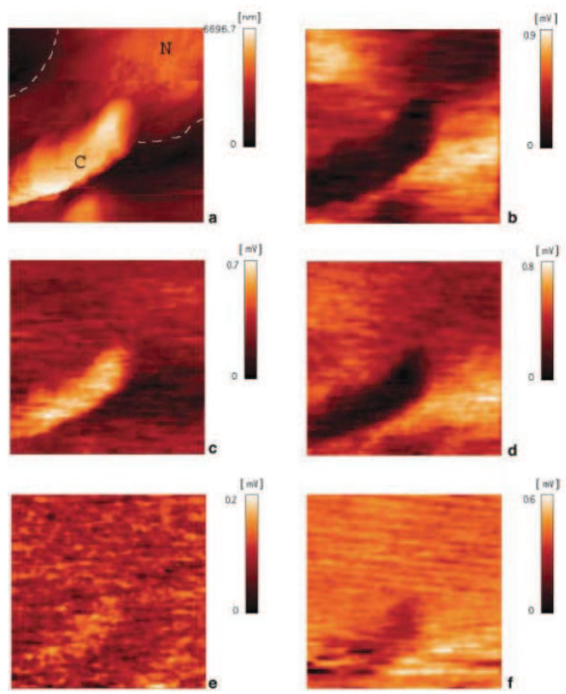


FIGURE 4 (*a*) $20 \times 20\ \mu m^2$ shear-force (topographic) image of a COS-7 cell in PBS. The cell body and nucleus (*upper right*) are seen and a crystal of PBS (*left side*) can also be seen on the cell. SNOM reflection images of the same field were obtained under illumination with (*b*) $\lambda = 8.05\ \mu m$, (*c*) $\lambda = 7.6\ \mu m$, (*d*) $\lambda = 6.95\ \mu m$, (*e*) $\lambda = 6.45\ \mu m$, and (*f*) $\lambda = 6.1\ \mu m$.

IR images of a cell with sub-micron resolution.

The images are produced by illuminating a specimen with light from an IR FEL, and scanning across the specimen with a near-field optical microscope (SNOM).

Comparison of images at different wavelengths allows determination of the chemical structure of the cell.

A. Cricenti *et al*, Biphysical Journal 85, 2705–2710 (2003).

A useful reference:

- P. Schmüser, M. Dohlus, Jörg Rossbach, "Ultraviolet and Soft X-Ray Free-Electron Lasers," Springer Tracts in Modern Physics, Volume 229 (2008).

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