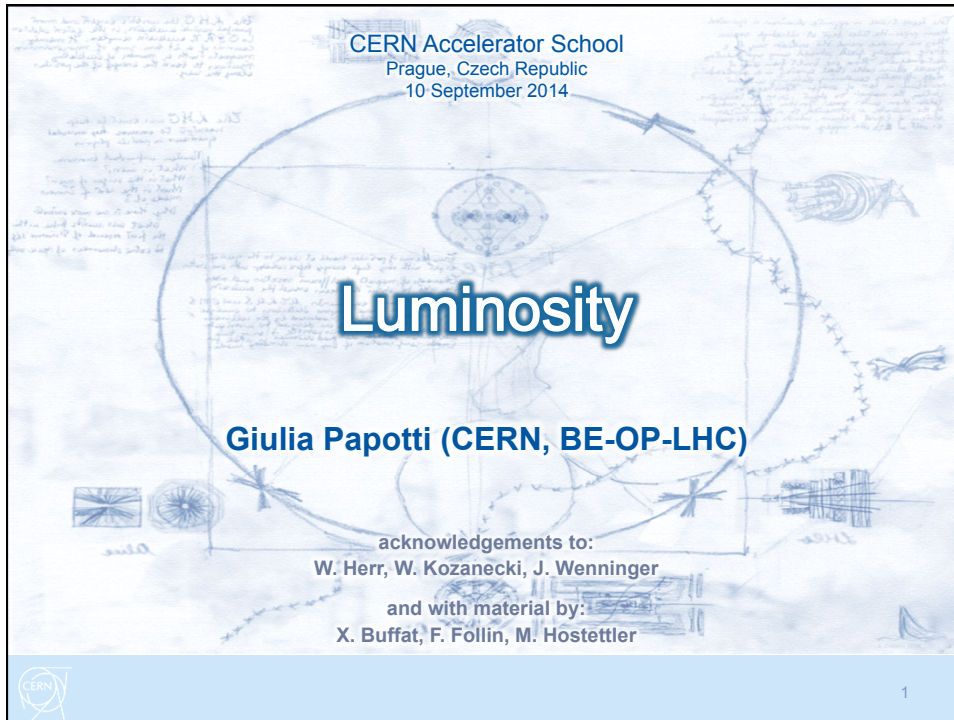



CERN Accelerator School  
Prague, Czech Republic  
10 September 2014








# Luminosity

**Giulia Papotti (CERN, BE-OP-LHC)**


acknowledgements to:  
W. Herr, W. Kozanecki, J. Wenninger  
and with material by:  
X. Buffat, F. Föllin, M. Hostettler

 1

## Bibliography

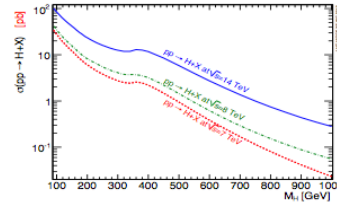
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-  J. Wenninger, “Luminosity diagnostics”, CAS on Beam Diagnostics, Dourdan (France), June 2008.
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-  A. Chao and M. Tigner, “Handbook of accelerator physics and engineering”, World Scientific, 2002.

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# collider

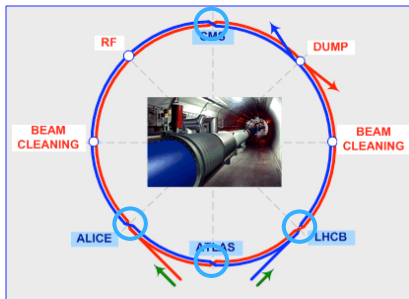
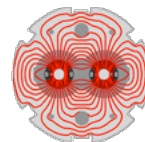
- at high energy to probe smaller scales or to produce heavier particles
    - lighter particles were studied in older machines
      - "to boldly go where no man has gone before"
    - some events only possible at higher energies
    - collider as last stage of the accelerator chain
      - e.g. at CERN: Linac+PSB+PS+SPS+LHC
  - particle colliders use two beams
    - higher available energy by colliding two beams ( $-\vec{p}_1 = \vec{p}_2, E_1 = E_2 = E+m_0$ )
    - than using a fixed target ( $p_2=0, E_2=m_0$ )
      - see *W. Herr, "Relativity"*
  - need many interactions to explore and prove rare events
    - luminosity measures the number of events for the experiments
- figures of merit of a collider: energy  $E_{cm}$  and luminosity  $L$



$$E_{cm} = \sqrt{(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2}$$

# e.g.: the Large Hadron Collider

- main example in this lecture
- choice of beam particle:
  - for a discovery machine, need hadrons
  - use proton-proton to have many events
- same particles to counter-rotate: need two rings
  - 2-in-1 magnet design



- LHC layout
  - 8 arcs and 8 straight sections (SS)
    - 4 SS for machine equipment
    - 4 SS for experiments
      - Alice, ATLAS, CMS, LHCb
  - common vacuum chamber in 4 interaction points only
  - note: also single ring colliders exist
    - e.g. SpqS, LEP, Tevatron

LHC	
$E_{cm} = 14 \text{ TeV}$	
$L = 10^{34} \text{ cm}^{-2}\text{s}^{-1}$	

## diversion: a CMS slice

or “what the experiments do with the collisions”

Key:

- Muon
- Electron
- Charged Hadron (e.g. Pion)
- Neutral Hadron (e.g. Neutron)
- Photon

4T

Silicon Tracker

Electromagnetic Calorimeter

Hadron Calorimeter

Superconducting Solenoid

2T

Iron return yoke interspersed with Muon chambers

Transverse slice through CMS

...but that is another story and shall be told another time

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## outline

- (motivation)
- luminosity
  - definition and derivation from machine parameters
  - head-on and offset collisions
  - reduction factors
    - crossing angles and crab cavities, hourglass
  - lifetime, contributions
  - luminosity scans and luminosity levelling
- integrated luminosity and ideal run time
- measurements and optimizations
  - vdM scans, high beta runs
- linear colliders
  - no fixed target
  - no coasting beams

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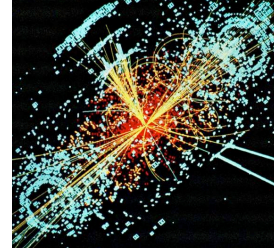
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## definition: cross section

- **process**: a particle encounters a target
  - e.g. another beam
  - the encounter produces a certain final state composed of various particles (with a certain probability)
- **cross-section**  $\sigma_{\text{event}}$  expresses the likelihood of the process
  - $\sigma_{\text{event}}$  represents the “area” over which the process occurs
  - units: [m<sup>2</sup>]
    - in nuclear and high energy physics: 1 barn (1 b = 10<sup>-24</sup> cm<sup>2</sup>)



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## L: definition

$$R = \frac{dN}{dt} = L(t)\sigma_{\text{event}}$$

- luminosity L relates cross-section  $\sigma$  and event rate  $R = dN/dt$  at time t :
  - quantifies performance (“brilliance”) of collider
  - relativistic invariant and independent of physical reaction

$$N = \sigma_{\text{event}} \int L(t) dt$$

- accelerator operation aims at maximizing the total number of events N for the experiments
  - $\sigma_{\text{event}}$  is fixed by Nature
  - aim at maximizing  $\int L(t) dt$

- units : [m<sup>-2</sup> s<sup>-1</sup>]
  - $\int L dt$  is frequently expressed in pb<sup>-1</sup> = 10<sup>36</sup> cm<sup>-2</sup> or fb<sup>-1</sup> = 10<sup>39</sup> cm<sup>-2</sup>
- e.g.: from LHC run 1, ATLAS+CMS got 1400 Higgs events in total
  - in ~30 fb<sup>-1</sup> each: 6.1 fb<sup>-1</sup> in 2011, 23.3 fb<sup>-1</sup> in 2012

LHC
N = 5
$\sigma_{\text{event}} = 0.5 \text{ fb} = 10^{-39} \text{ cm}^2$
$\int L(t) dt = 10 \text{ fb}^{-1}$



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# circular colliders

Machine	Years in operation	Beam type	Beam energy [GeV]	Luminosity [cm <sup>-2</sup> s <sup>-1</sup> ]
ISR	1971-'84	p p	31	>2x10 <sup>31</sup>
LEP I	1989-'95	e+ e-	45	3x10 <sup>30</sup>
LEP II	1995-2000	e+ e-	90-104	10 <sup>32</sup>
KEKB	1999-2010	e+ e-	8 x 3.5	2x10 <sup>34</sup>
SppS	1981-'84	p anti-p	270	6x10 <sup>30</sup>
TEVATRON	1983-2011	p anti-p	980	2x10 <sup>32</sup>
LHC	2008-?	p p	7000	10 <sup>34</sup>



# L from machine parameters -1-

- intuitively: more L if there are more protons and more tightly packed

$$L \propto N_1 N_2 \Omega_{x,y}$$



$$L \propto N_1 N_2 K \int_{x,y,s,s_0} \rho_1(x,y,s,-s_0) \rho_2(x,y,s,s_0) dx dy ds ds_0$$

- K = 2 c: kinematic factor (see W. Herr, "Relativity")
- N<sub>1</sub>, N<sub>2</sub>: bunch population
- ρ<sub>1,2</sub>: density distribution of the particles (normalized to 1)
- x,y: transverse coordinates
- s: longitudinal coordinate
- s<sub>0</sub>: "time variable", s<sub>0</sub> = c t
- Ω<sub>x,y</sub>: overlap integral



## L from machine parameters -2-

- for a circular machine can reuse the beams  $f$  times per second (storage ring)
- for  $k$  colliding bunch pairs per beam
- for uncorrelated densities in all planes:  $\rho(x, y, s, t) = \rho_x(x)\rho_y(y)\rho_s(s - vt)$

$$L = 2fkN_1N_2 \int_{x,y,s,s_0} \rho_{1x}(x)\rho_{1y}(y)\rho_{1s}(s - s_0)\rho_{2x}(x)\rho_{2y}(y)\rho_{2s}(s + s_0) dx dy ds ds_0$$

- for Gaussian bunches:  $\rho_u(u) = \frac{1}{\sigma_u\sqrt{2\pi}} \exp\left\{-\frac{(u-u_0)^2}{2\sigma_u^2}\right\}$   $u = x, y$
- for equal beams in  $x$  or  $y$ :  $\sigma_{1x} = \sigma_{2x}$ ,  $\sigma_{1y} = \sigma_{2y}$

- can derive a closed expression: 
$$L = \frac{kN_1N_2f}{4\pi\sigma_x\sigma_y}$$

- $f$ : revolution frequency
- $k$ : number of colliding bunch pairs at that Interaction Point (IP)
- $N_1, N_2$ : bunch population
- $\sigma_{x,y}$ : transverse beam size at the collision point

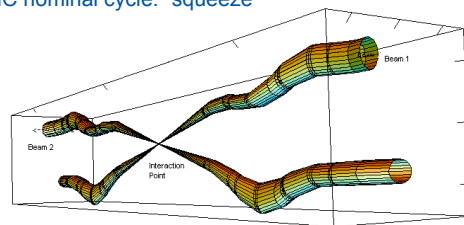
LHC
$k = 2808$
$N_1, N_2 = 1.15 \cdot 10^{11}$ ppb
$f = 11.25$ kHz
$\sigma_x, \sigma_y = 16.6 \mu\text{m}$
$L = 1.2 \cdot 10^{34}$ cm <sup>2</sup> s <sup>-1</sup>



## need for small $\beta^*$

- expand physical beam size  $\sigma_{x,y}$ :  $\sigma_x^* = \sigma_y^* = \sqrt{\frac{\beta^* \epsilon}{\gamma}}$   $\rightarrow L = \frac{kN_1N_2f\gamma}{4\pi\beta^*\epsilon}$

- try and conserve low  $\epsilon$  from injectors
  - explicit dependence on energy ( $\gamma$ )
- intensity pays more than  $\epsilon$  and  $\beta^*$
- design low  $\beta^*$  insertions
  - limits by triplet aperture, protection by collimators
  - in LHC nominal cycle: "squeeze"



Relative beam sizes around IP1 (Atlas) in collision

LHC
$\beta^* = 18 \rightarrow 0.55$ m
$\epsilon = 3.75 \mu\text{m}$
$\gamma = 7463$
$\sigma_{x,y} = 16.6 \mu\text{m}$



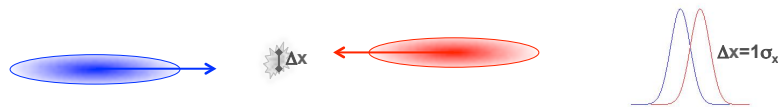
# reduction factors (F)

- transverse offsets
- crossing angles and crab cavities
- hourglass effect



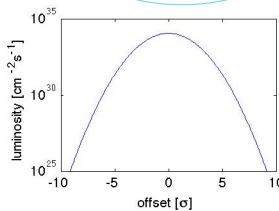
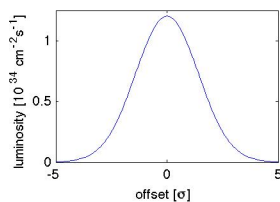
# transverse offsets

- in case the beams do not overlap in the transverse plane (e.g. in x)



more generally

$$L = \frac{kN_1N_2f}{4\pi\sigma_x\sigma_y} \exp\left\{-\frac{\Delta x^2}{4\sigma_x^2} - \frac{\Delta y^2}{4\sigma_y^2}\right\} F$$



$\Delta x$	F
0	1
1 $\sigma$	0.779
2 $\sigma$	0.368
3 $\sigma$	0.105
4 $\sigma$	0.018
5 $\sigma$	0.002



## transverse offsets -2-

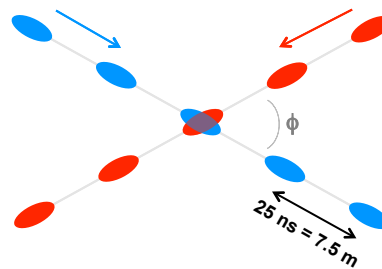
- more general expression including different beam sizes:
  - $\sigma_{1x} \neq \sigma_{2x}, \sigma_{1y} \neq \sigma_{2y}$

$$L = \frac{kN_1N_2f}{2\pi\sqrt{(\sigma_{x,1}^2 + \sigma_{x,2}^2)(\sigma_{y,1}^2 + \sigma_{y,2}^2)}} \exp\left\{-\frac{(\Delta x)^2}{2(\sigma_{x,1}^2 + \sigma_{x,2}^2)} - \frac{(\Delta y)^2}{2(\sigma_{y,1}^2 + \sigma_{y,2}^2)}\right\}$$



## crossing angles -1-

- to avoid parasitic collisions when there are many bunches
  - otherwise collisions elsewhere than in interaction point only
  - e.g.: CMS experiment is 21 m long, common vacuum pipe is 120 m long
- luminosity is reduced as the particles no longer traverse the entire length of the counter-rotating bunch



$$L = \frac{kN_1N_2f}{4\pi\sigma_x\sigma_y} \frac{1}{\sqrt{1 + \left(\frac{\sigma_s}{\sigma_x} \tan \frac{\phi}{2}\right)^2}} F$$

$\frac{\sigma_s}{\sigma_x} \tan \frac{\phi}{2}$  is called the Piwinski angle

valid for small  $\phi$  and  $\sigma_s \gg \sigma_x, \sigma_y$

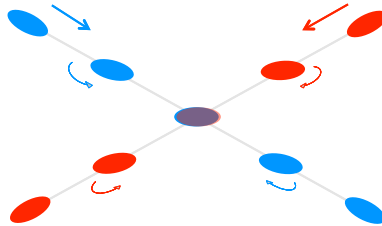
LHC
$\phi = 285 \mu\text{rad}$
$\sigma_s = 7.5 \text{ cm}$
$F = 0.84$





## crossing angles -2-

- for very small  $\beta^*$ , need big crossing angle: big reduction in L
  - e.g. for LHC upgrade (HL-LHC):  $\beta^* = 15$  cm,  $\phi = 590$   $\mu$ rad,  $F \sim 0.35$
- “crab crossing” scheme being considered



- use fast RF cavities for bunch rotation (transverse deflection)
  - used at KEKB, but with leptons and “global” scheme
  - at LHC, need “local” scheme due to collimators, need compact cavities
    - feasibility to be demonstrated, studies on-going



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## hourglass effect

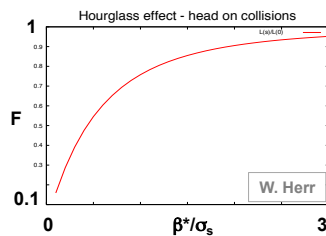
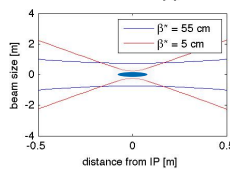
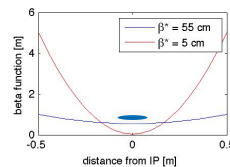


- $\beta$  depends on longitudinal position  $s$ 
  - see B. Holzer, chapter on *Insertions in “Transverse Beam Dynamics”*

$$\beta(s) \approx \beta^* \left( 1 + \left( \frac{s}{\beta^*} \right)^2 \right)$$

- then beam size  $\sigma_{x,y}$  depends on  $s$ 
  - if  $\beta^* \gg \sigma_s$ , effect is negligible
  - if  $\beta^* \sim \sigma_s$ , bunch samples bigger  $\beta$  than  $\beta^*$

$$\sigma_{x,y}(s) \approx \sigma_{x,y}^* \sqrt{1 + \left( \frac{s}{\beta_{x,y}^*} \right)^2}$$



- L reduction is non-negligible for long bunches and small  $\beta$

LHC	HL-LHC
$\beta^*/\sigma_s > 7$	$\beta^*/\sigma_s \sim 2$
$F \sim 1$	$F \sim 0.90$



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## LHC parameters

Parameter	Nominal	2010	2011	2012
beam energy [TeV]	7.0	3.5	3.5	4.0
bunch spacing [ns]	25	150	75 / 50	50
k [no. bunches]	2808	368	1380	1380
$N_b$ [ $10^{11}$ p/bunch]	1.15	1.2	1.45	1.6
$\epsilon$ [mm mrad]	3.75	2.2	2.3	2.5
$\beta^*$ [m]	0.55	3.5	1.5 $\rightarrow$ 1	0.6
half crossing angle [ $\mu$ rad]	142.5	100	120	145
L reduction factor	$\sim 0.84$	$\sim 1$	0.95/0.91	$\sim 0.8$
L [ $\text{cm}^{-2}\text{s}^{-1}$ ]	$10^{34}$	$2 \times 10^{32}$	$3.5 \times 10^{33}$	$7.7 \times 10^{33}$

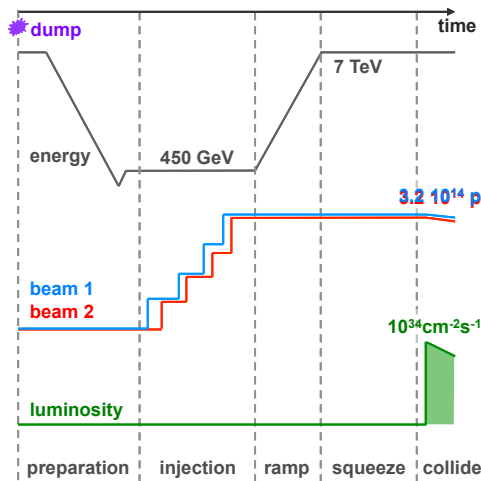


## L evolution during a fill

natural decay, components  
luminosity levelling



## diversion: what is a fill?



- **fill**: a complete machine cycle
  - includes all phases needed to get to luminosity production
  - customarily: starts at dump
  - also called "luminosity run"
    - note: "LHC run 1" is 2010-13
- need time to prepare before producing luminosity!
  - ramp-down, inject, ramp, squeeze...
  - efficiency is not 100%, even with 100% availability!

2012	typ. time
prep	>50 min.
inj	~60 min.
ramp	~15 min.
squ.	~20 min.
coll.	0-20 h



## L natural decay during a fill

$$L = \frac{kN_1N_2f\gamma}{4\pi\beta^*\varepsilon} F$$

- not changing during the fill:
  - $\gamma$  (set by magnetic field in bends)
  - $f$  (set by beam energy and tunnel length)
  - $\beta^*$  (set up during beam commissioning, compromise between aperture, collimator settings, tolerances)
    - with a couple of exceptions...
  - $k$  (set at injection)
- changing during a fill (and naming only a few causes):
  - $\varepsilon$  increases
    - Intra Beam Scattering
    - noise in power converters
  - $N_1, N_2$  decrease
    - luminosity burn-off (i.e. particle loss from collisions)
    - scattering on residual gas
  - $F$  changes
    - imperfect overlap from orbit drifts, can be corrected by orbit corrections

LHC
$\tau_{IBS,x} \sim 105$ h
$\tau_{IBS,s} \sim 63$ h
$\tau_{B.O.} \sim 45$ h
$\tau_{gas} > 100$ h



## max peak L is not all...

- might need luminosity control
  - if too high can cause high voltage trips then impact efficiency
  - might have event size or bandwidth limitations in read-out
  - too many simultaneous event cause loss of resolution
- ...experiments also care about:
  - time structure of the interactions: *pile up*  $\mu$ 
    - average number of inelastic interactions per bunch crossing

$$\langle R \rangle = \left\langle \frac{dN}{dt} \right\rangle = \mu f$$

	design	2010	2011	2012	HL-LHC
$\mu$	21	4	17	37	140

- spatial distribution of the interactions: *pile-up density*
  - e.g. HL-LHC: accept max pile up density of 1.3 events/mm
- quality of the interactions (e.g. background)
- size of luminous region
  - e.g. need constant length (input to MonteCarlo simulations)

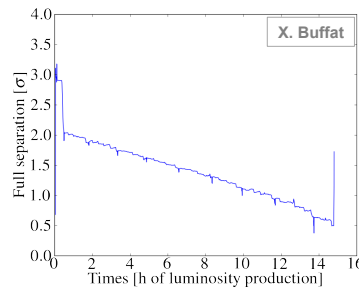
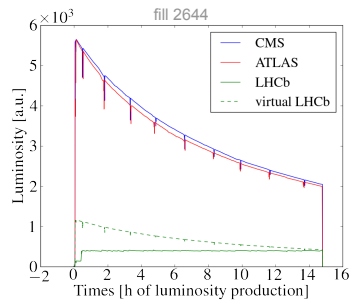


## L levelling

- some experiments need to limit the pile-up
  - thus luminosity per bunch pair
    - e.g.  $\mu < 2.1$  at LHCb in 2012
- stay as long as possible at the maximum value that experiment can manage
  - which is lower than what the machine could provide
- maintain the luminosity constant over a period of time (i.e. the fill)
- possible techniques:
  - by transversely offsetting the beams at the IP
  - by  $\beta^*$



## L levelling by separation



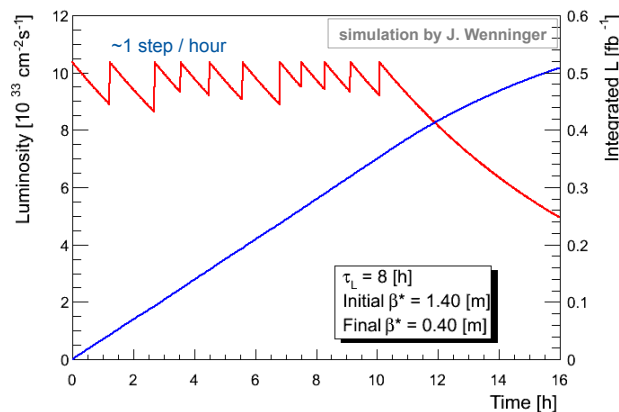
$$\frac{\Delta x}{\sigma_x} = \sqrt{-4 \log \frac{L}{L_0}}$$

- worked beautifully in LHC run 1 for LHCb and ALICE
  - while ATLAS and CMS fully head-on
- can't use it for all experiments at the same time
  - Landau damping from beam-beam helps stability
- might need different solutions for run 2 or HL-LHC



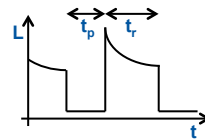
## L levelling with $\beta^*$

- reduce  $\beta^*$  in steps while keeping beams in collisions
- tested successfully at LHC in 2012 Machine Developments
  - more to do with controls than beam physics



## ideal run time -1-

- so far talked about instantaneous L
- but need integrated luminosity  $N \propto \int L(t) dt$ 
  - gives the number of events
- need to account for extra time to prepare a fill ( $t_p$ )
  - inject, ramp, squeeze, ...
  - plus downtime (an accelerator is a very complex system!)
- exercise: assume exponential decay for L:  $L(t) = L_0 e^{-\frac{t}{\tau}}$



- calculate optimum run time ( $t_r$ ) to maximize the average luminosity  $\langle L \rangle$

$$\langle L \rangle = \frac{\int_{t_r} L(t) dt}{t_r + t_p}$$

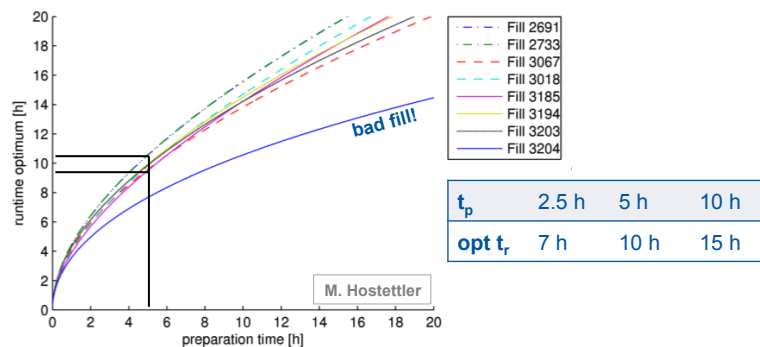
- need
  - good peak luminosity  $L_0$
  - good luminosity lifetime  $\tau$
  - short preparation time
    - "turnaround": jargon for "from dump to stable beams"
  - good machine availability (little downtime, that goes into average preparation time)

LHC	
$\tau \sim$	15 h
$t_p \sim$	5 h
$t_r \sim$	10 h



## ideal run time -2-

- from 2012 LHC data
  - based on more complicated and accurate model for L decay
  - numerical integration to find optimum  $t_r$
- derive optimum fill length: good agreement with previous simple model



## L calibration

van der Meer scans  
high beta runs  
Bhabha scattering



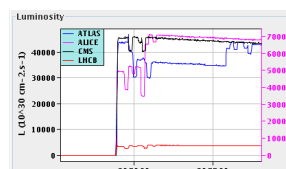
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## L measurements

- relative and absolute L
  - relative: based on an arbitrary scale
    - good enough to monitor variations
      - e.g. for optimizing the rates in CCC
  - absolute: mandatory to measure a process cross section
    - reminder:  $N = \sigma_{event} \int L(t) dt$
    - needs to be calibrated at some point in time
- calibrations
  - from machine parameters
    - not directly from  $\epsilon_{x,y}$ ,  $\beta^*$ ,  $N_{1,2}$ , ... (gives 5-10% precision only)
  - from optical theorem
  - from reactions with well known cross sections



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# vdM scans

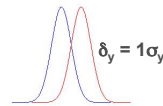
- first done by S. van der Meer at the ISR (1968) in one plane
  - generalized to bunched beams by C. Rubbia at SpS

- recall:  $L_b = f N_1 N_2 \Omega_x \Omega_y$ 
  - assumes uncorrelated densities in all planes

- key: calculate overlap from ratio of rates
  - by measuring rates for different overlaps and integrating over the whole range
  - can measure rates R in arbitrary units!

$$\Omega_y = \frac{R_y(0)}{\int R_y(\delta_y) d\delta_y}$$

- what it takes
  - accurate bunch-by-bunch intensities
  - dedicated fill: no crossing angle, few bunches
  - scans in x, y to get the overlaps  $\Omega_x, \Omega_y$ 
    - need a few steps of  $\delta_y$  for  $\int R_y(\delta_y) d\delta_y$



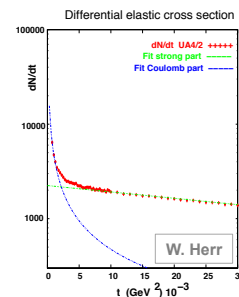
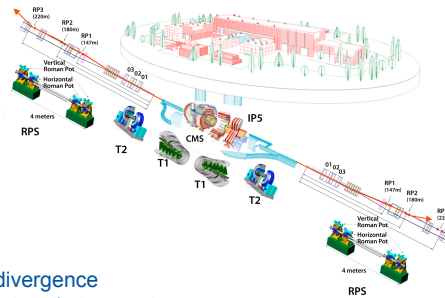
# high beta runs

- optical theorem allows to link:
  - total cross section
  - forward elastic scattering

$$\sigma_{tot}^2 = \frac{16\pi}{1+\rho^2} \left( \frac{d\sigma_{el}}{dt} \right)_{t=0}$$

- “forward” means “at small angle”
  - use high  $\beta^*$  optics to get small beam divergence
    - use Roman Pots: include silicon detectors that can get as close as 1-4 mm to the beam
    - e.g. TOTEM experiment at LHC
  - use small emittance beams

- can also study the Coulomb region,  $t \rightarrow 0$ 
  - $t$  = squared momentum transfer in particle scattering
  - see *W. Herr, "Relativity"*
  - Coulomb scattering can be computed reliably
    - don't need to measure the inelastic rate
  - need  $\beta^* \sim 2.5$  km at LHC
  - e.g. ALFA experiment at ATLAS





## from known cross section

- use reactions with well known cross sections
  - $\sigma$  can be calculated with high precision
  - high event rates for low statistical error
  - background processes identified and/or subtracted

$$L(t) = \frac{\dot{N}(t)}{\sigma}$$

- lepton machines:  $e^+e^-$  elastic scattering (Bhabha scattering)

$$e^+e^- \rightarrow e^+e^-$$

- have to go to small angles ( $\sigma_{el} \propto \Theta^{-3}$ )

$$\sigma = k \left( \frac{1}{\theta_{\min}^2} - \frac{1}{\theta_{\max}^2} \right)$$

- small rates at high energy ( $\sigma_{el} \propto 1/E^2$ )



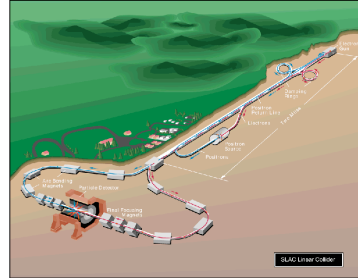
## linear colliders

disruption, pinch effect  
enhancement factor  
beamstrahlung



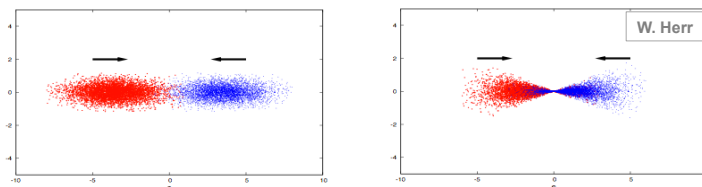
## linear colliders

- e.g.:
  - SLC at SLAC, operated in the 90's
  - being designed: CLIC and ILC
- with electron-positron collisions (e+e-)
- linear: particles collide only once
  - from "revolution" to "repetition" frequency ( $f_{\text{rep}}$ )
    - e.g. 120 Hz at SLC, 5 Hz at ILC, 50 Hz at CLIC
  - thus need intense beams to reach high luminosity
- intense beams cause intense electromagnetic fields affecting the particles in the opposing beam
  - disruption effects
  - beamstrahlung effects



## disruption effects -1-

- strong field by one beam bends the opposing particle trajectories
- quantified by disruption parameter 
$$D_{x,y} = \frac{2r_e N \sigma_z}{\gamma \sigma_{x,y} (\sigma_x + \sigma_y)}$$
- nominal beam size is reduced by the disruptive field (*pinch effect*)
  - additional focusing for the opposing beam



- $r_e$ : electron classical radius
- $N$ : bunch population
- $\sigma_{x,y,z}$ : transverse beam size at the collision point
- $\gamma$ : relativistic factor



## disruption effects -2-

- define an “enhancement factor”  $H_D$ :  $H_D = \frac{\sigma_x \sigma_y}{\bar{\sigma}_x \bar{\sigma}_y}$

- so luminosity can be re-written:

$$L = \frac{N_1 N_2 k f_{rep}}{4\pi \bar{\sigma}_x \bar{\sigma}_y} \rightarrow L = \frac{H_D N_1 N_2 k f_{rep}}{4\pi \sigma_x \sigma_y}$$

- for round beams ( $D_x = D_y$ ) and weak disruption ( $D \ll 1$ ):

$$H_D = 1 + \frac{2}{3\sqrt{\pi} D} + O(D^2)$$

- beyond  $D \ll 1$ , need simulations

- $D$ : disruption parameter
- $\sigma_{x,y,z}$  [ $\bar{\sigma}_{x,y,z}$ ]: transverse beam size at the collision point [resp.: effective beam size]



## beamstrahlung

- disruption at the interaction point is a strong bending:
- results in synchrotron radiation (*beamstrahlung*)
  - causes spread of centre-of-mass energy
  - high energy photons increase detector background
- quantified by beamstrahlung parameter  $Y$

$$Y = \gamma \frac{\langle E + B \rangle}{B_C} \approx \frac{5}{6} \frac{r_e^2 \gamma N}{\alpha \sigma_z (\sigma_x + \sigma_y)}$$

- with  $B_C \equiv \frac{m^2 c^3}{e \hbar} \approx 4.4 \cdot 10^{13} \text{ Gauss}$



## wrap-up

