Collective Effect II

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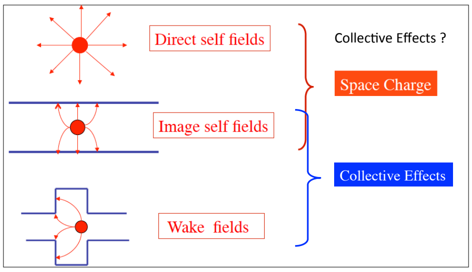
CAS - Prague

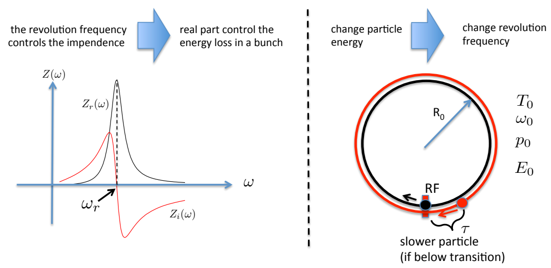
Type of fields



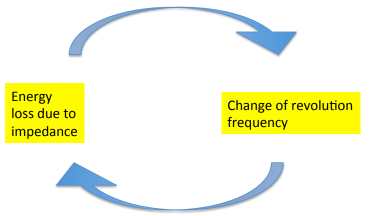
Robinson Instability

Robinson Instability

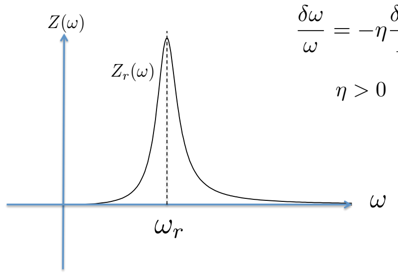
It is an instability arising from the coupling of the impendence and longitudinal motion



The coupling of two effects



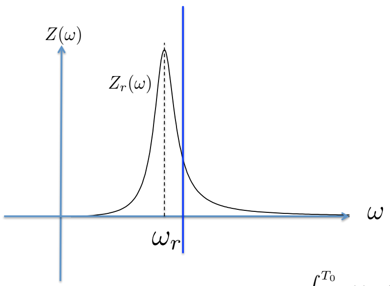
Below transition



$$\frac{\delta\omega}{\omega} = -\eta\frac{\delta E}{E}$$

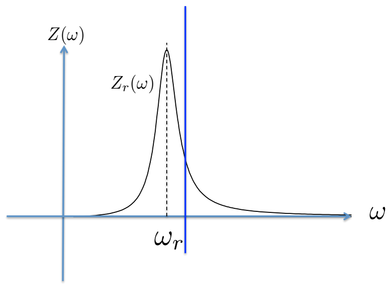
$$\eta > 0$$

Below transition

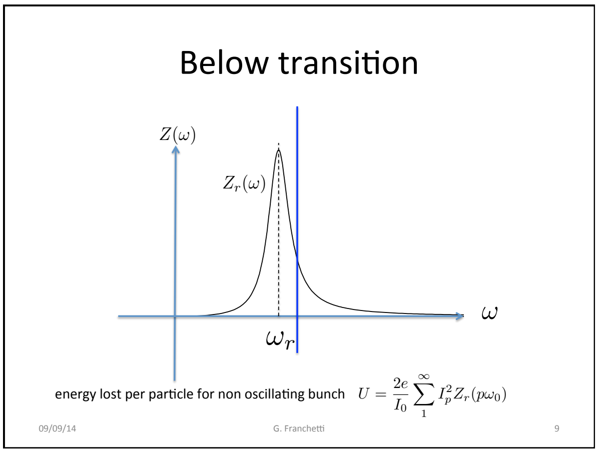


Energy lost in one turn $W_b = \int_0^{T_0} I(t)V(t)dt$

Below transition



Where $V(t)$ is given by the impedance $Z_r(\omega)$


Below transition
$Z(\omega)$
$Z_r(\omega)$
ω
ω_r
energy lost per particle for non oscillating bunch $U = \frac{2e}{I_0}\sum_1^{\infty} I_p^2 Z_r(p\omega_0)$
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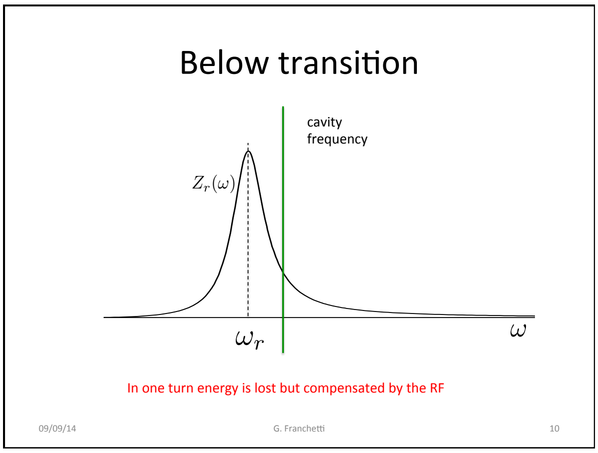
Below transition
cavity
frequency
$Z_r(\omega)$
ω_r
ω
In one turn energy is lost but compensated by the RF
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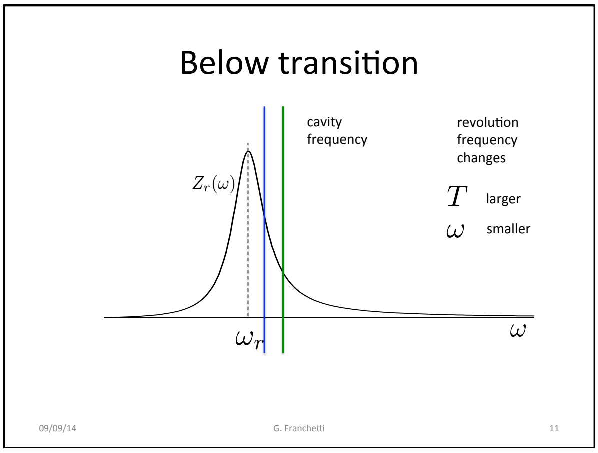
Below transition
cavity
frequency
revolution
frequency
changes
T larger
ω smaller
$Z_r(\omega)$
ω_r
ω
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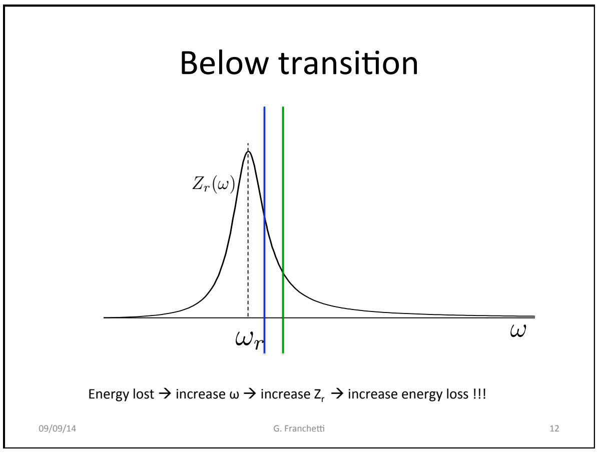
Below transition
$Z_r(\omega)$
ω_r
ω
Energy lost → increase ω → increase Z_r → increase energy loss !!!
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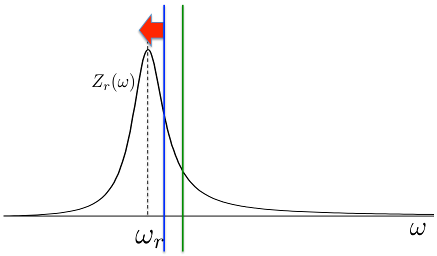
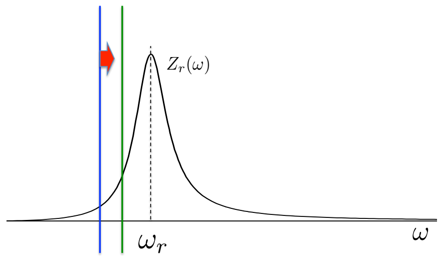
Below transition

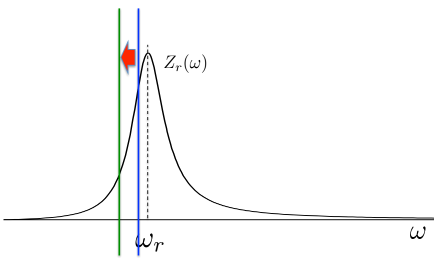


Below transition



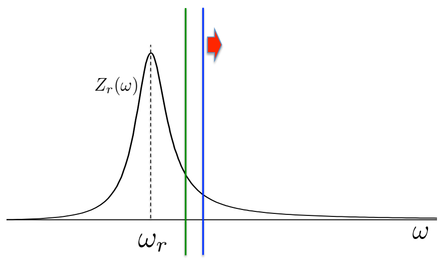
Stable

Below transition

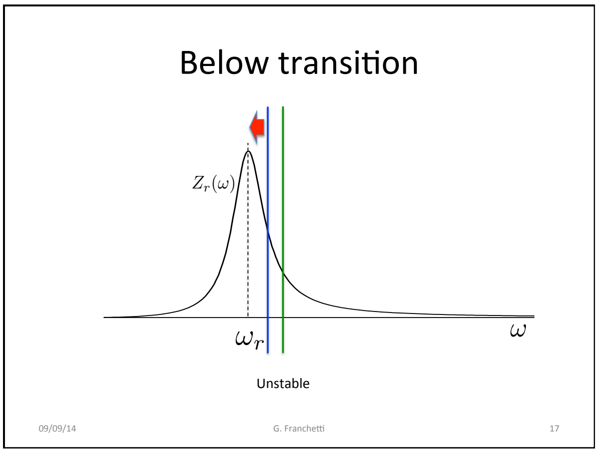


Stable

Below transition



Unstable


Below transition
$Z_r(\omega)$
ω_r
ω
Unstable
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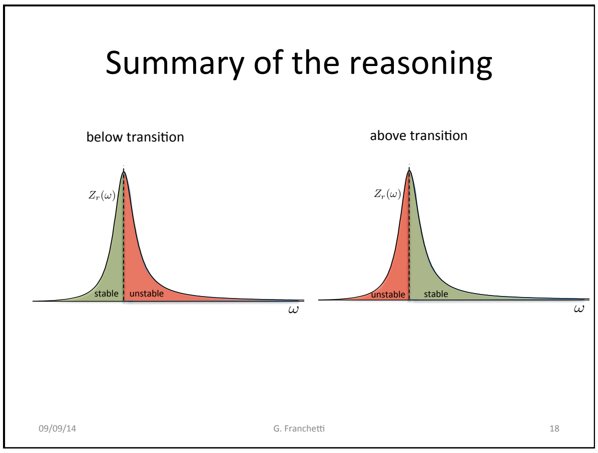
Summary of the reasoning
below transition
above transition
$Z_r(\omega)$
stable
unstable
ω
$Z_r(\omega)$
unstable
stable
ω
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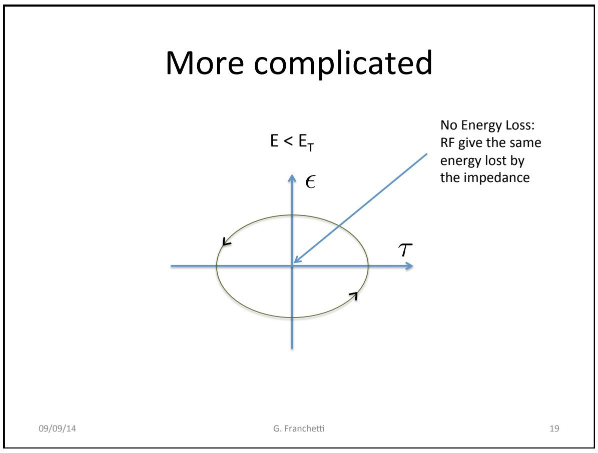
More complicated
E < E$_T$
ϵ
τ
No Energy Loss:
RF give the same
energy lost by
the impedance
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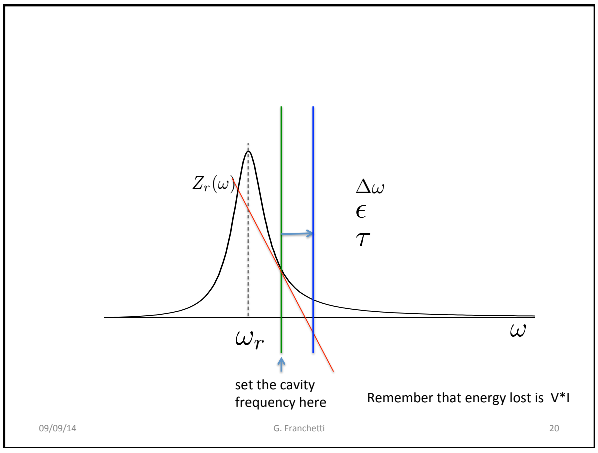
$Z_r(\omega)$
$\Delta\omega$
ϵ
τ
ω_r
ω
set the cavity
frequency here
Remember that energy lost is V*I
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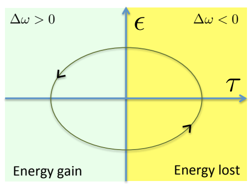
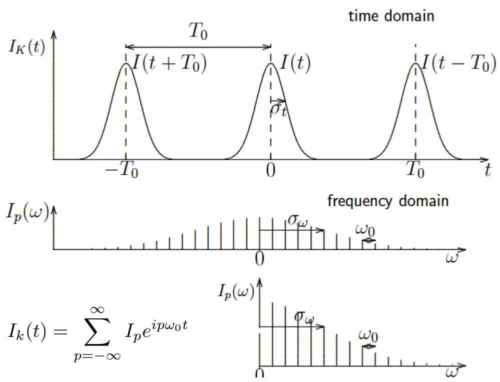
Source of difficulty

$E < E_T$

Impedance effect →

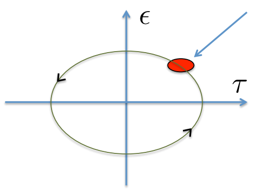




$$I_k(t) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0 t}$$

Still we neglect something

$E < E_T$



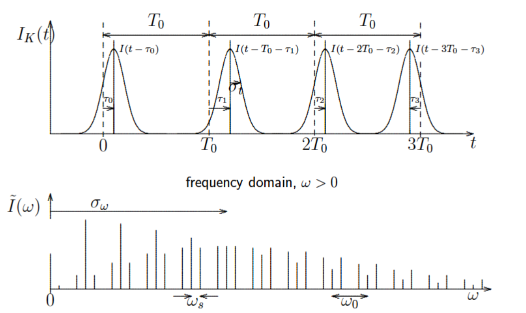
position of the bunch at turn "k"

$$\tau_k = \hat{\tau}\cos(2\pi Q_s k)$$

Q_s is the synchrotron tune

$$\tau_k = \hat{\tau}\cos(\omega_s t)$$

time domain

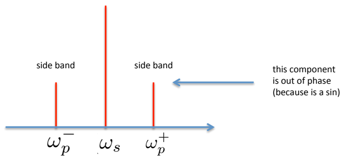


$$t \to t + \hat{\tau}\cos(\omega_s t) \quad \Rightarrow \quad I_k(t) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0[t+\hat{\tau}\cos(\omega_s t)]}$$

Current

$$I_k(t) \simeq \sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\tau}{2}\underbrace{\sin((p+Q_s)\omega_0 t)}_{\omega_p^+} + \frac{p\omega_0\tau}{2}\underbrace{\sin((p-Q_s)\omega_0 t)}_{\omega_p^-}\right]$$

The bunch current can be described by 3 components with frequency very close



That means that the energy loss due to the impedance has to be computed on the 3 currents…

Voltage created by the resistive impedance

Main component $$V = 2\sum_{\omega>0}^{\infty} I_p Z_r(p\omega_0)\cos(p\omega_0 t)$$

1st sideband $$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 \hat{\tau} Z_r(\omega_p^+)\sin(\omega_p^+ t)$$

2nd sideband $$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 \hat{\tau} Z_r(\omega_p^-)\sin(\omega_p^- t)$$

Prosthaphaeresis formulae

$$\sin(\omega_p^- t) = \sin(p\omega_0 t)\cos(\omega_s t) - \cos(p\omega_0 t)\sin(\omega_s t)$$

$$\sin(\omega_p^+ t) = \sin(p\omega_0 t)\cos(\omega_s t) + \cos(p\omega_0 t)\sin(\omega_s t)$$

But $\tau = \hat{\tau}\cos(\omega_s t)$ ⇒ $$\begin{cases}\cos(\omega_s t) = \dfrac{\tau}{\hat{\tau}} \\ \sin(\omega_s t) = -\dfrac{\dot{\tau}}{\hat{\tau}\omega_s}\end{cases}$$

$$\sin(\omega_p^+ t) = \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} - \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}$$

$$\sin(\omega_p^- t) = \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} + \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}$$

Voltage created by the resistive impedance

Main component $$V = 2\sum_{\omega>0}^{\infty} I_p Z_r(p\omega_0)\cos(p\omega_0 t)$$

1st sideband $$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 Z_r(\omega_p^+)[\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}]$$

2nd sideband $$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 Z_r(\omega_p^-)[\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}]$$

Therefore the induced Voltage depends on $\tau, \dot{\tau}$

Energy lost in one turn

$$E_l = \int_0^{T_0} V(t)I(t)dt$$

energy lost per particle per turn

$$U = \frac{2e}{I_0}\left[I_p^2 Z_r(p\omega_0) - \frac{I_p^2 p\omega_0}{2}(Z_r(\omega_p^+) - Z_r(\omega_p^-))\frac{\dot\tau}{\omega_s}\right]$$

this term can give rise to a constant loss, or a constant gain of energy

In terms of the energy of a particle

$$U = \frac{2e}{I_0}\left[I_p^2 Z_r(p\omega_0) - \frac{I_p^2 p\omega_0}{2}(Z_r(\omega_p^+) - Z_r(\omega_p^-))\frac{\eta\epsilon}{\omega_s}\right]$$

$$\frac{\partial U}{\partial \epsilon} = -\frac{e}{I_0}\sum_{\omega>0}\frac{I_p^2 p\omega_0}{2}(Z_r(\omega_p^+) - Z_r(\omega_p^-))\frac{\eta}{\omega_s}$$

This is a slope in the energy, and the sign of the slope depends on

$Z_r(\omega_p^+) - Z_r(\omega_p^-)$ and η

The longitudinal motion now!

$$\ddot\tau + 2\alpha_s\dot\tau + \omega_{s0}^2\tau = 0$$

$$\alpha_S = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E}\frac{\partial U}{\partial \epsilon} = \frac{\omega_s \sum p I_p^2 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2I_0 h\hat V\cos\phi_s}$$

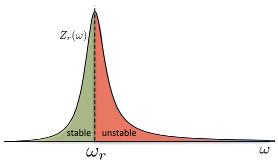
Robinson Instability

If $\alpha_S > 0$ there is a damping

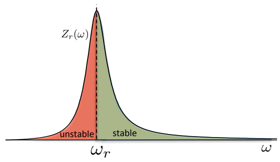
If $\alpha_s < 0$ there is an instability

Robinson Instability

below transition



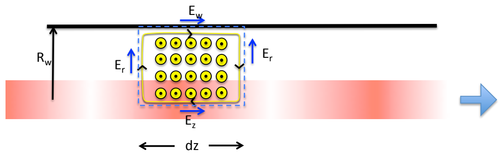
above transition



Longitudinal space charge and resistive wall impedance

Space charge longitudinal field

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \qquad \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{a}$$



$$\oint \vec{E} \cdot d\vec{l} = \int E_r(z) dr + E_w \Delta z - \int E_r(z + \Delta z) dr - E_z \Delta z$$

For a KV beam

Electric Field

$$E_r = \begin{cases} \frac{\lambda(z)}{2\epsilon_0} r & \text{if} \quad r < r_0 \\ \frac{\lambda(z) r_0^2}{2\epsilon_0} \frac{1}{r} & \text{if} \quad r > r_0 \end{cases}$$

$$\int_0^{r_w} E_r(z) dr = \int_0^{r_0} \frac{\lambda(z)}{2\epsilon_0} r dr + \int_{r_0}^{r_w} \frac{\lambda(z) r_0^2}{2\epsilon_0} \frac{1}{r} dr$$

$$\int_0^{r_w} E_r(z) dr = \frac{\lambda(z) r_0^2}{4\epsilon_0} \left[1 + 2 \ln \left(\frac{r_w}{r_0} \right) \right]$$

Therefore

$$\int E_r(z) dr - \int E_r(z + \Delta z) dr = -\frac{r_0^2}{4\epsilon_0} \left[1 + 2 \ln \left(\frac{r_w}{r_0} \right) \right] \frac{\partial \lambda(z)}{\partial z} \Delta z$$

$$\oint \vec{E} \cdot d\vec{l} = (E_w - E_z) \Delta z - \frac{r_0^2}{4\epsilon_0} \left[1 + 2 \ln \left(\frac{r_w}{r_0} \right) \right] \frac{\partial \lambda(z)}{\partial z} \Delta z$$

Magnetic Field

$$B_\perp = \begin{cases} \dfrac{\mu_0 v_z \lambda(z)}{2} r & \text{if} \quad r < r_0 \\ \dfrac{\mu_0 v_z \lambda(z) r_0^2}{2} \dfrac{1}{r} & \text{if} \quad r > r_0 \end{cases}$$

$$\int B_\perp da = \int_0^{r_0} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z)}{2} r + \int_{r_0}^{r_w} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z) r_0^2}{2} \frac{1}{r}$$

$$\int B_\perp da = \frac{\mu_0 v_z r_0^2 \lambda \Delta z}{4} \left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]$$

Maxwell-Faraday Law $\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{a}$

$$(E_w - E_z)\Delta z - \frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial \lambda(z)}{\partial z}\Delta z = +\frac{\mu_0 v_z r_0^2 \Delta z}{4}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial \lambda}{\partial t}$$

from the equation of continuity $\frac{\partial \lambda}{\partial t} + v_z \frac{\partial \lambda}{\partial z} = 0$

$$E_z = E_w - \frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{1}{\gamma^2}\frac{\partial \lambda}{\partial z}$$

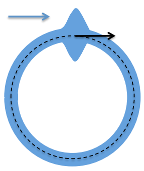
again we find the factor $1/\gamma^2$!

Space charge impedance

$$\lambda(\theta, t) = \sum_n \lambda_n e^{i(n\theta - \omega_n t)} \qquad \theta = 2\pi \frac{z}{L}$$

$$\omega_n = n\omega_0$$

Local density



$$V_{z0} = 2\pi R E_{zw} - i \sum_n \frac{I_n}{4\pi\epsilon_0} \frac{2\pi n}{\beta c \gamma^2}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right] e^{i(n\theta - \omega_n t)}$$

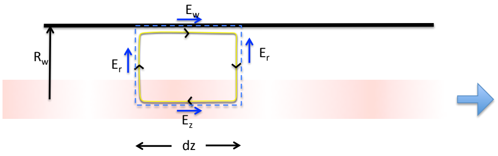
Perfect vacuum chamber $E_{zw} = 0$

$$I = I_n e^{i(n\theta - \omega_n t)} \quad \Rightarrow \quad V = -i\frac{I_n}{4\pi\epsilon_0}\frac{2\pi n}{\beta c \gamma^2}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right] e^{i(n\theta - \omega_n t)}$$

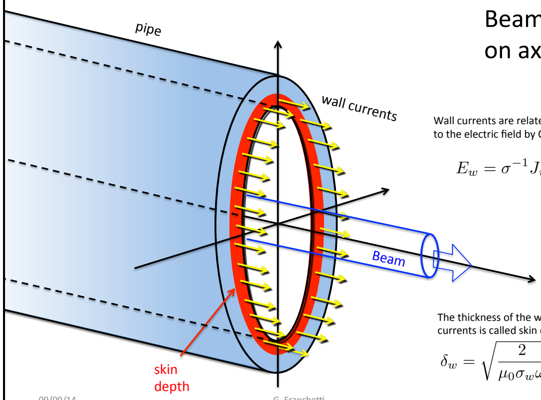
$$Z_{||sc} = \frac{\hat{V}}{\hat{I}} \quad \Rightarrow \quad Z_{||sc} = -i\frac{1}{\epsilon_0 c}\frac{n}{2\beta\gamma^2}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]$$

Resistive Wall impedance

Do not take into account B



$E_w = E_z$



Beam
on axis

Wall currents are related to the electric field by Ohm' law

$$E_w = \sigma^{-1} J_w$$

The thickness of the wall currents is called skin depth

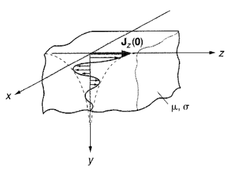
$$\delta_w = \sqrt{\frac{2}{\mu_0 \sigma_w \omega}}$$

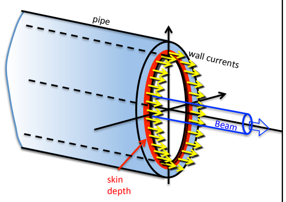
Impedance of the surface (pipe)

$$Z_{surf} = \frac{1+i}{\sigma \delta_w}$$

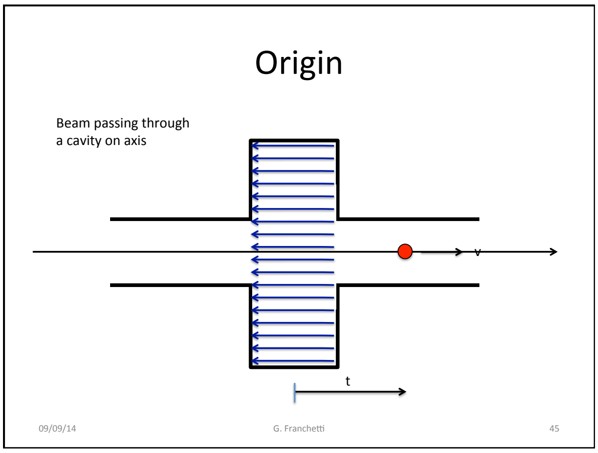
Longitudinal impedance (beam)

$$Z_{||} = \frac{2\pi R}{2\pi r_p} \frac{1+i}{\sigma \delta_w}$$





Transverse impedance


Origin
Beam passing through
a cavity on axis
v
t
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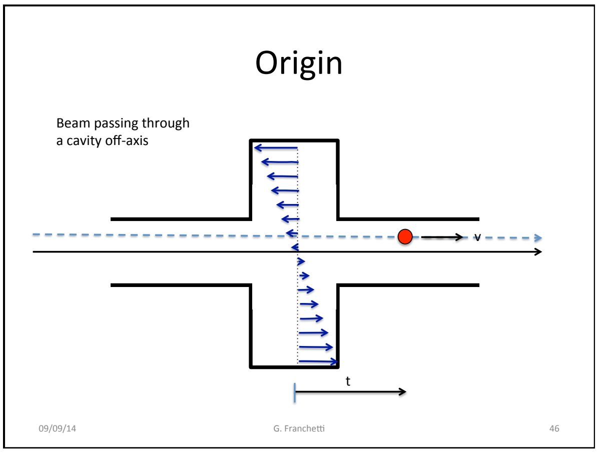
Origin
Beam passing through
a cavity off-axis
v
t
09/09/14
G. Franchetti
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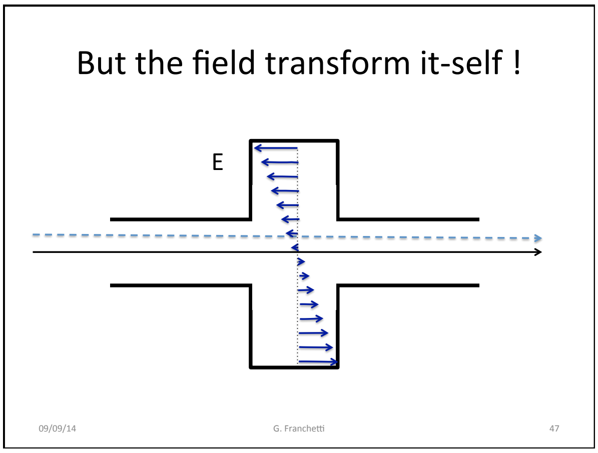
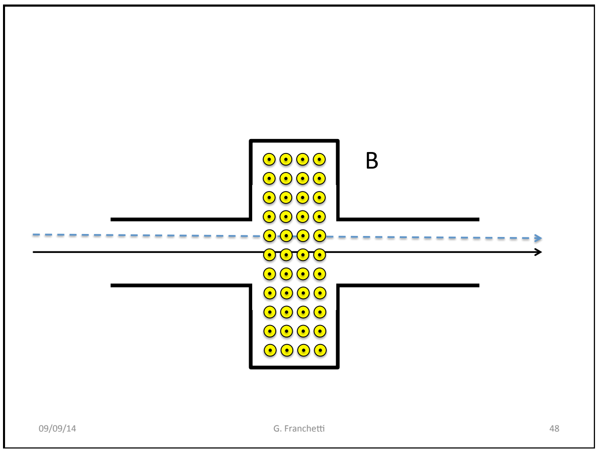
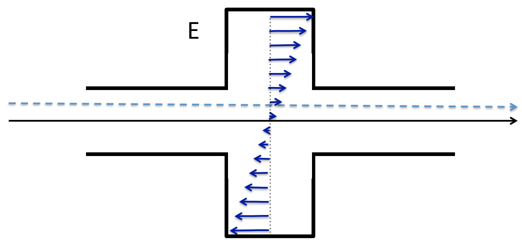
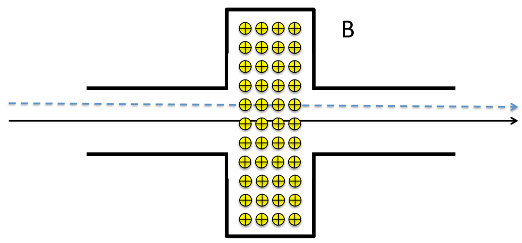

But the field transform it-self !
E
09/09/14
G. Franchetti
47


B
09/09/14
G. Franchetti
48





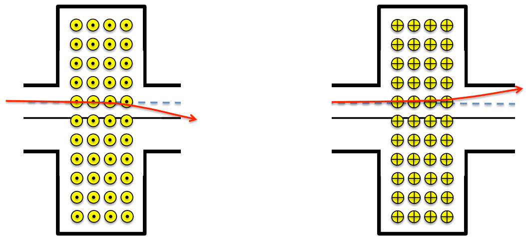
Effect on the dynamics

The dynamics is much more affected by B, than E because

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

this speed is high

The beam creates its own dipolar magnetic field !



(dipolar errors create integer resonances.... we expect the same...)

Transverse impedance

Definition of longitudinal impedance (classical)

$I = \hat{I}e^{i\omega t}$ → System → $V = \hat{V}e^{i\omega t}$

Impedance

$$Z(\omega) = \hat{V}/\hat{I}$$

For a displaced beam

source of the effect Ix_0 → Effect $\vec{E} + \vec{v} \times \vec{B}$

this field acts on a single particle

It means that in the equation of motion we have to add this effect

$$\frac{d^2x}{ds^2} + k_x x = \frac{q}{m\gamma v_0^2}[E_x + (\vec{v} \times \vec{E})_x]$$

therefore for a weak effect or distributed we find

$$\frac{d^2x}{ds^2} + \left(\frac{Q_x}{R}\right)^2 x = \frac{q}{m\gamma v_0^2}\frac{1}{2\pi R}\int_0^{2\pi R}[E_x + (\vec{v} \times \vec{E})_x]ds$$

In the time domain

$$\frac{d^2x}{dt^2} + (Q_x\omega_0)^2 x = \frac{q}{m\gamma}\frac{1}{2\pi R}\int_0^{2\pi R}[E_x + (\vec{v} \times \vec{E})_x]ds$$

But $\int_0^{2\pi R}[E_x + (\vec{v} \times \vec{E})_x]ds$ is like a "strange" voltage

$$V = -\int_0^{2\pi R}[\vec{E} + \vec{v} \times \vec{B}]_\perp ds$$

Now the situation is the following:

Ix_0 → System → $V = -\int_0^{2\pi R}[\vec{E} + \vec{v} \times \vec{B}]_\perp ds$

it depends on frequency

Transverse beam coupling impedance

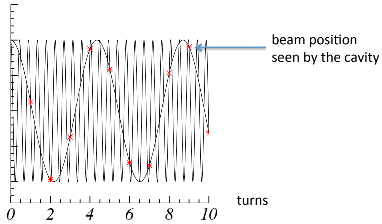
$$Z_\perp(\omega) = i\frac{\int_0^{2\pi R}[\vec{E} + \vec{v} \times \vec{B}]_\perp ds}{\beta I x_0}$$

now the question is
what is ω ?

What is it ω ?

It is given by the fractional tune, as this is the frequency seen in a cavity

Example: Q = 2.23 fractional tune q = 0.23



B-field induced by beam displacement

From $\frac{\partial E_z}{\partial x} = kIx_0$ ⇒ $E_z = kIx_0x$

electric field at the position of beam x_0 is

$$E_z(x_0) = kIx_0^2$$

Longitudinal impedance

$$Z_{||} = -\frac{E_z(x_0)l}{I} = -kx_0^2 l$$

The magnetic field comes from Maxwell $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$

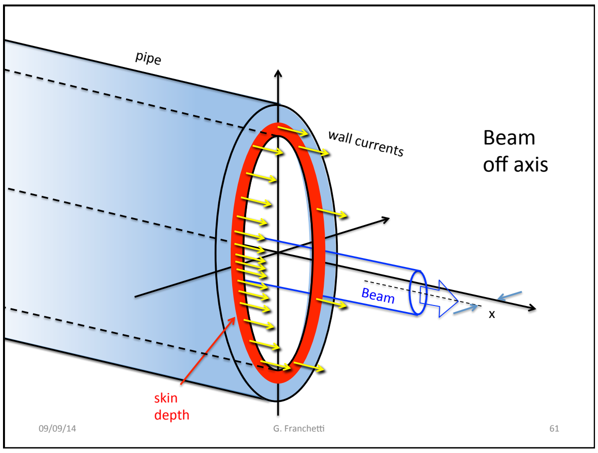
$\frac{\partial B_y}{\partial t}|_{x_0} = kIx_0$ taking $Ix_0 = I\hat{x}e^{i\omega t}$

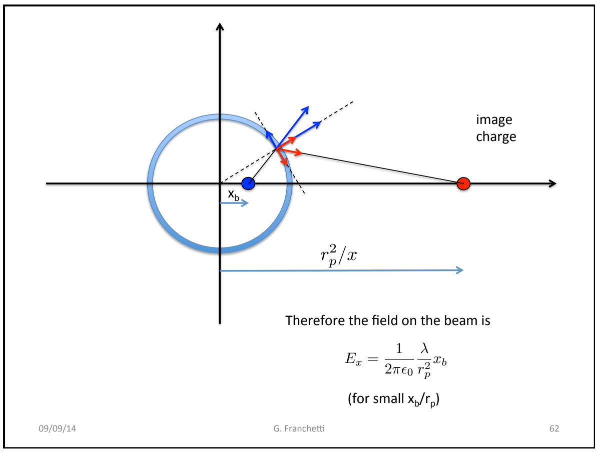
⇒ $B_y = \frac{kI\hat{x}}{i\omega}e^{i\omega t} = \frac{kIx_0}{i\omega}$

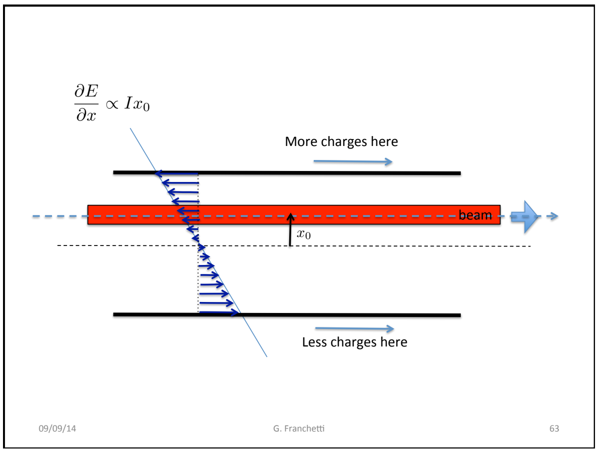
Transverse impedance

$Z_\perp = i\frac{\int_0^l[\vec{v} \times \vec{B}]_\perp ds}{Ix_0}$ ⇒ $Z_\perp = -\frac{v_z k l}{\omega}$

$$Z_\perp = \frac{v_z}{2\omega}\frac{d^2 Z_{||}(\omega)}{dx}$$







Transverse resistive Wall impedance

$$Z(\omega_n)_\perp = \frac{2R}{r_p^2}\frac{Z_{||}(\omega_n)}{n}|_{res}$$

09/09/14 G. Franchetti 64

Transverse instability

Coasting beam instability

Force due to the impedance (in the complex notation)

$$F_\perp = i\frac{qZ_\perp I_0}{2\pi R}x_b$$

+

Equation of motion of one particle for a beam on axis

$$\ddot{x} + Q^2\omega_0^2 x = 0$$

=

Equation of motion of a beam particle when the beam is off-axis

$$\ddot{x} + Q^2\omega_0^2 x = -i\frac{qZ_\perp I_0}{2\pi Rm\gamma}x_b$$

Collective motion

On the other hand the beam center is $x_b = \int x\, n(x,y,s)dxdy$

with $\int \tilde{n}dV = 1$

therefore

$$\int \ddot{x}\tilde{n}dV + \int Q^2\omega_0^2 x\tilde{n}dV = -i\frac{qZ_\perp I_0}{2\pi Rm\gamma}x_b$$

If all particles have the same frequency, i.e. each particle experience a force

$$Q^2\omega^2 x$$

then $\ddot{x}_b + Q^2\omega_0^2 x_b = -i\frac{qZ_\perp I_0}{2\pi Rm\gamma}x_b$

$$\ddot{x}_b + Q^2\omega_0^2 x_b = -i\frac{qZ_\perp I_0}{2\pi Rm\gamma}x_b$$

We can define a coherent "detuning" because this is a linear equation

$$Q^2\omega_0^2 + i\frac{qZ_\perp I_0}{2\pi Rm\gamma} = (Q + \Delta Q^c)^2\omega_0^2$$

$$\Delta Q^c = i\frac{1}{2Q\omega_0^2}\frac{qZ_\perp I_0}{2\pi Rm\gamma}$$

$$\ddot{x}_b + Q^2\omega_0^2 x_b = -2Q\omega^2 \Delta Q^c x_b$$

that is

$$\ddot{x}_b + (Q^2\omega_0^2 + 2Q\omega_0^2\Delta Q^c)x_b = 0$$

But now ΔQ^c is a complex number !!

Solution $x_b = A\exp[-\omega_0 I_m(\Delta Q^c)t + i\omega_0[Q + Re(\Delta Q^c)]t]$

$\tau_I^{-1} = \omega_0 Im(\Delta Q^c)$ Is the growth rate of the transverse resistive wall instability

$$\frac{1}{\tau} = \frac{qRe\{Z_\perp\}I_0}{4\pi R m\gamma Q\omega_0}$$

This instability always take place

Instability suppression
→ Landau damping

An important assumption

We assumed that all particles have the same frequency so that

$$\int Q^2\omega_0^2 x\tilde{n}dV = Q^2\omega_0^2 \int x\tilde{n}dV = Q^2\omega_0^2 x_b$$

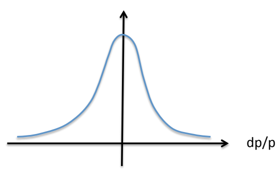
This assumption means that each particle of the beam respond in the same way to a change of particle amplitude

Coherent motion → drive particle motion, which is again coherent

Chromaticity ?

What happened if the incoherent force created by the accelerator do not allow a coherent build=up

Momentum spread



one particle with off-momentum dp/p has tune

$$\delta Q = \xi \frac{\delta p}{p} \quad \Rightarrow \quad Q = Q_0 + \delta Q = Q + \xi \frac{\delta p}{p}$$

chromaticity

If each particle of the beam has different dp/p then the force that the lattice exert on a particle depends on the particle !

$$F_x = \left(Q_0 + \xi \frac{\delta p}{p}\right)^2 \omega^2 x$$

Incoherent motion damps x_b

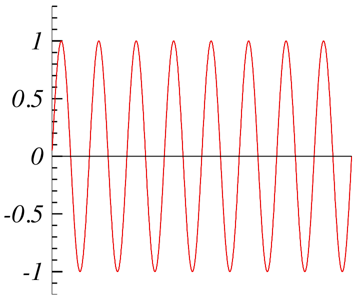
Equation of motion without impedances

$$\ddot{x} + \left(Q_0 + \xi \frac{\delta p}{p}\right)^2 \omega^2 x = 0$$

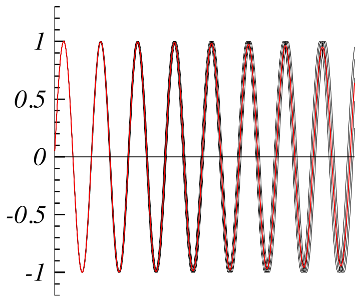
Motion of center of mass as an effect of the spread of the frequencies of oscillation

The momentum compaction also provides a spread of the betatron oscillations

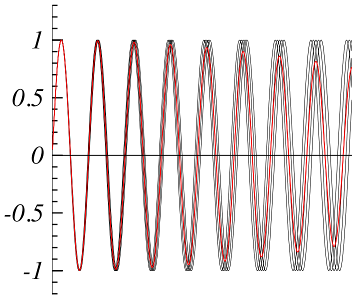
Example:
N. particles = 5
dq/q = 0



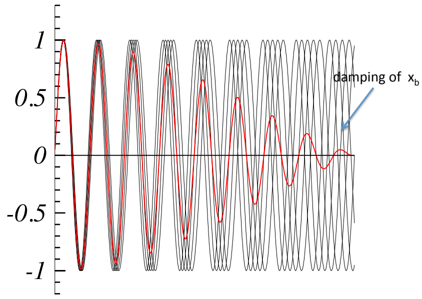
Example:
N. particles = 5
dq/q = 5E-3



Example:
N. particles = 5
dq/q = 1E-2

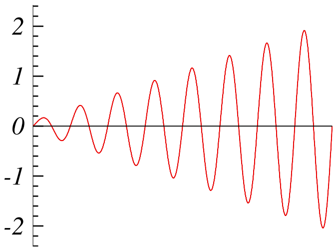


Example:
N. particles = 5
dq/q = 2.5E-2



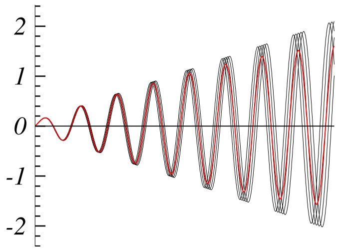
But incoherent motion reduces x_b

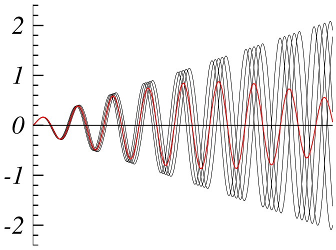
Example:
these are 5 sinusoid
with amplitude linearly
growth



Example:
now a spread dq/q of
1E-2 is added to the
5 curves

The center of mass
growth slower





Example:
now a spread dq/q of 1E-2 is added to the 5 curves

the spread of the particles damps the oscillations of the center of mass → the instability cannot develop

Situation

Coherent effect

Incoherent effect

Growth rate

τ_I

The faster wins

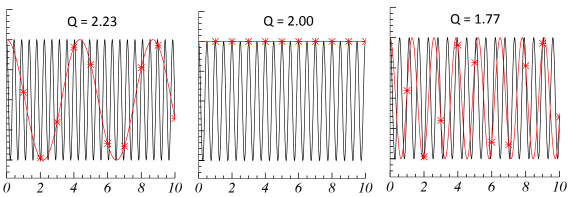
Damping rate

τ_D

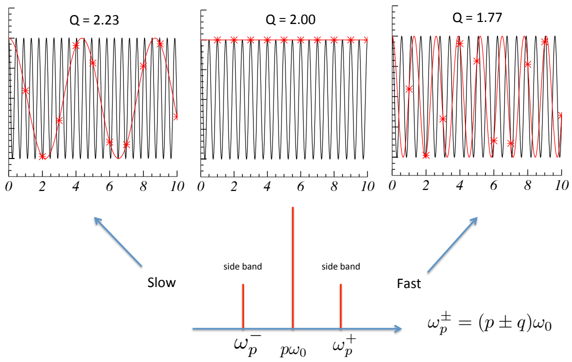
instability of a single bunch

Example

beam position at the cavity



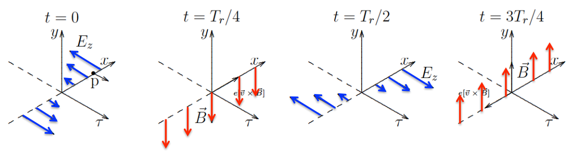
No oscillations →
$\omega = 0$



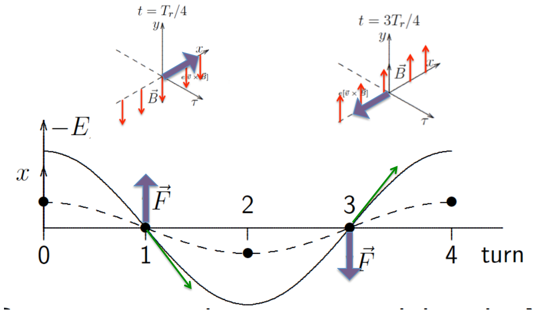
$\omega_p^{\pm} = (p \pm q)\omega_0$

behavior of the field in the cavity

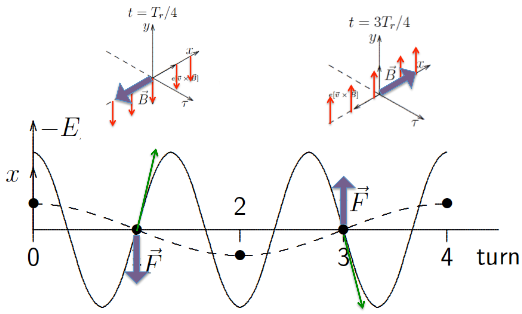
T_r = time of oscillation of the field in the cavity



Cavity tuned upper sideband



Cavity tuned upper sideband



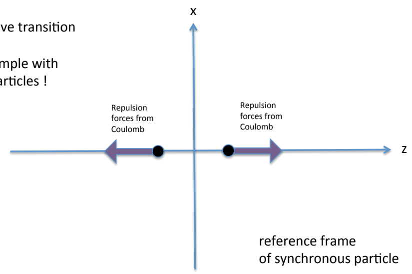
As for the Robinson Instability

$$\alpha_s = \frac{1}{\tau} \propto \sum_p I_p^2 [Z_\perp(\omega_p^+) - Z_\perp(\omega_p^-)]$$

Negative mass instability

Above transition

Example with
2 particles !

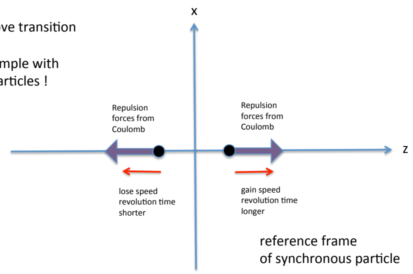


reference frame
of synchronous particle

Negative mass instability

Above transition

Example with
2 particles !

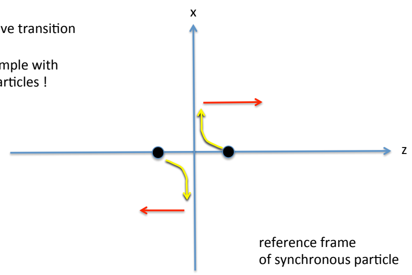


reference frame
of synchronous particle

Negative mass instability

Above transition

Example with
2 particles !



reference frame
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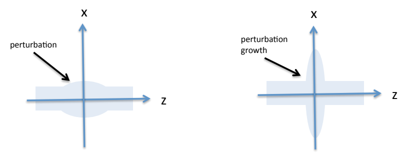
Negative mass instability

Above transition



repulsive forces attract particles as if their mass were negative

Head-Tail instability

coherent effect by direct space charge (!)

Summary

Dear Participants to the CAS – Prague,

This handout is not finished !!
I will add the HEAD-TAIL
and reduce the formulas (maybe)

Giuliano