Very high precision theoretical challenges

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Multi-TeV Probes of Standard Model Physics with the LHC

Today's high energy colliders

Today's high energy physics program relies mainly on results from

Collider	Process	status
LEP/LEP2	e^+e^-	1989-2000
Hera	$e^{\pm}p$	1992-2007
Tevatron	pp	1983-2011
LHC	pp	started 2010

- LEP high precision measurements of masses, couplings, EW parameters ...
- Hera: mainly measurements of parton densities and diffraction
- Tevatron: mainly discovery of top and many QCD measurements
- LHC designed to
  - discover the Higgs [done]
  - unravel possible BSM physics [elusive up to now]

Future high-energy colliders ?

- Future colliders are of course already under discussion: ILC (international linear collider,) CLIC, FCC (Future Circular Collider)...
- However no decision has been taken yet (collider type, beams, energy, location ...)
- The typical time-scale to build a collider is about 30 years. Still, given the huge scale of such a project decisions will happen only after LHC results from Run II

No matter what happens, for the next twenty years collider precision phenomenology will be LHC phenomenology

These lectures

These lectures will try to give you an overview of today's theoretical challenges when seeking for high precision analysis and interpretation of LHC data

Mains aims of today's LHC phenomenology are to (stress-) test the Higgs mechanism (precision Higgs measurements) and discover BSM physics. For this purpose one needs to

✓measure cross-sections
✓measure particle properties (spin, masses, couplings, ...)

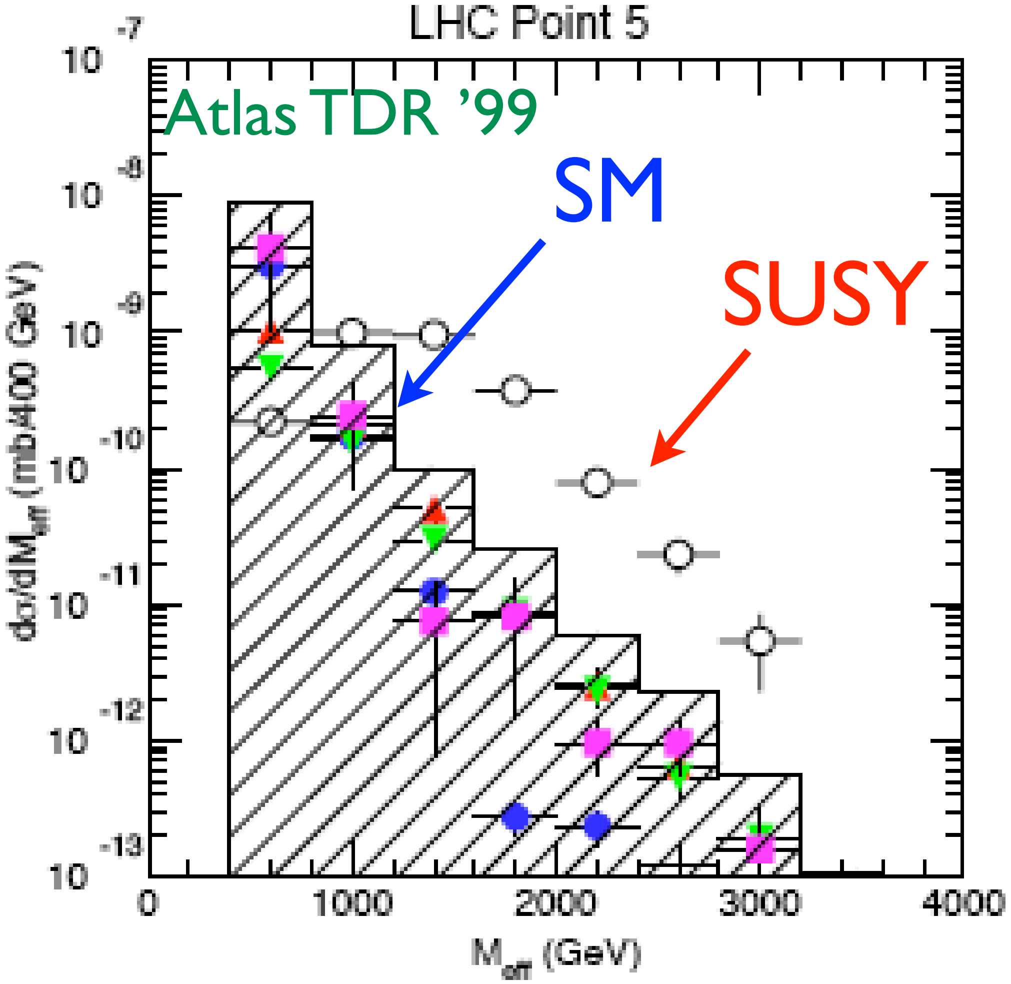
- Inclusive cross-section measurements can be done purely with data (no need for theory really)
- However, the extraction of properties requires theoretical predictions for cross-sections as a function of the "property to be measured"

These lectures will be about how we make precise theory predictions

These lectures

For a correct data interpretation it is crucial to

1. understand how much a given approximation can be trusted
2. know how to improve on it if necessary



How reliable is the SM prediction?
If an excess is seen in the M_{eff} distribution, can one safely conclude that it is because of New Physics?

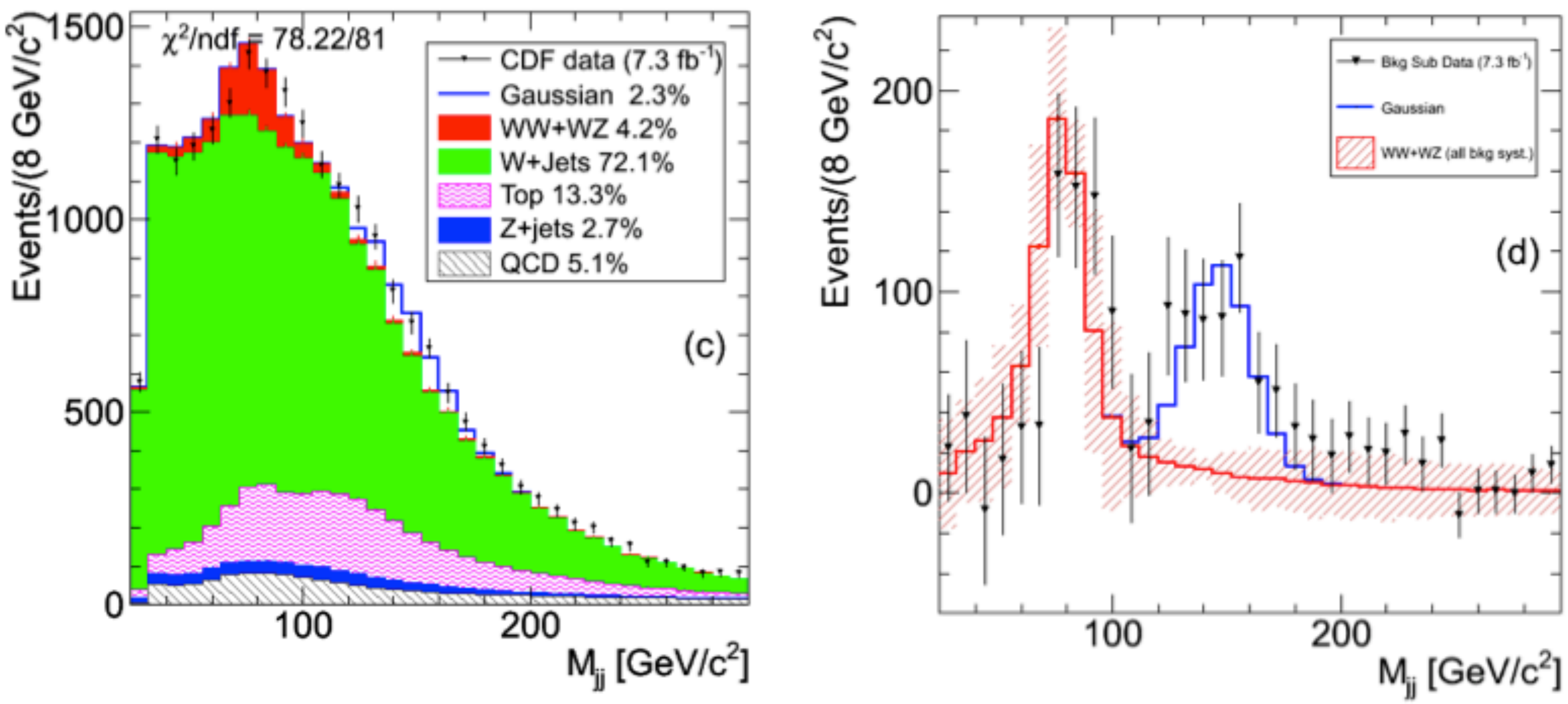
These lectures will also be about understanding how reliable some of the commonly used theoretical predictions are

One example

In 2011 CDF reported seeing a peak in M_{jj} for W + dijet events: first claim based on 4.3 fb^{-1} was of 3.2σ

CDF 1104.0699

Update to include 7.3 fb^{-1} $\Rightarrow$ 4.1σ



http://www-cdf.fnal.gov/physics/ewk/2011/wjj

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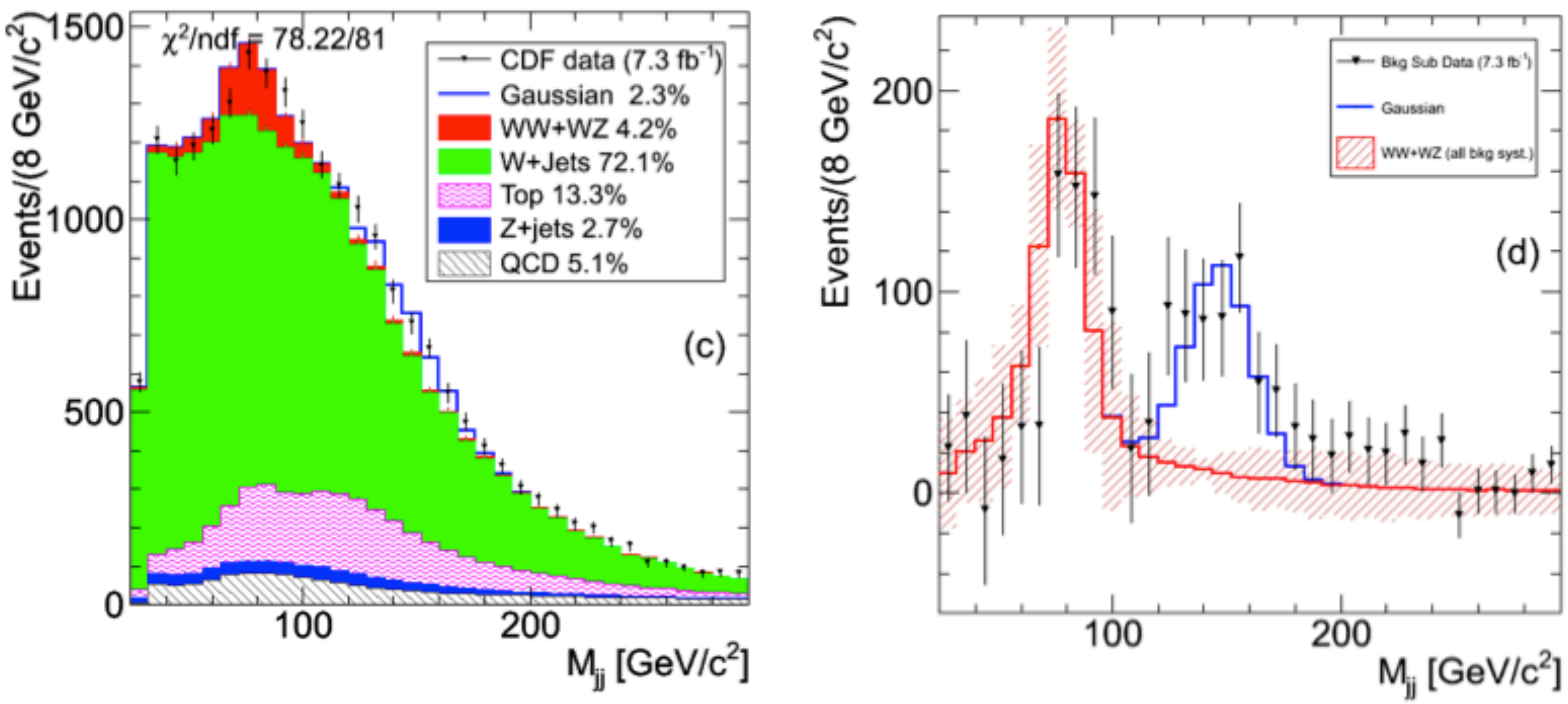

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Subsequently:

- a large numbers of tentative BSM explanations

[...]

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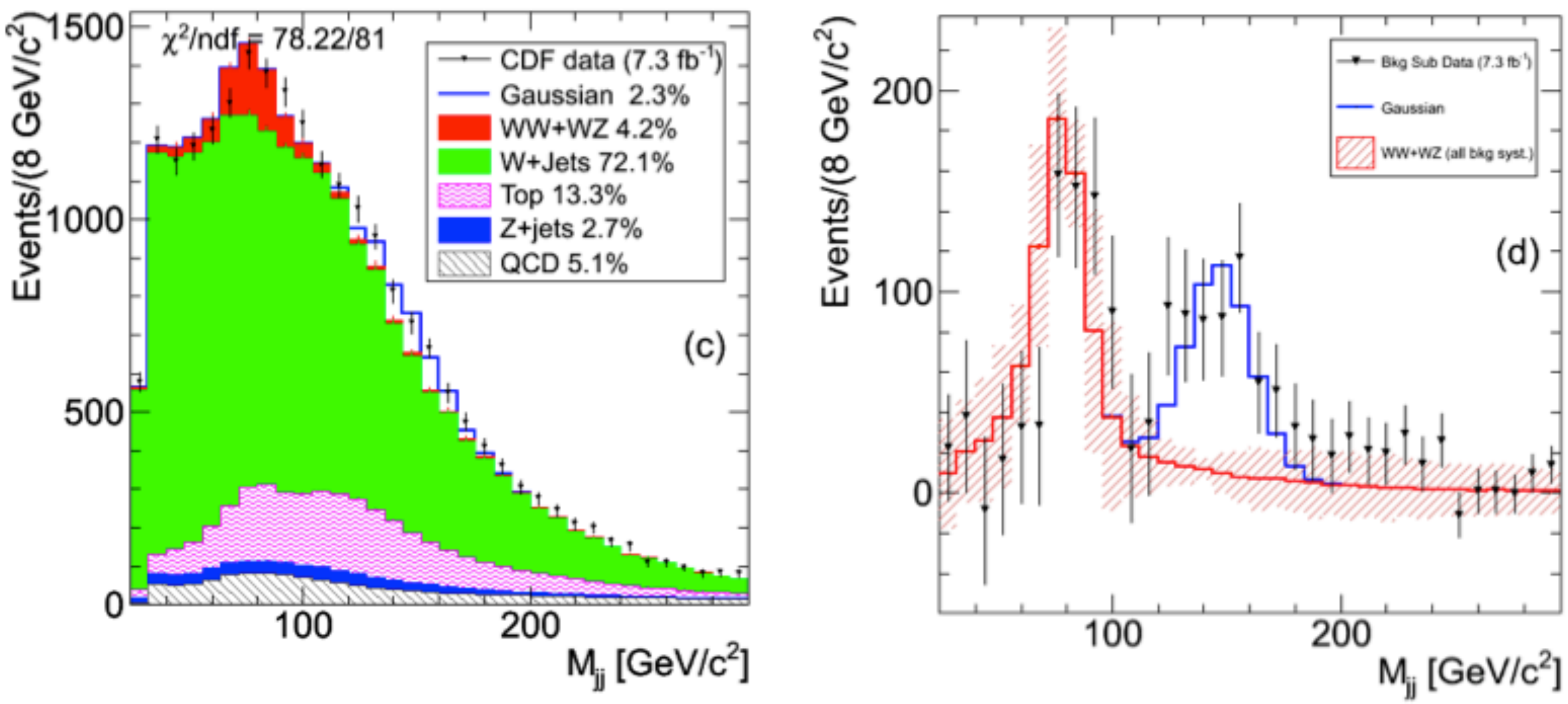

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- SM re-analysis (i.e. can this be due to poor modeling of QCD?) [...]

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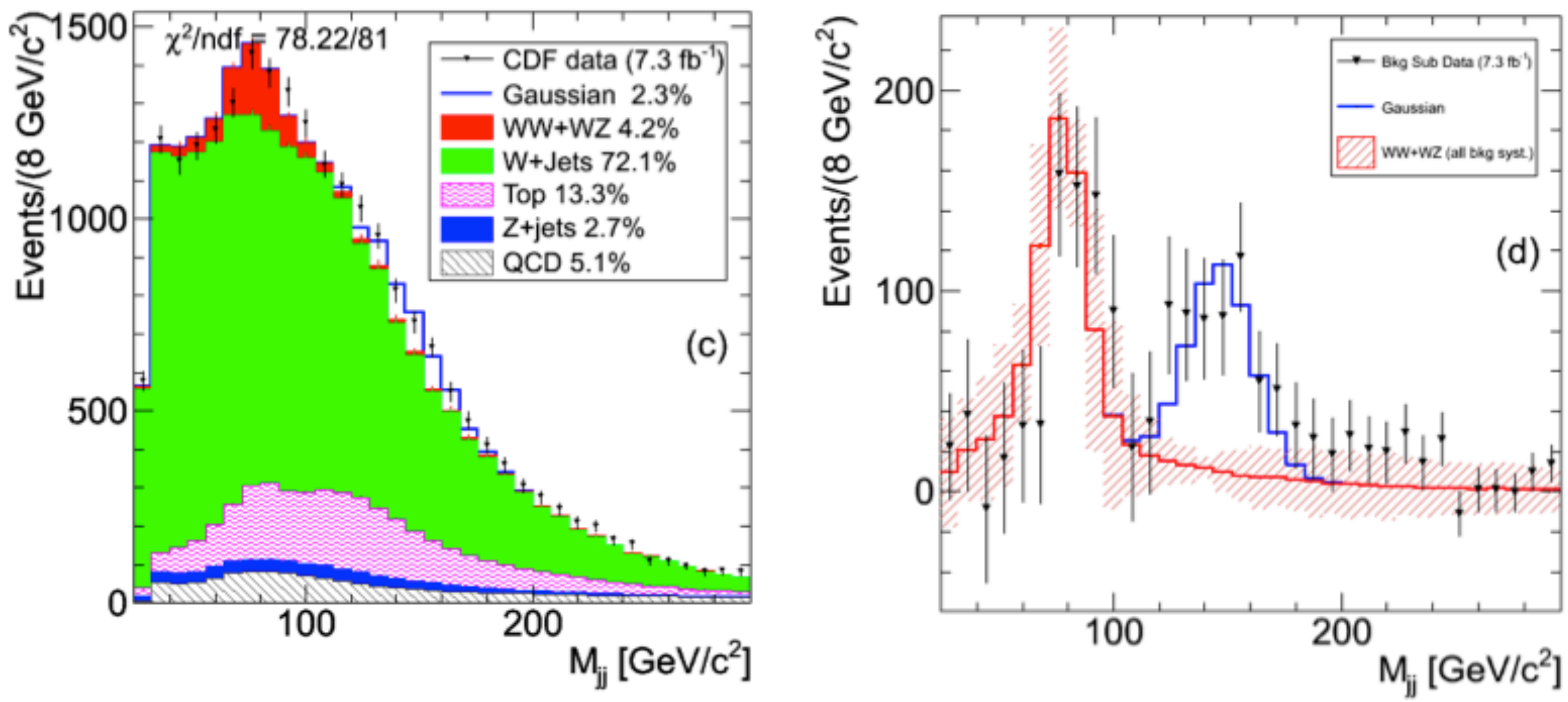

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Subsequently:

- a large numbers of tentative BSM explanations [...]
- SM re-analysis (i.e. can this be due to poor modeling of QCD?) [...]
- D0 data do *not* support excess seen by CDF D0 col. 1106.1921

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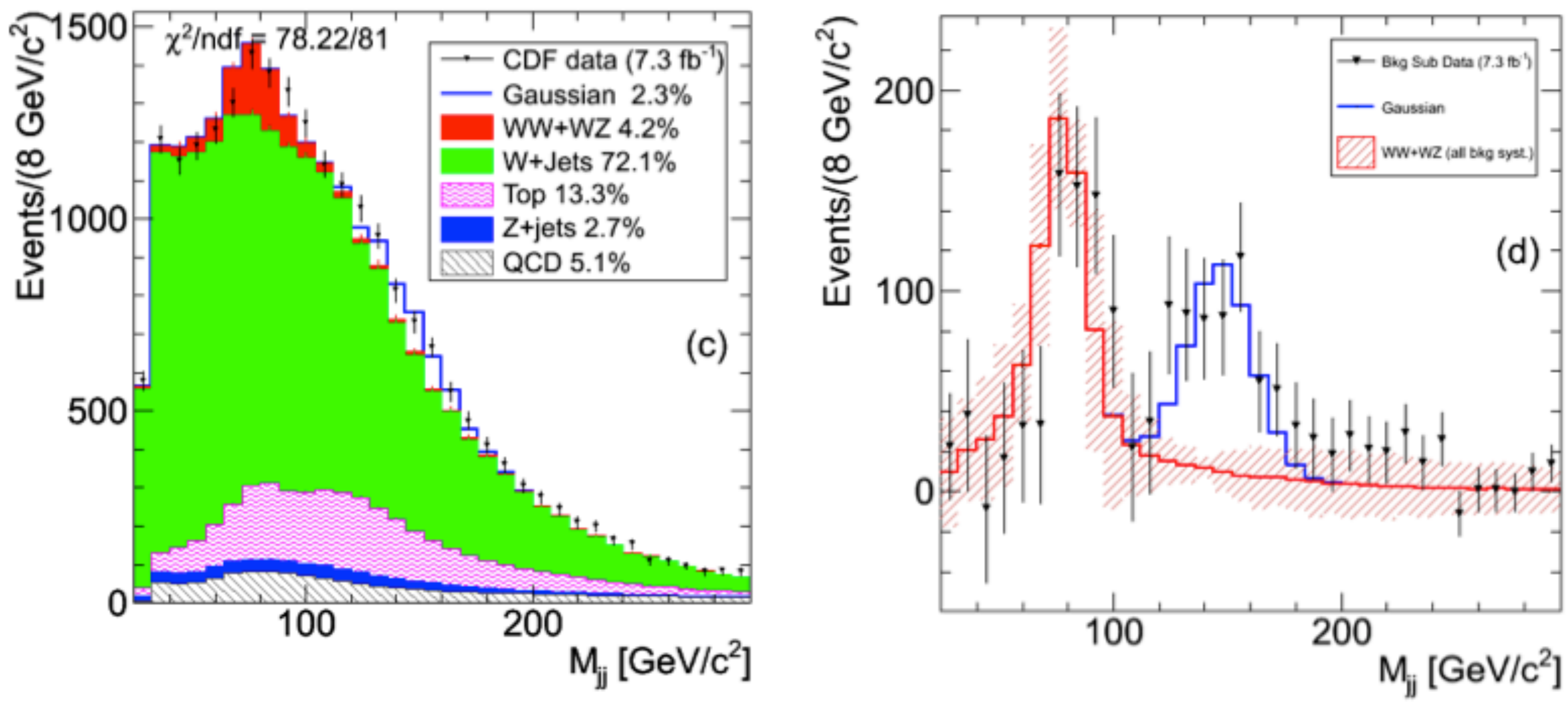

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The statement that

"Once we see a resonant peak on top of smooth background it's New Physics and we don't need precise SM predictions"

Is not true

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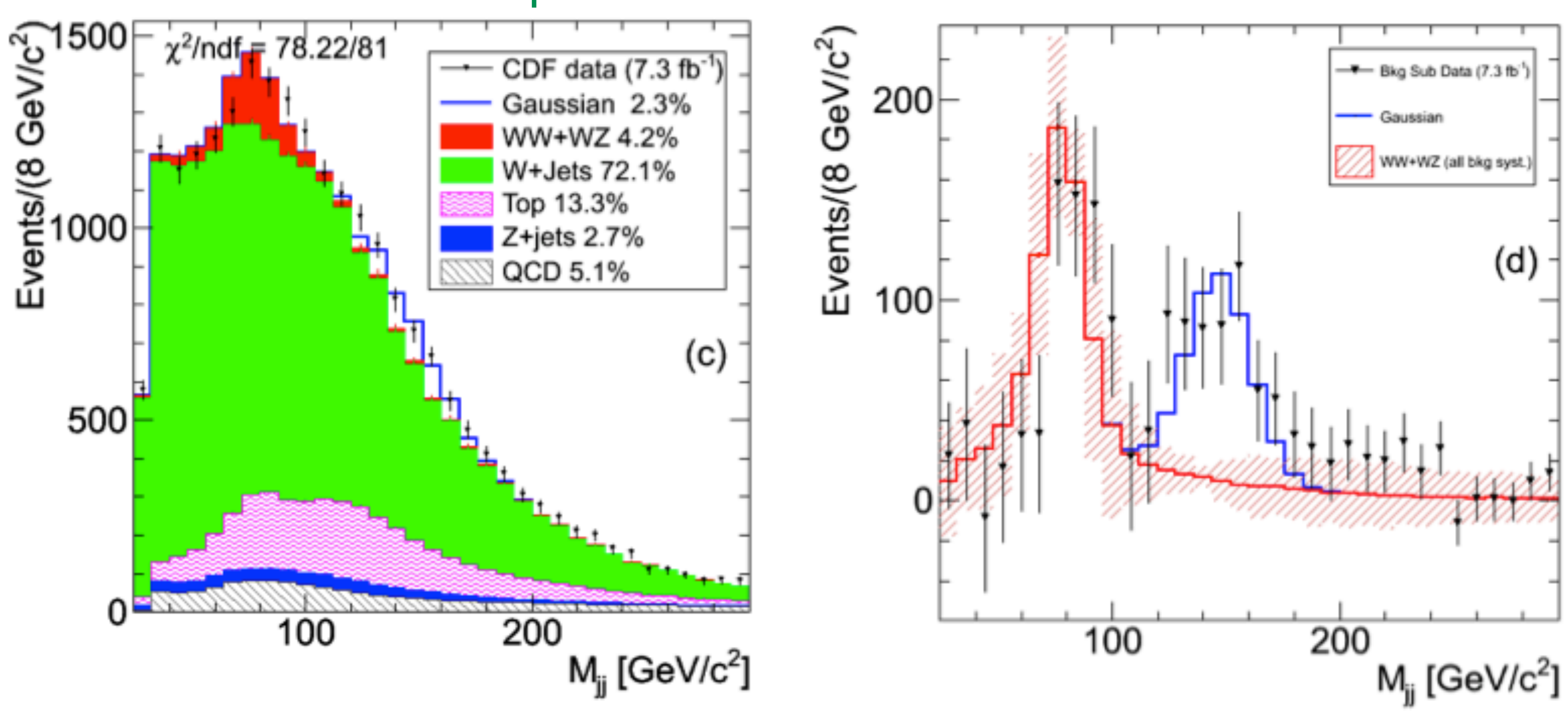


Conclusion

http://www-cdf.fnal.gov/physics/ewk/2011/wjj

- confirmation or not by a different experiment very important (re-analysis of new data not sufficiently independent)
- need robust SM predictions with reliable errors

This means that one needs solid understanding of Standard Model processes

Precision collider phenomenology

Precision is achieved by computing quantum corrections (EW, QCD)

By far dominant corrections are QCD ones because

- QCD coupling is larger
- QCD radiation from initial state
- color enhancement

We will start discussing some basic QCD. This will give us the elements to discuss what I consider the most pressing high-precision theoretical challenges today.

To give you a taste, my top ten high-precision challenges in collider phenomenology include ...

My top ten high-precision theory challenges

	Theory challenge
1.	automated NLO
2.	reliable PDF error
3.	PDF with EW effects
4.	NNLO for generic $2 \to 2$ processes
5.	analytic understanding of jet-substructure
6.	NNLO + parton shower
7.	N^3LO for Higgs and Drell Yan (differential?)
8.	multi-jet merging
9.	automated NNLL resummations
10.	improve Monte Carlo (w reliable error estimate)

QCD

Satisfactory model for strong interactions: non-abelian gauge theory SU(3)

$$U^\dagger U = UU^\dagger = 1 \quad \det(U) = 1$$

Hadron spectrum fully classified with the following assumptions

- hadrons (baryons,mesons): made of spin 1/2 quarks
- each quark of a given flavour comes in N_c=3 colors
- SU(3) is an exact symmetry
- hadrons are colour neutral, i.e. colour singlet under SU(3)
- observed hadrons are colour neutral ⇒ hadrons have integer charge

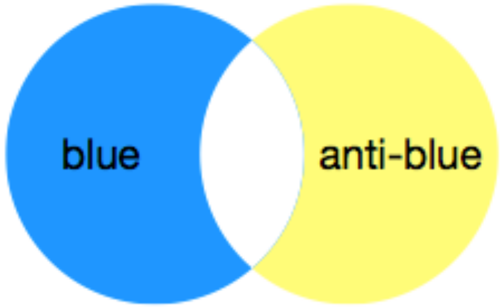
$$\begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$$

Color singlet hadrons

Quarks can be combined in 2 elementary ways into color singlets of the $SU_c(3)$ group

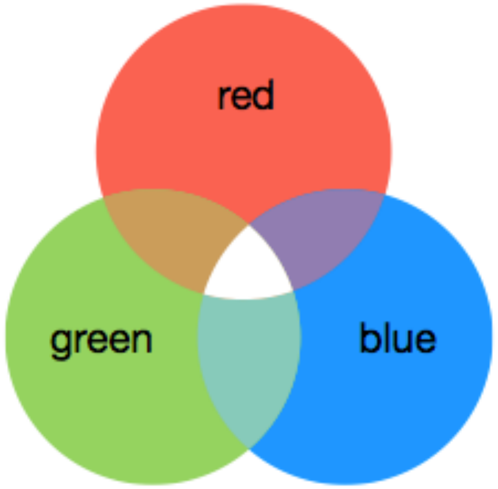
Mesons (bosons, e.g. pion ...)

$$\sum_i \psi_i^* \psi_i \to \sum_{ijk} U_{ij}^* U_{ik} \psi_j \psi_k = \sum_k \psi_k^* \psi_k$$



Baryons (fermions, e.g. proton, neutrons ...)

$$\sum_{ijk} \epsilon_{ijk} \psi_i \psi_j \psi_k \to \sum_{ii'jj'kk'} \epsilon_{ijk} U_{ii'} U_{jj'} U_{kk'} \psi_{i'} \psi_{j'} \psi_{k'} = \sum_{i'j'k'} \epsilon_{i'j'k'} \det(U) \psi_{i'} \psi_{j'} \psi_{k'}$$



First experimental evidence for colour

1. Existence of Δ^{++} particle: particle with three up quarks of the same spin and with symmetric spacial wave function. Without an additional quantum number Pauli's principle would be violated
   $\Rightarrow$ color quantum number

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I. Existence of Δ^{++} particle: particle with three up quarks of the same spin and with symmetric spacial wave function. Without an additional quantum number Pauli's principle would be violated
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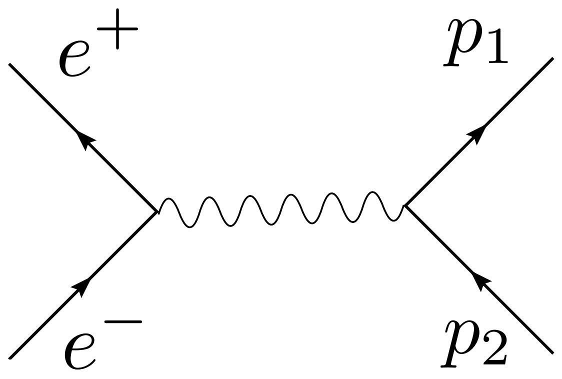
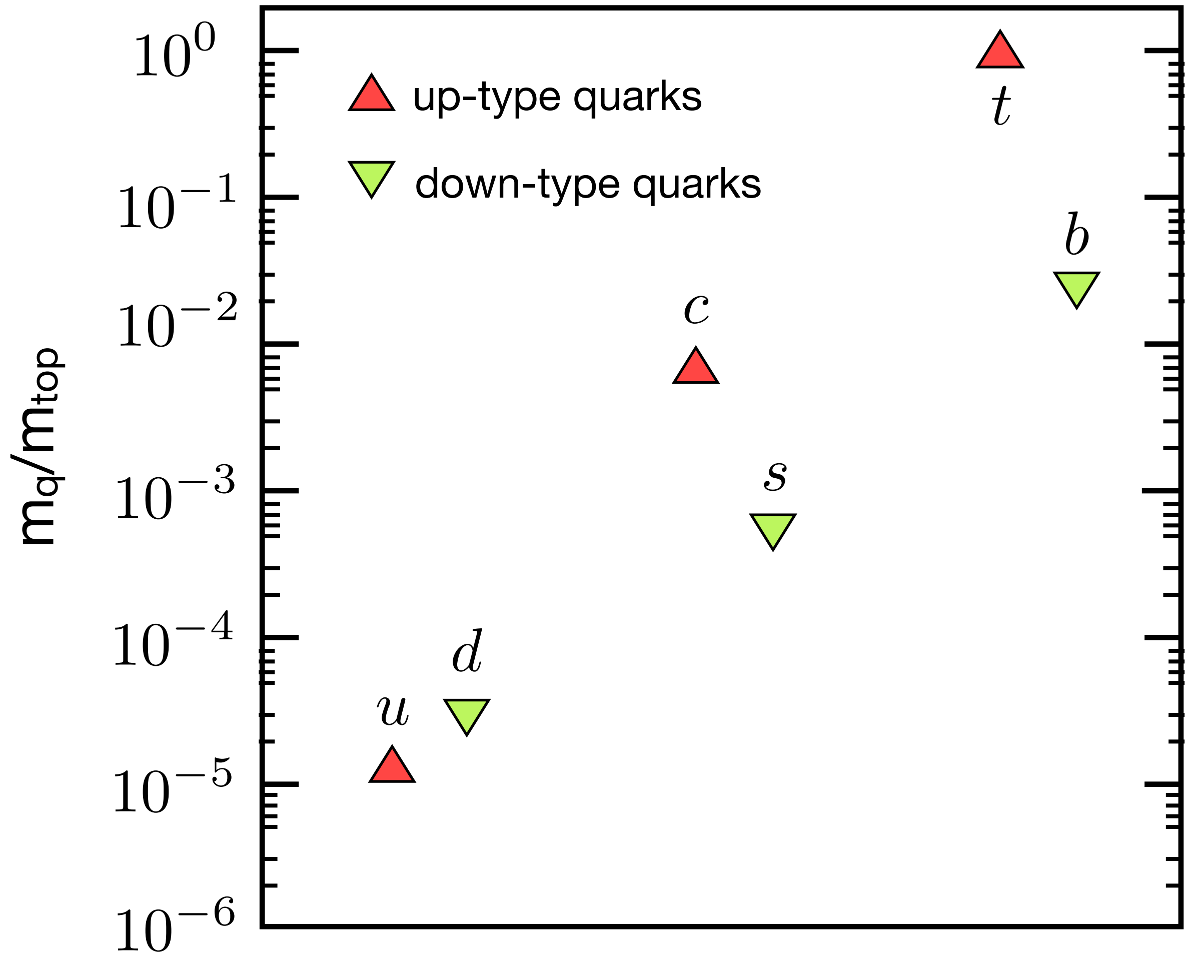
II. R-ratio: ratio of ($e^+e^- \to$ hadrons)/($e^+e^- \to \mu^+\mu^-$)

$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$



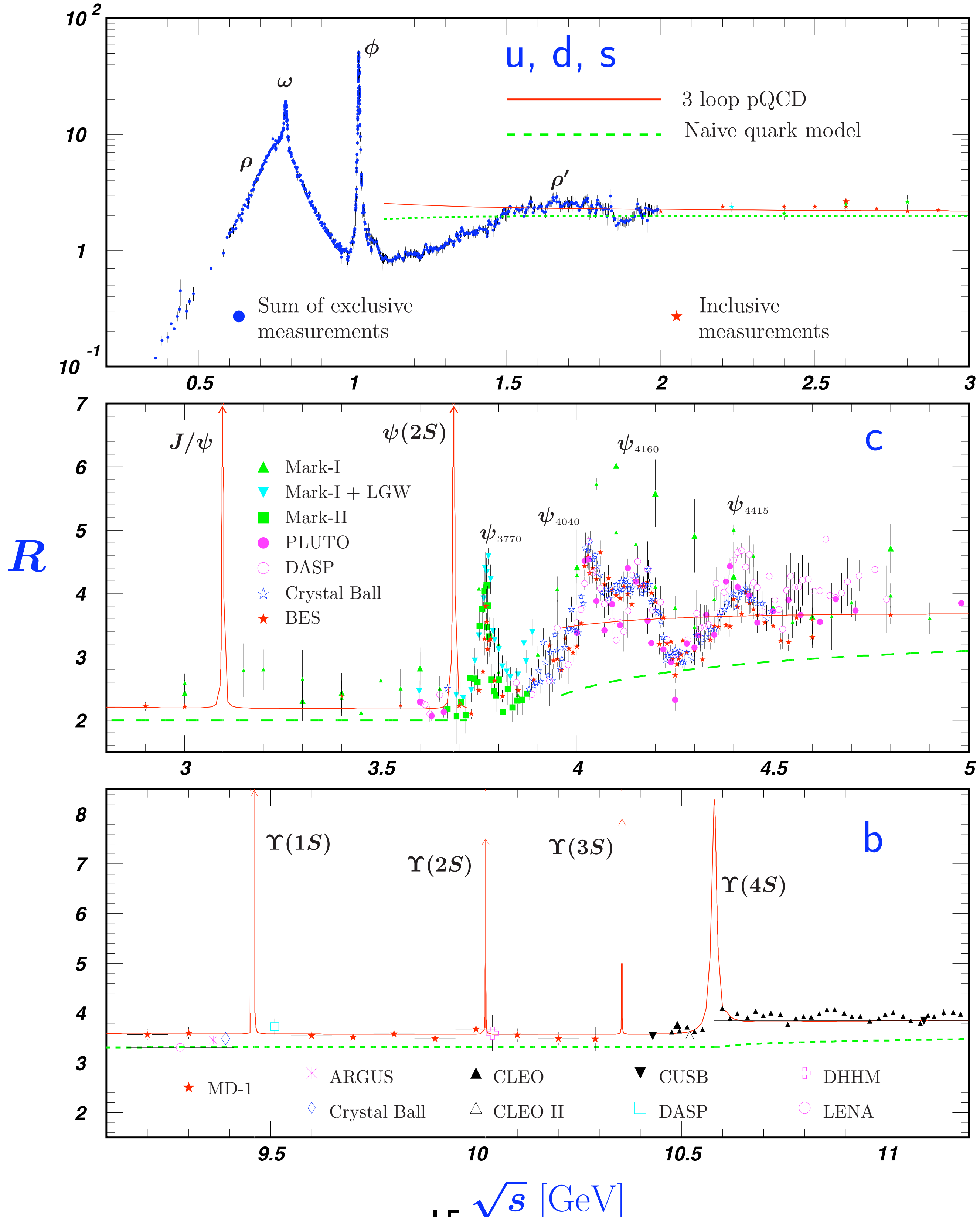
Data compatible with $N_c = 3$. Will come back to R later.

Quark mass spectrum

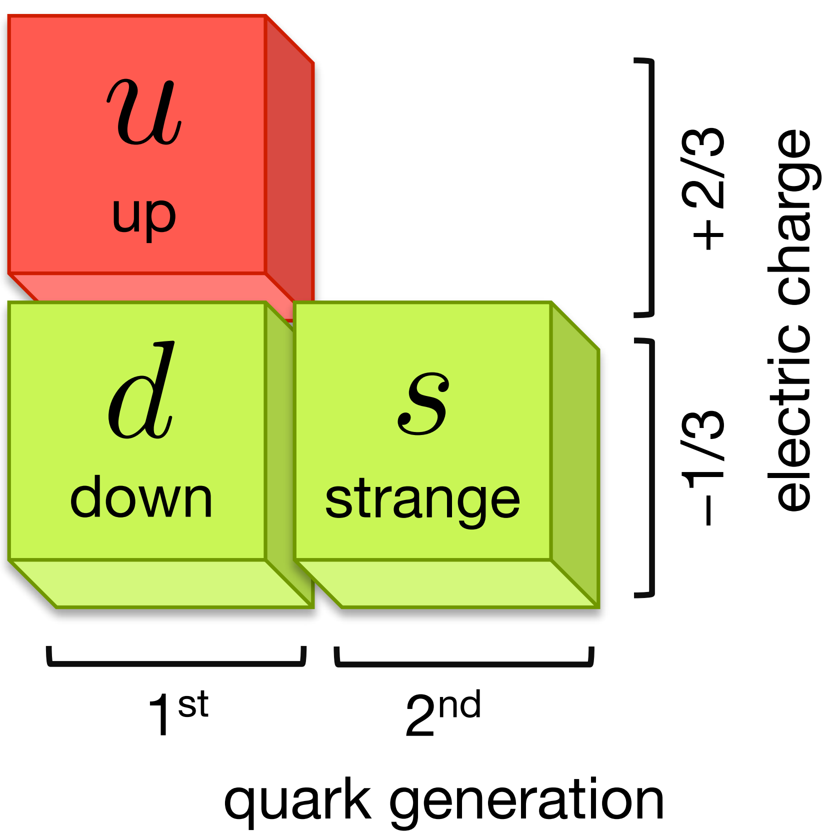


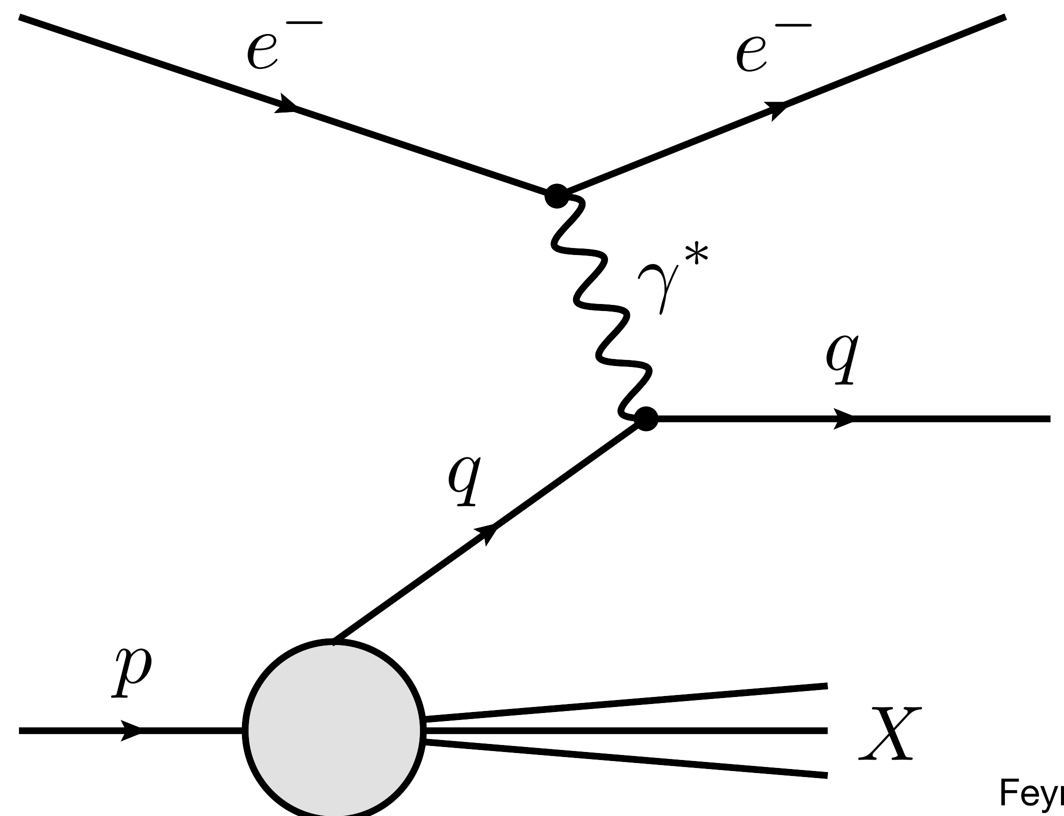
charge 2/3 mass=	up few MeV	charm ~1.6 GeV	top ~172 GeV
charge -1/3 mass =	down few MeV	strange ~100 MeV	bottom ~5 GeV

The R-ratio: comparison to data



QCD matter sector

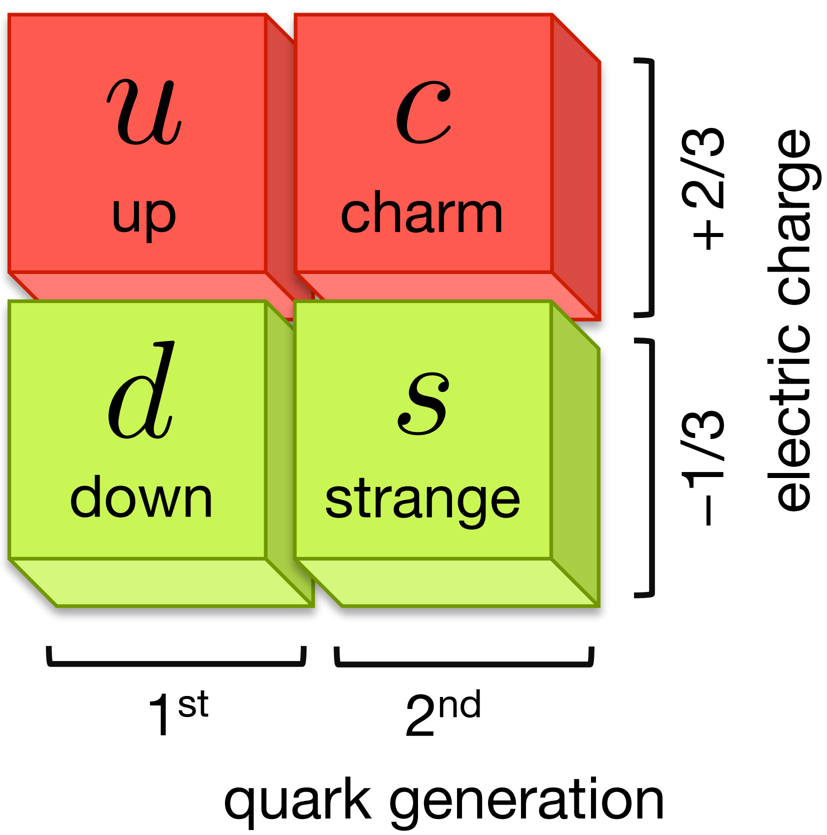




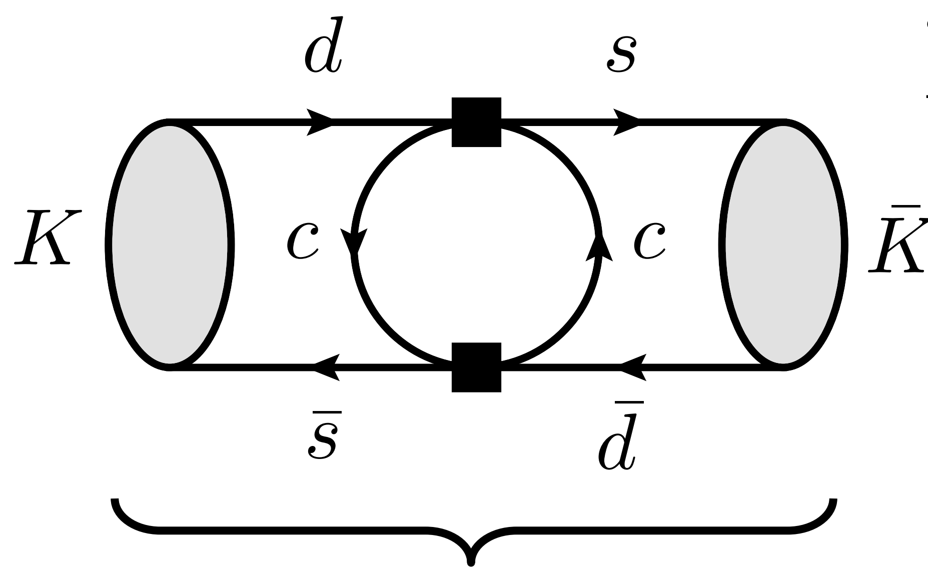
Feynman diagram describing DIS of an electron on a proton

- The light quark's existence was validated by the SLAC's deep inelastic scattering (DIS) experiments in 1968: strange was a necessary component of Gell-Mann and Zweig's three-quark model, it also provided an explanation for the kaon and pion mesons discovered in cosmic rays in 1947

QCD matter sector



Feynman diagram describing the mixing of a kaon into its anti-particle. The black boxes indicate weak effective four-fermion interactions

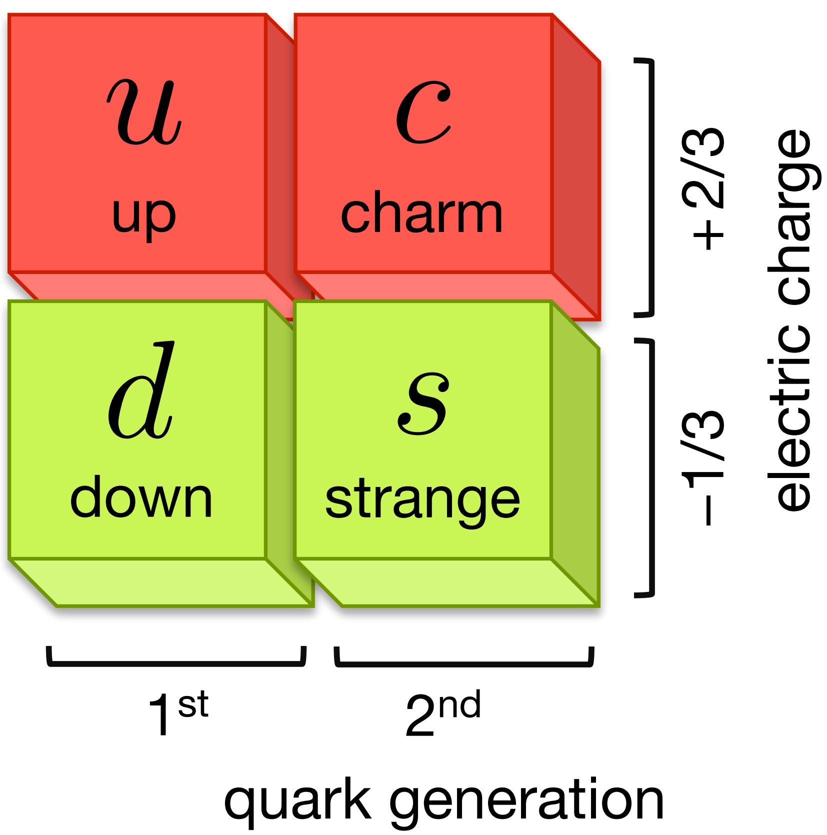


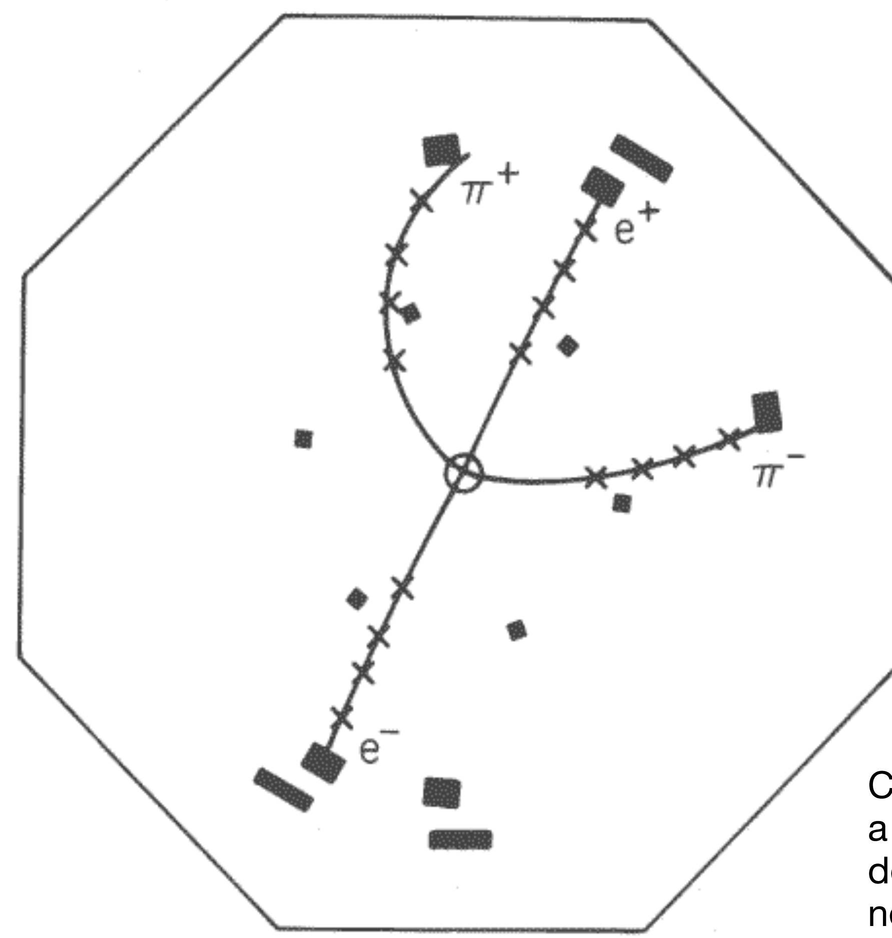
$$\propto G_F^2 \, \sin^2 \theta_c \, \frac{m_c^2}{M_W^2}$$

- In 1970 Glashow, Iliopoulos, and Maiani (GIM mechanism) presented strong theoretical arguments for the existence of the as-yet undiscovered charm quark, based on the absence of flavor-changing neutral currents

[S. L. Glashow, J. Iliopoulos and L. Maiani, *Phys. Rev.* D **2** (1970) 2]

QCD matter sector

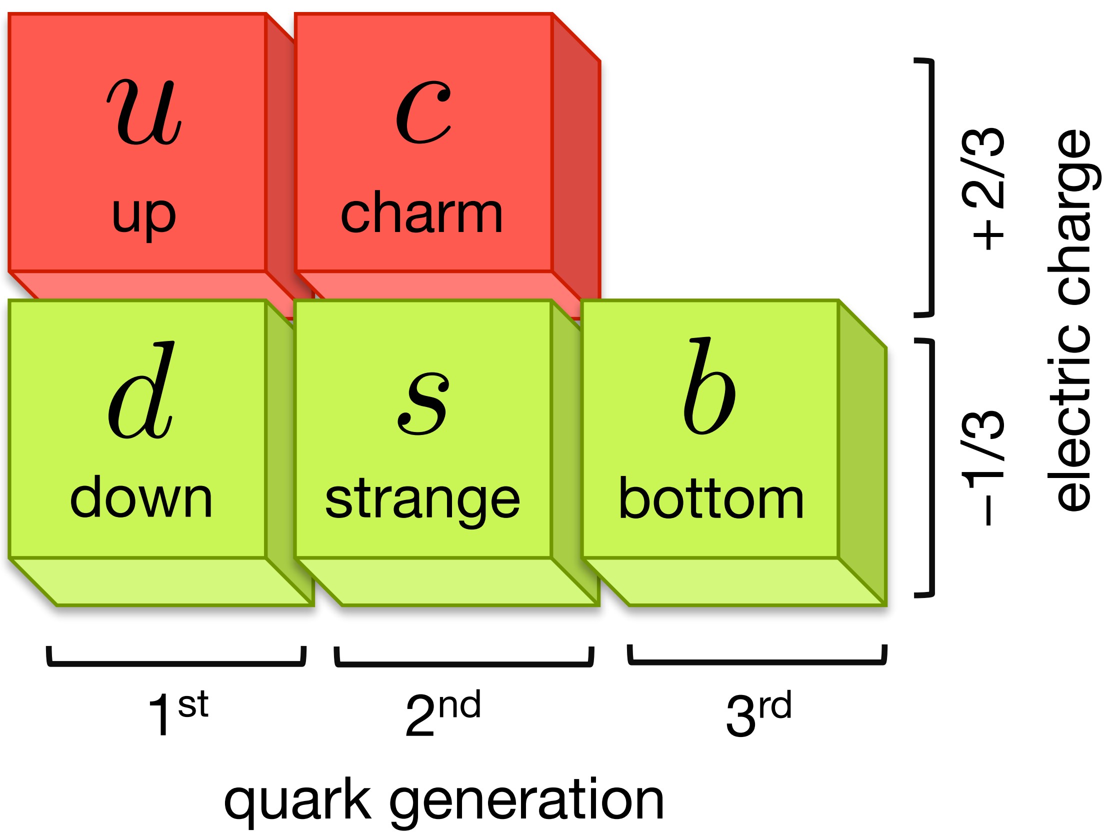


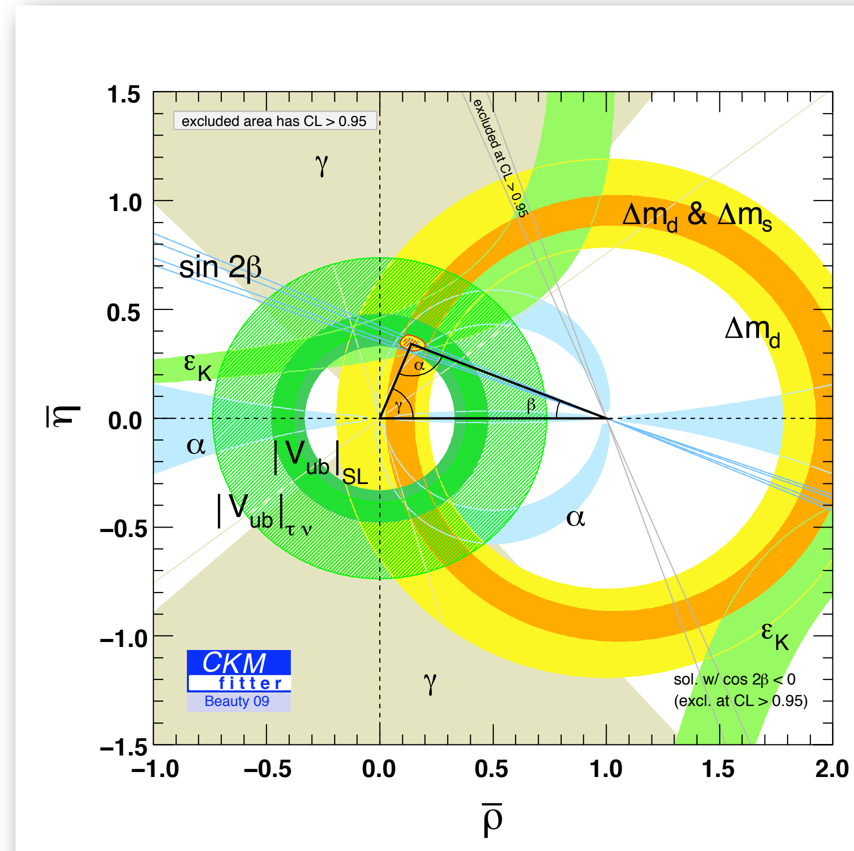


Computer reconstruction of a ψ′ decay in the Mark I detector at SLAC, making a near-perfect image of the Greek letter ψ

- Charm quarks were observed almost simultaneously in November 1974 at SLAC and at BNL as charm anti-charm bound states (<u>charmonium</u>). The two groups had assigned the discovered meson two different symbols, *J* and ψ. Thus, it became formally known as the *J*/ψ meson (Nobel Prize 1976)

QCD matter sector



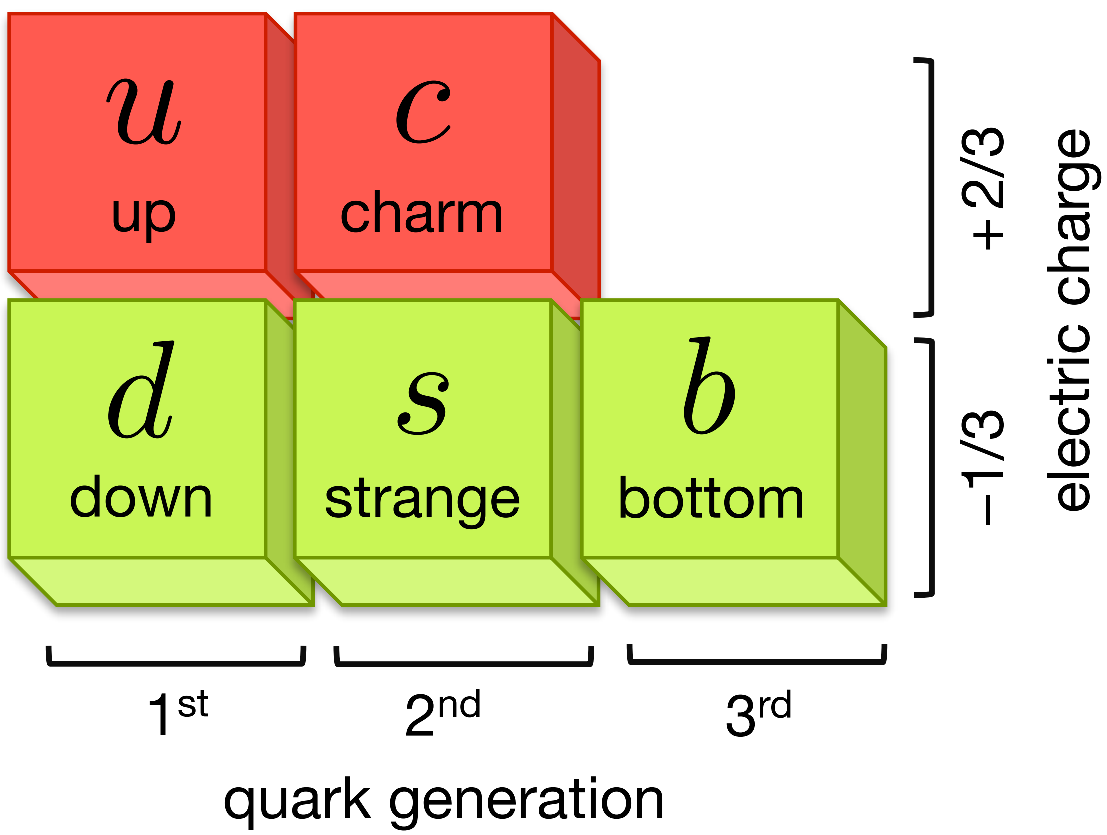


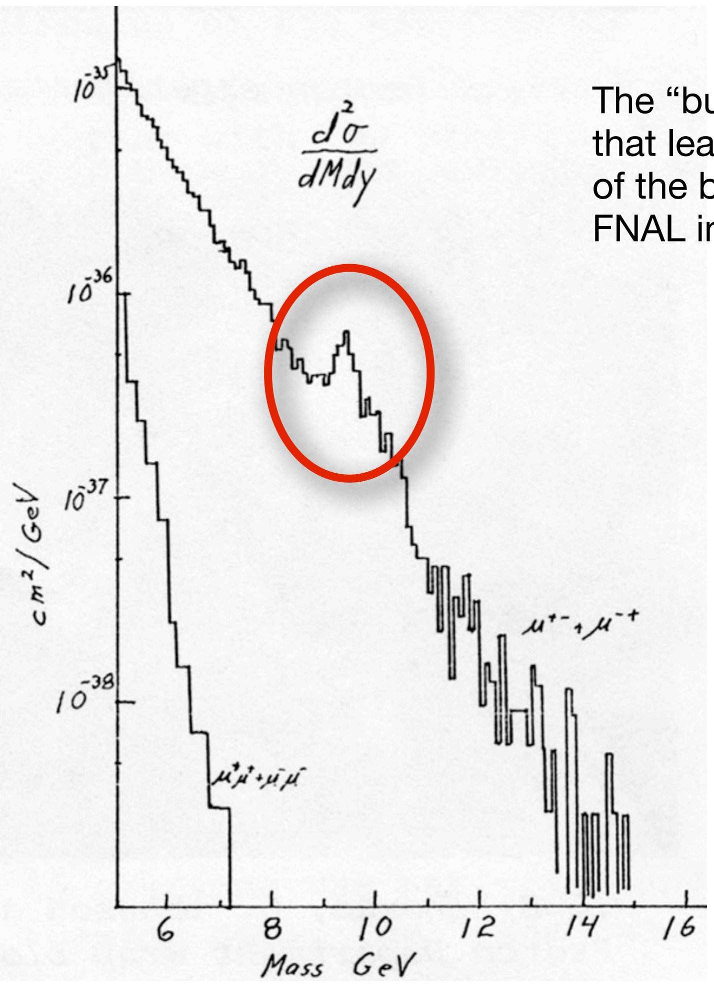
Unitarity triangle measuring the amount of CP violation in the standard model

- The bottom quark was theorized in 1973 by Kobayashi and Maskawa in order to accommodate the phenomenon of CP violation, which requires the existence of at least three generations of quarks in Nature (Nobel Prize 2008)

[M. Kobayashi and T. Maskawa, *Prog. Theor. Phys.* **49** (1973) 652]

QCD matter sector





The “bump” at 9.5 GeV that lead to the discovery of the bottom quark at FNAL in 1977

- In 1977, physicists working at the fixed target experiment E288 at FNAL discovered the ϒ (Upsilon) meson. This discovery was eventually understood as being the bound state of the bottom and its anti-quark (bottomonium)

QCD matter sector

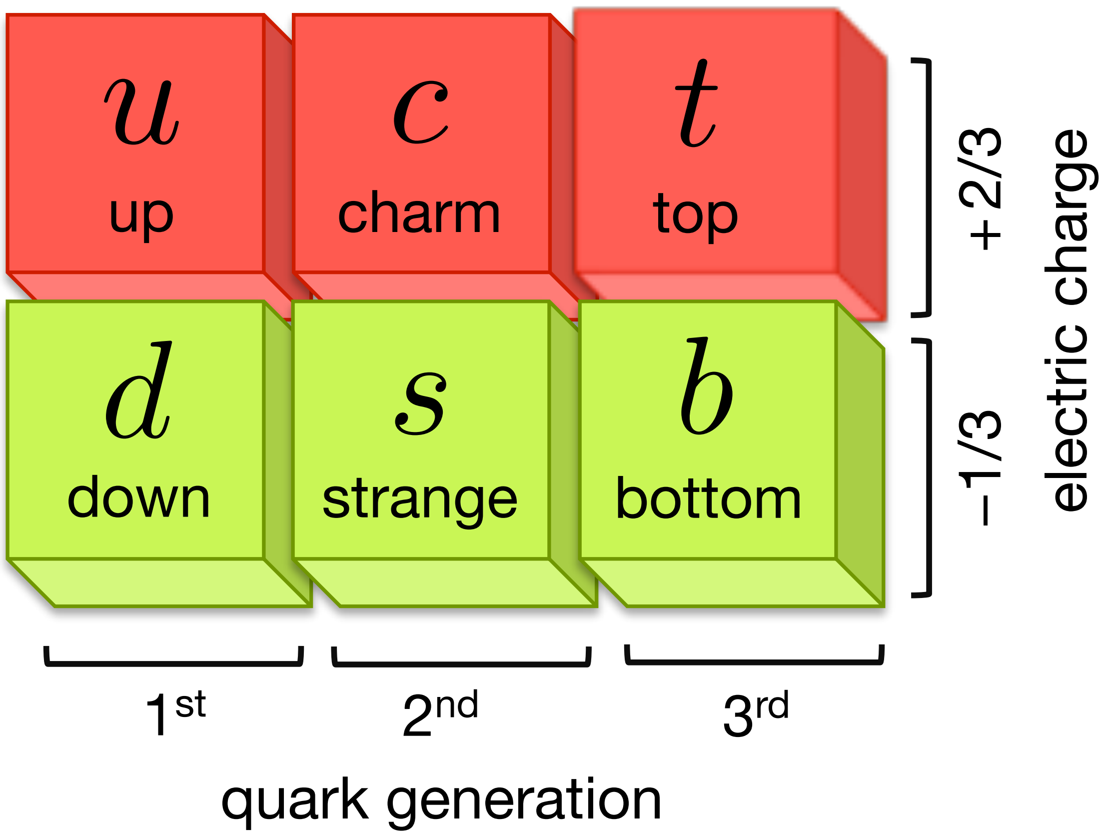
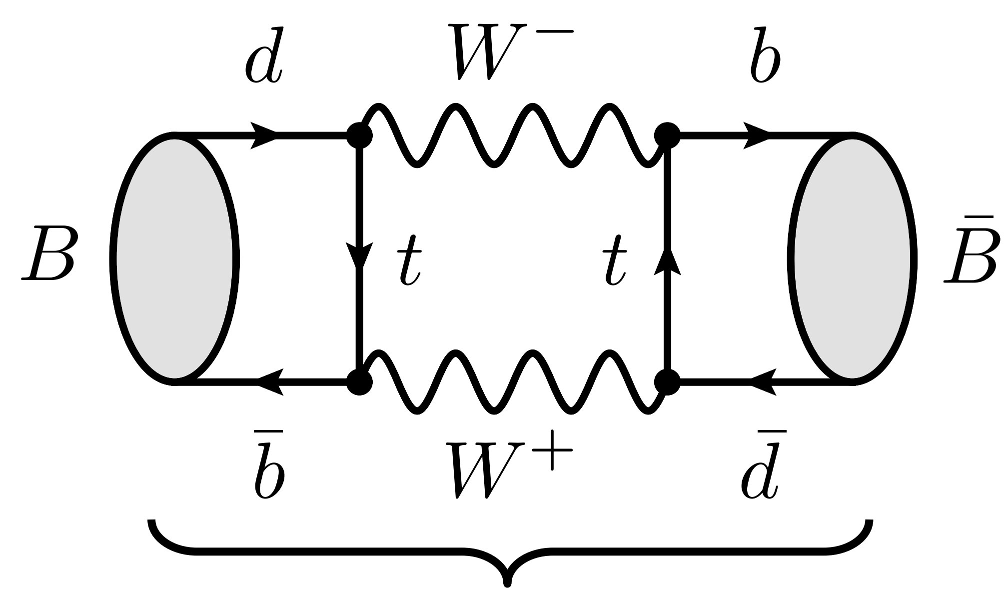


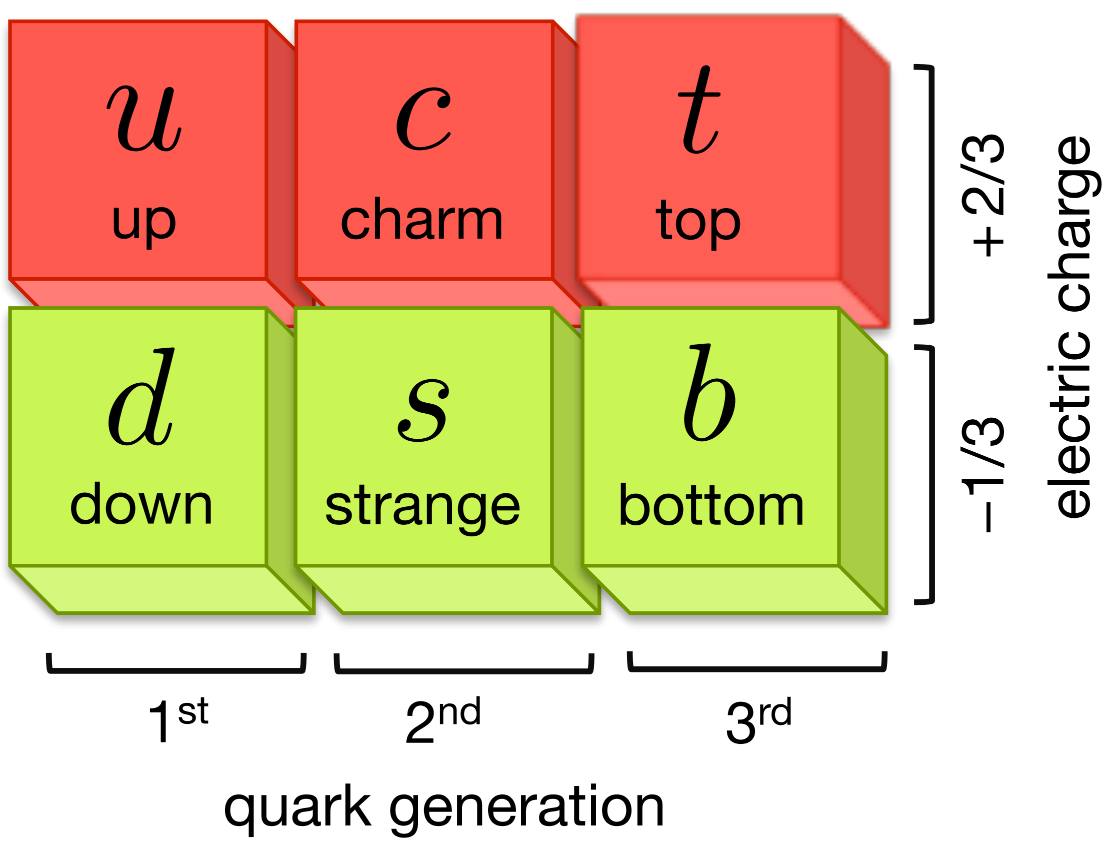
Diagram involving the virtual exchange of top quarks that induces a mass difference in the *B* meson system

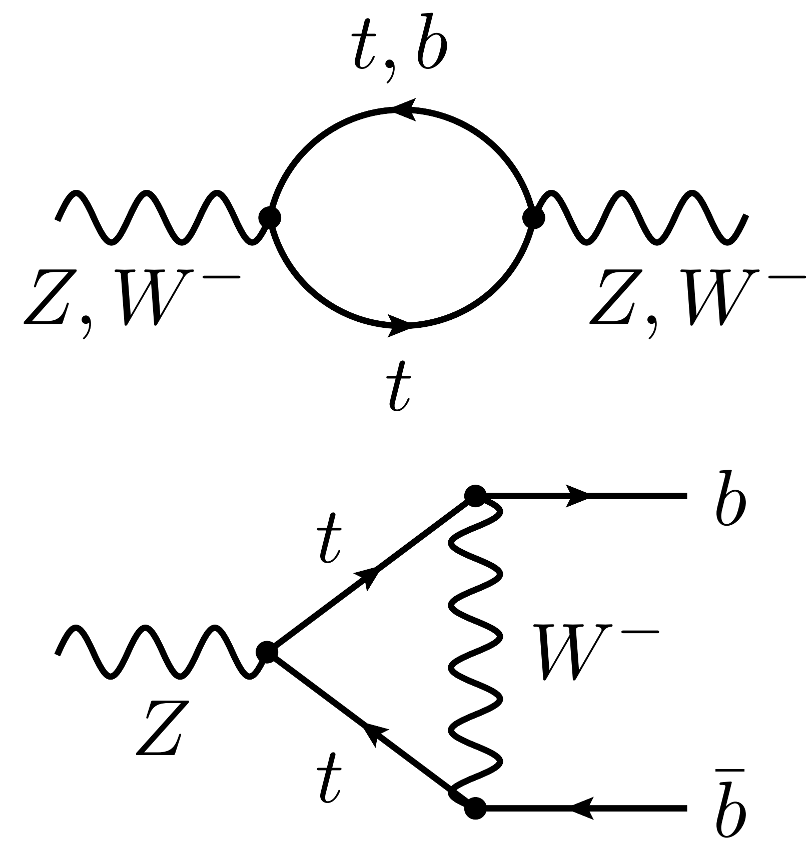


$$\Delta M_B \propto G_F^2 \, m_B \, f_B^2 \, |V_{td}|^2 \, m_t^2$$

- The measurement of the oscillations of *B* mesons into its own anti-particles in 1987 by ARGUS led to the conclusion that the top-quark mass has to be larger than 50 GeV. This was a big surprise at that time, because in 1987 the top quark was generally believed to be much lighter

QCD matter sector

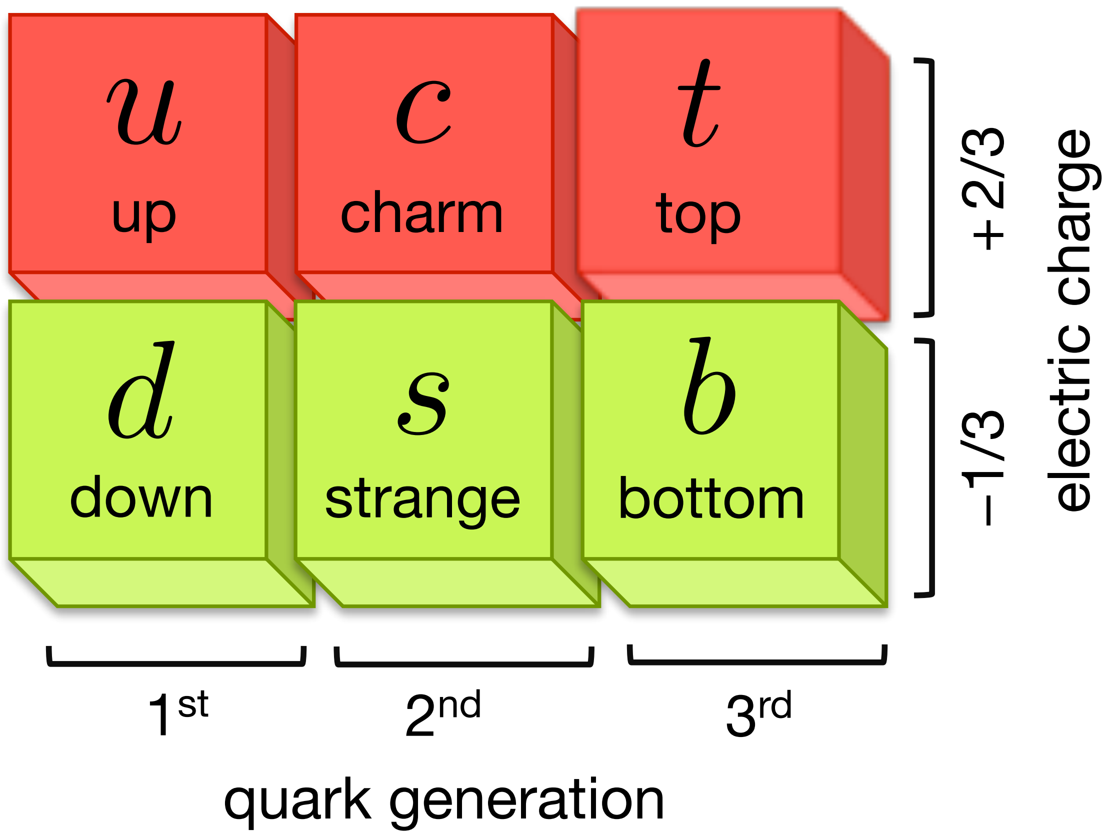


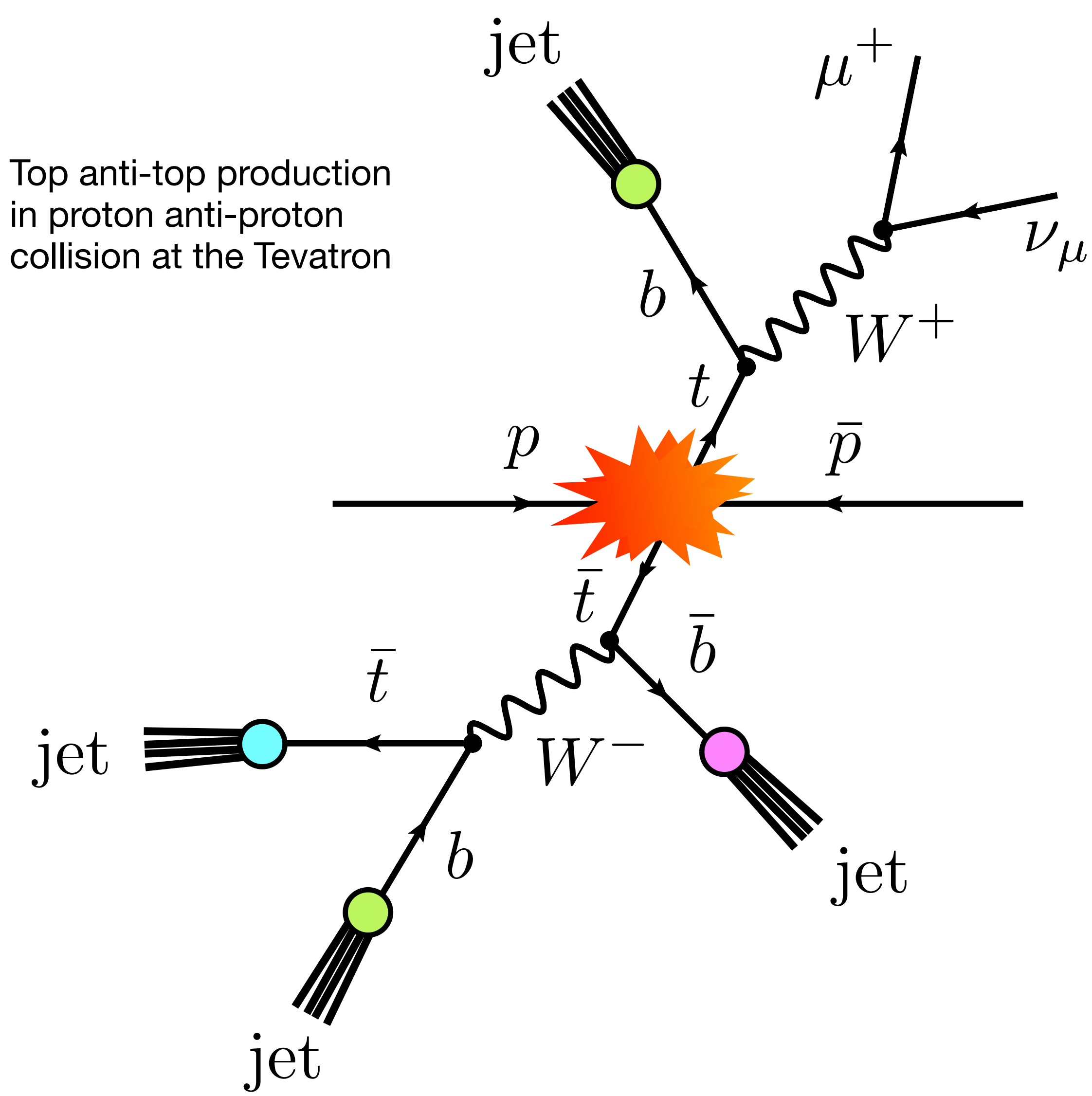


Diagrams that feature a quadratic dependence on the top-quark mass

- It was also realized that certain precision measurements of the electroweak vector-boson masses and couplings are very sensitive to the value of the top-quark mass. By 1994 the precision of these indirect measurements led to a prediction of the top-quark mass between 145 GeV and 185 GeV

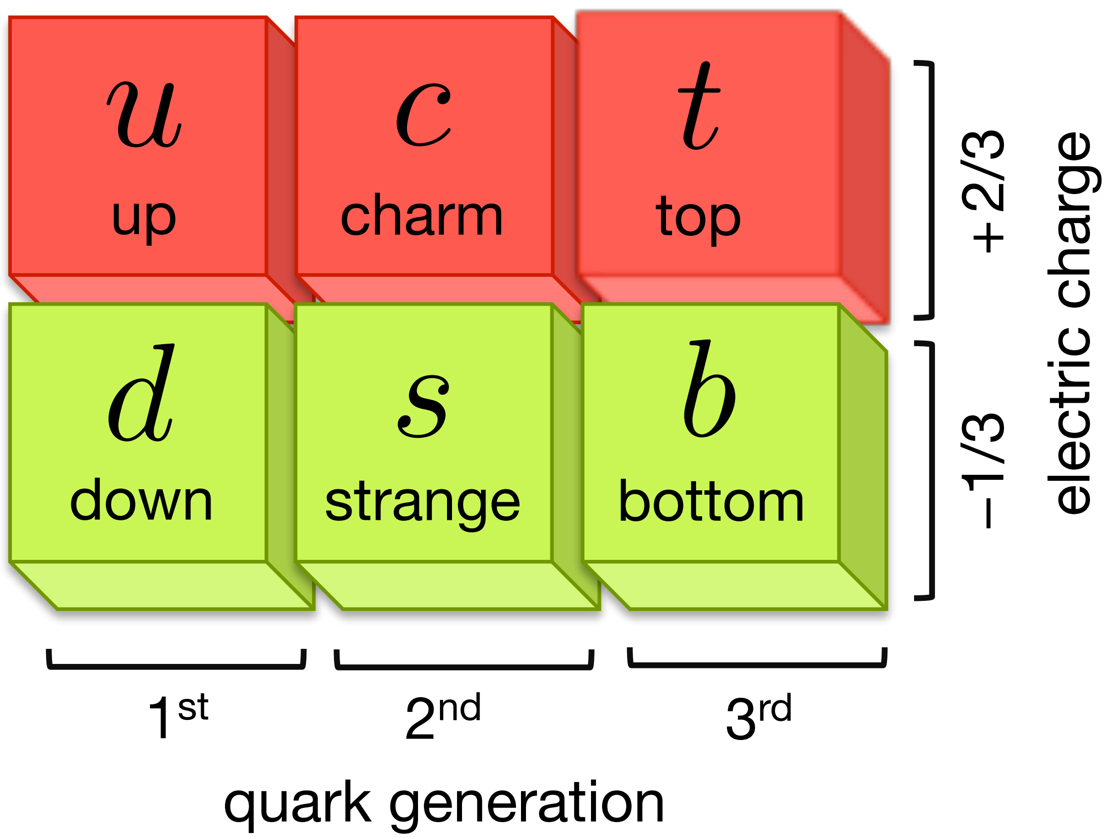
QCD matter sector

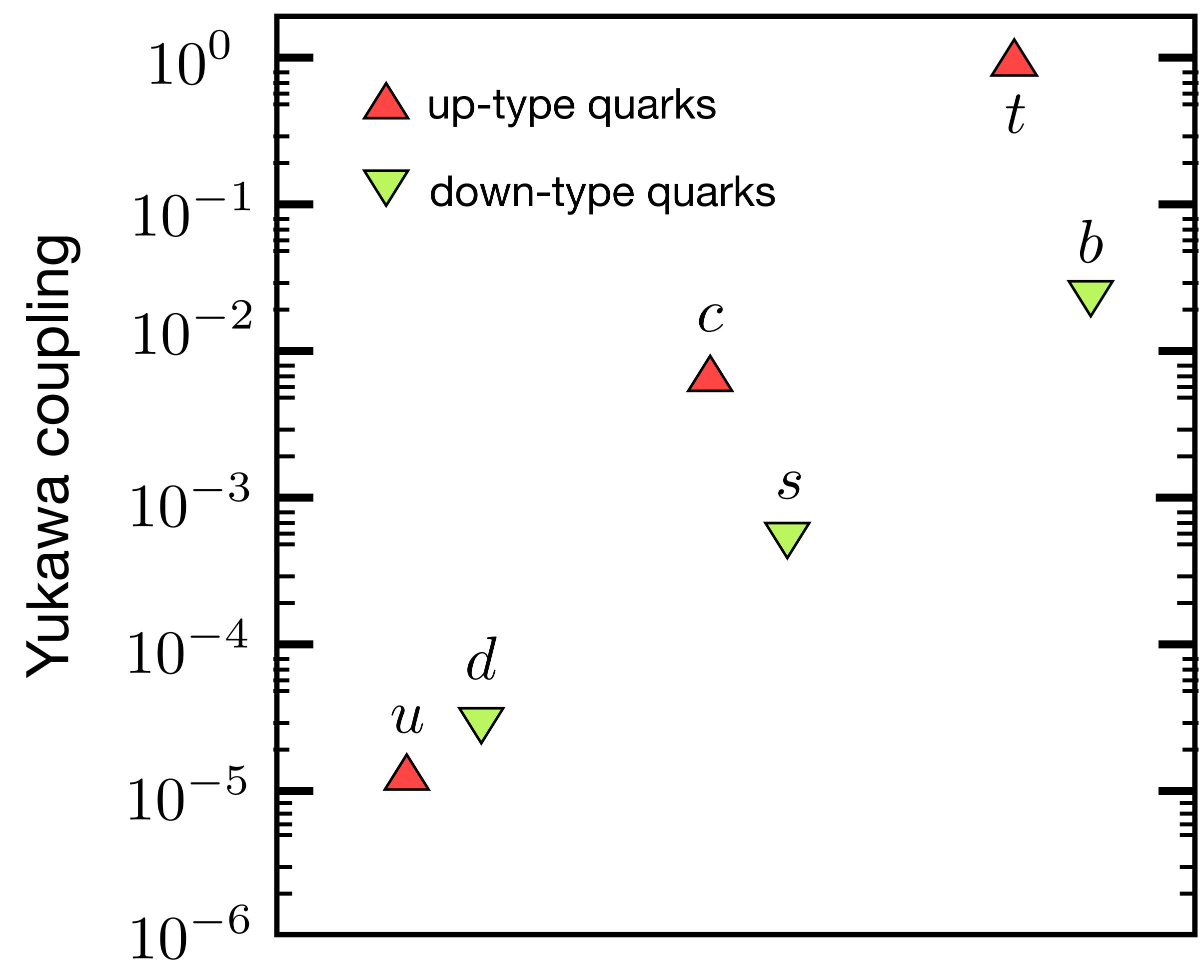




- The top quark was finally discovered in 1995 by CDF and D0 at FNAL. While the mass of the top quark is today quite well known, m_t = (173.0 ± 0.6 ± 0.8) GeV, other properties like its charge (2/3) are much less constrained

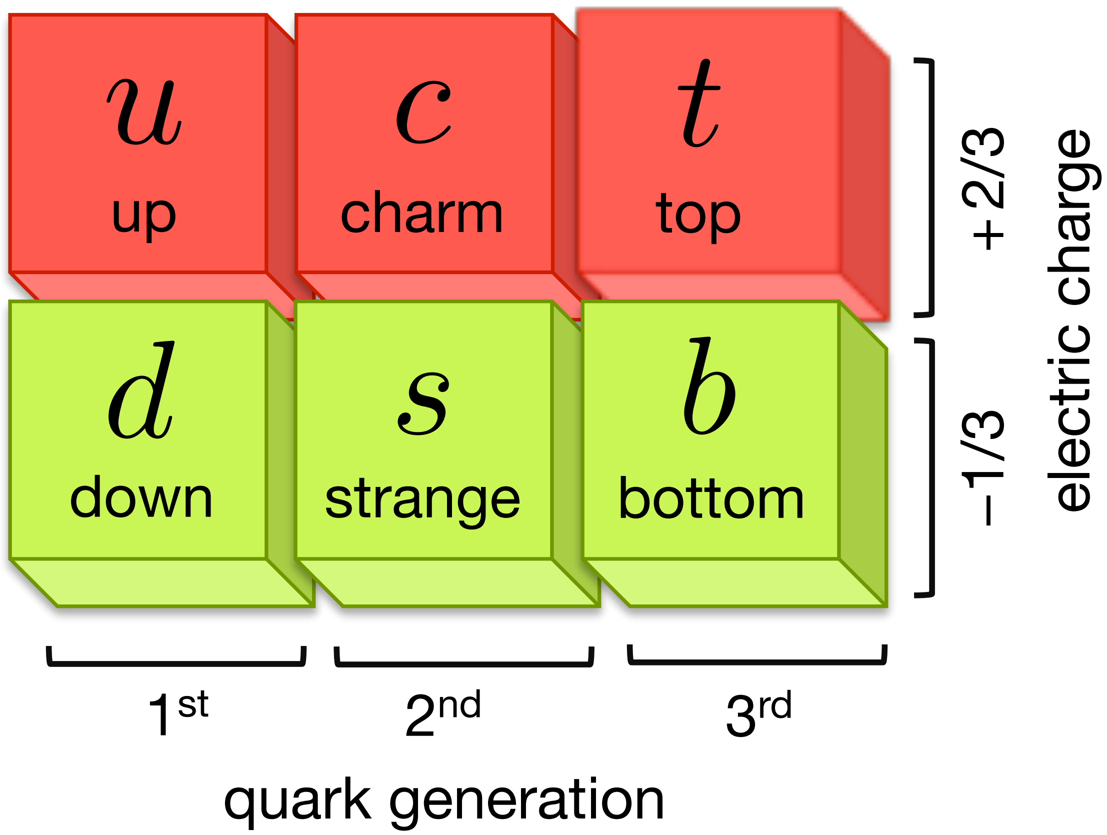
QCD matter sector

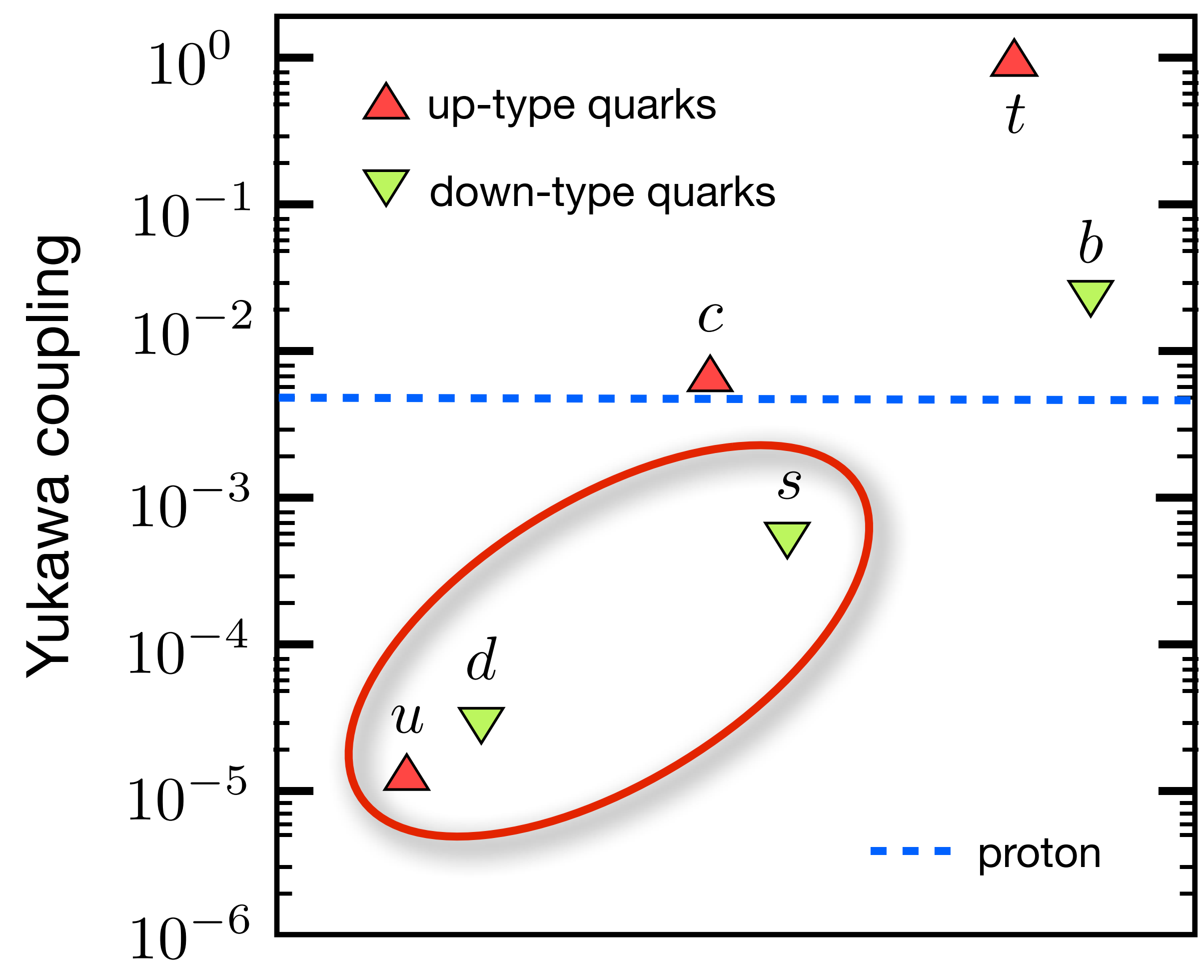




- The masses of the six different quark flavors range from around 2 MeV for the up quark to around 173 GeV for the top. Why these masses are split by almost six orders of magnitude is one of the big mysteries of particle physics

QCD matter sector





- The masses of the up, down, and strange are much lighter than the proton. If one takes these light flavors to have an identical mass, the quarks become indistinguishable under QCD, and one obtains an effective $SU(3)_f$ symmetry

QED and QCD

- QED and QCD are very similar, yet very different theories
- quarks are a bit like leptons, but there are three of each
- gluons are a bit like photons, but there are eight of them
- gluons interact with themselves
- the QCD coupling is also small at collider energies, but larger then the QED one
- the similarities and differences are evident from the two Lagrangians

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So, let's start by looking at the QED Lagrangian

The QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}}$$

$$= \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$= \bar{\psi}\,(iD\!\!\!\!/ - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2$$

- electromagnetic vector potential A_μ
- field strengh tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- covariant derivative $D_\mu = \partial_\mu + ieA_\mu$

QED Feynman rules

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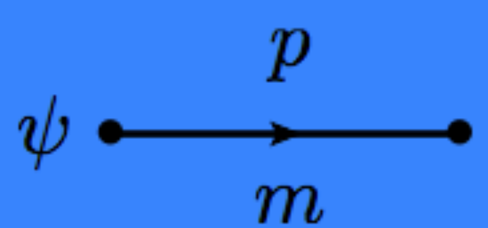
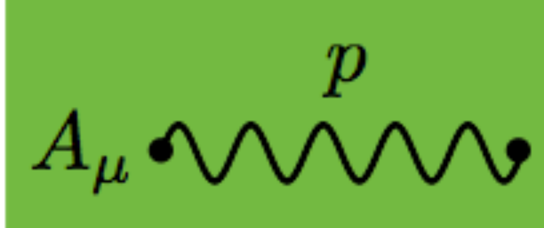
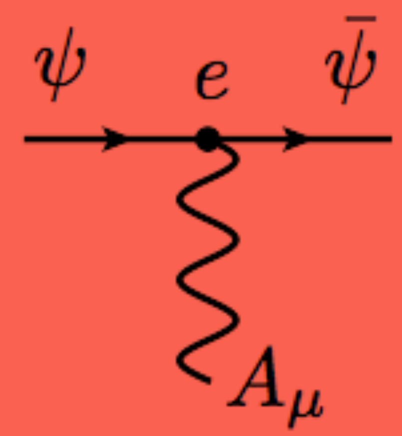


$$= \frac{i\,(p\!\!\!/ + m)}{p^2 - m^2}$$



$$= \frac{-ig^{\mu\nu}}{p^2}$$



$$= ie\gamma^\mu$$

QED gauge invariance

$$\mathcal{L}_{\text{QED}} \;=\; \bar{\psi}\,(i\not{D} - m)\,\psi - \frac{1}{2}(F_{\mu\nu})^2$$

A crucial property of the QED Lagrangian is that it is invariant under

$$\psi(x) \to e^{i\alpha(x)}\,\psi(x)\,, \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

which acts on the Dirac field as a local phase transformation

<u>Exercise:</u>
Check that the QED Lagrangian is invariant under the above transformations

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Yang and Mills (1954) proposed that the local phase rotation in QED could be generalized to invariance under any continuos symmetry

[C. N. Yang and R. L. Mills, Phys. Rep. 96 (1954) 191]

The QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_f \bar{\psi}_i^{(f)} \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)}$$

$$D^\mu_{ij} \equiv \partial^\mu \delta_{ij} + i g_s t^a_{ij} A^\mu_a \,, \qquad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

⇒ covariant derivative

⇒ field strength

The QCD Lagrangian

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⇒ covariant derivative ⇒ field strength

- only one QCD parameter g_s regulating the strength of the interaction (quark masses have EW origin)

The QCD Lagrangian

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_f \bar{\psi}_i^{(f)} \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)}$$

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- color matrices t^a_{ij} are the generators of SU(3)
- QCD flavour blind (differences only due to EW)

The generators of SU(N)

The gauge group of QCD is SU(N) with N =3

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So, the fundamental representation of SU(N) has N^2-1 generators t^a :
N×N traceless hermitian matrices ⇒ N^2-1 gluons

$$U = e^{i\theta_a(x)t^a} \qquad a = 1, \cdots N^2 - 1$$

The Gell-Mann matrices

One explicit representation: $t^A = \frac{1}{2}\lambda^A$

λ^A are the Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Standard normalization: $\mathrm{Tr}(t^a t^b) = T_R\,\delta^{ab} \quad T_R = \frac{1}{2}$

Notice that the first three Gell-Mann matrices contain the three Pauli matrices in the upper-left corner

The generators of SU(N)

Infinitesimal transformations (close to the identity) give complete information about the group structure. The most important characteristic of a group is the commutator of two transformations:

$$\begin{aligned}[U(\delta_1), U(\delta_2)] &\equiv U(\delta_1)U(\delta_2) - U(\delta_2)U(\delta_1) \\ &= (i\delta_1^a)(i\delta_2^b)[t^a, t^b] + \mathcal{O}(\delta^3)\end{aligned}$$

The two matrices to not commute, therefore the transformations don't. Such a group is called non-abelian.

- Familiar abelian groups: translations, phase transformations U(1) ...
- Familiar non-abelian groups: 3D-rotations

The generators of SU(N)

Consider the commutator

$$Tr([t_a, t_b]) = 0 \quad \Rightarrow \quad [t_a, t_b] = i f_{abc} t^c$$

f_{abc} are the (real) structure constants of the SU(N_c) algebra, they generate a representation of the algebra called adjoint representation

Clearly, f_{abc} is anti-symmetric in (ab). It is easy to show that it is fully antisymmetric

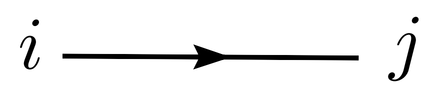
$$i f_{abc} = 2\,\mathrm{Tr}\left([t_a, t_b] t_c\right)$$

and that hence it is fully antisymmetric

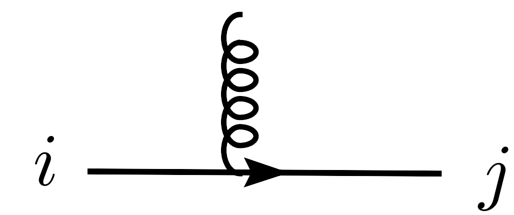
$$f_{abc} = -f_{bac} = -f_{acb}$$

Color algebra: fundamental identities

Fundamental representation 3:

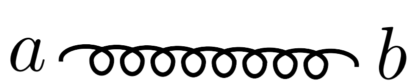


$= \delta_{ij}$

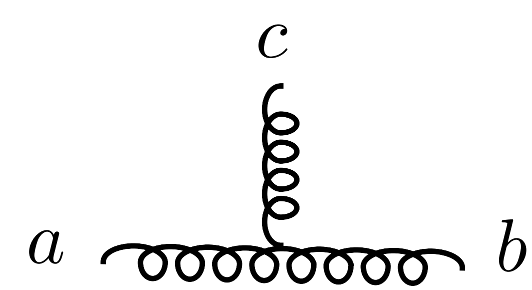


$= t^a_{ij}$

Adjoint representation 8:

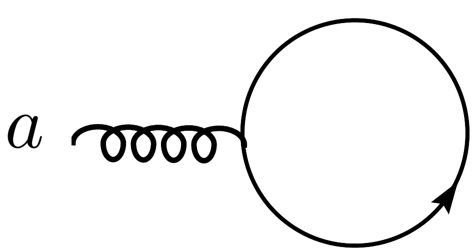


$= \delta_{ab}$



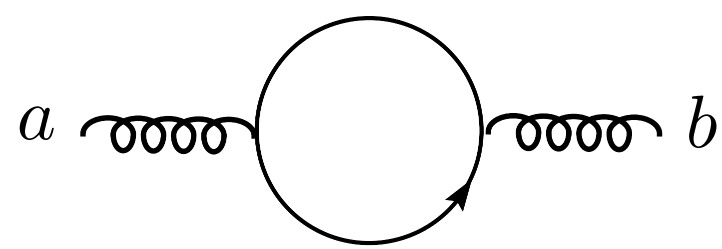
$= i f_{abc}$

Trace identities:

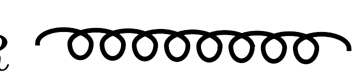


$= 0$

$\mathrm{Tr}(t^a) = 0$

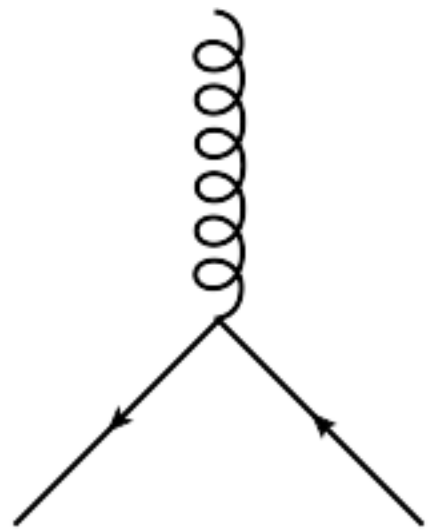


$= T_R$



$\mathrm{Tr}(t^a\, t^b) = \mathrm{T_R}\delta^{ab}$

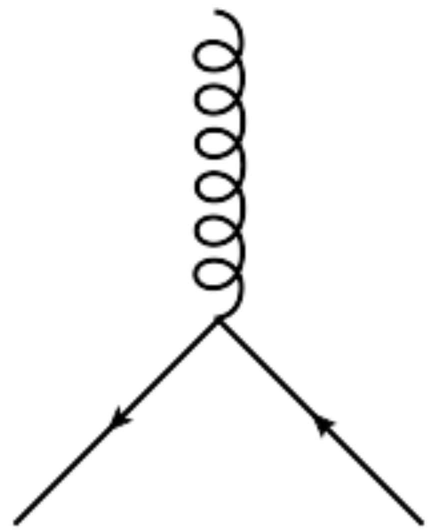
What do color identities mean physically



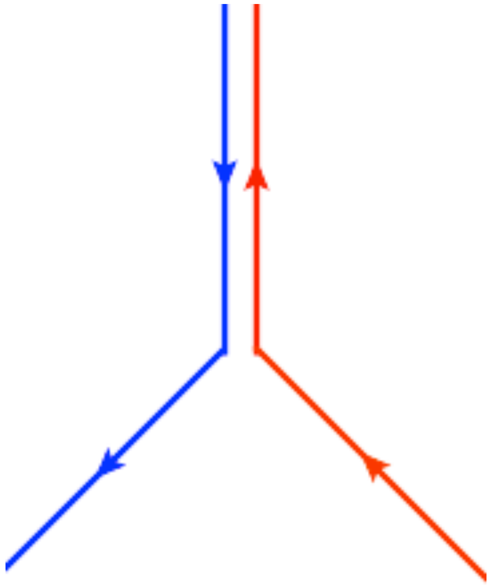
$$\bar{\psi}_i \, t^A_{ij} \, \psi_j$$

What does this really mean?

What do color identities mean physically



$$\bar{\psi}_i \, t^A_{ij} \, \psi_j$$



$$(1, 0, 0) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

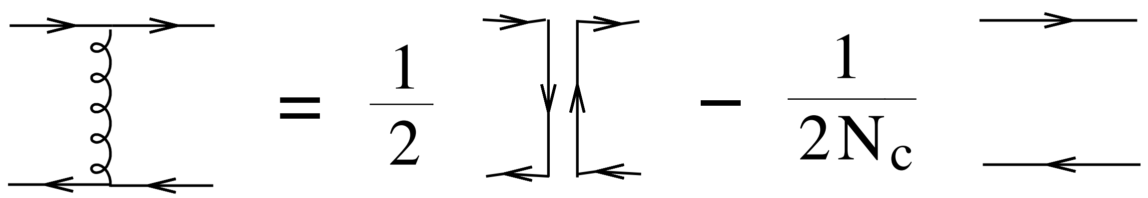
What does this really mean? $\bar{\psi}_i$ t^1_{ij} ψ_j

Gluons carry color and anti-color. They repaint quarks and other gluons.

Color algebra: Casimirs & Fierz identity

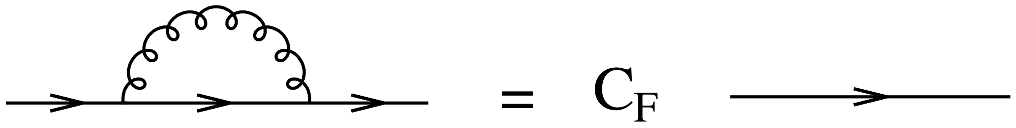
Fierz identity:

$$(t^a)^i_k (t^a)^l_j = \frac{1}{2}\delta^i_j \delta^l_k - \frac{1}{2N_c}\delta^i_k \delta^l_j$$



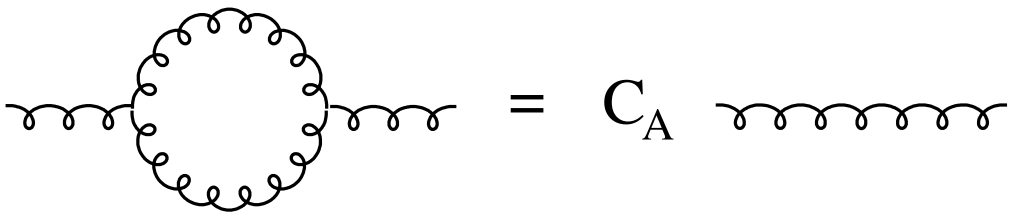
Fundamental representation 3:

$$\sum_a (t^a_{ij})(t^a_{kj}) = C_F \delta_{ij} \qquad C_F = \frac{N_c^2 - 1}{2N_c}$$



Adjoint representation 8:

$$\sum_{cd} f^{acd} f^{bdc} = C_A \delta^{ab} \qquad C_A = N_c$$



Exercises:

1) derive Fierz identity

2) use the Fierz identity to derive the value of C_F

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

- Gauge transformation for the quark field

$$\psi \to \psi' = U(x)\psi$$

- The covariant derivative $(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s t^a_{ij} A^\mu_a$ must transform as (covariant = transforms "with" the field)

$$D_\mu \psi \to D'_\mu \psi' = U(x) D_\mu \psi$$

- From which one derives the transformation property of the gluon field

$$t^a A_a \to t^a A'_a = U(x) t^a A_a U^{-1}(x) + \frac{i}{g_s} \left(\partial U(x)\right) U^{-1}(x)$$

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

- It follows that

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} U^\dagger(x)$$

$$t^a F^a_{\mu\nu} \to t^a F^{a'}_{\mu\nu} = U(x) t^a F^a_{\mu\nu} U^{-1}(x)$$

e.g. because $i\, g_s\, t^a F^a_{\mu\nu} = [D_\mu, D_\nu]$

- Therefore the QCD Lagrangian is indeed gauge invariant

$$-\frac{1}{4} F'^{\mu\nu}_a F'^a_{\mu\nu} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}$$

$$\sum_f \bar{\psi}'^{(f)}_i \left(i \not{D}'_{ij} - m_f \delta_{ij} \right) \psi'^{(f)}_j = \sum_f \bar{\psi}^{(f)}_i \left(i \not{D}_{ij} - m_f \delta_{ij} \right) \psi^{(f)}_j$$

Gauge invariance

The QCD Lagrangian is invariant under local gauge transformations, i.e. one can redefine the quark and gluon fields independently at every point in space and time without changing the physical content of the theory

<u>Remarks:</u>

- the field strength alone is not gauge invariant in QCD (unlike in QED) because of self interacting gluons (carries of the force carry colour, unlike the photon)

- a gluon mass term violate gauge invariance and is therefore forbidden (as for the photon). On the other hand quark mass terms are gauge invariant.

$$\cancel{m^2 A_\mu A^\mu}$$

Isospin symmetry

Isospin SU(2) symmetry: invariance under u ↔ d

Particles in the same isospin multiplet have very similar masses (proton and neutron, neutral and charged pions)

The QCD Lagrangian has isospin symmetry if $m_u = m_d$ or $m_u, m_d \to 0$

The fermionic Lagrangian becomes

$$\mathcal{L}_F = \sum_f \left(\bar{\psi}_L^{(f)} \not{D} \psi_L^{(f)} + \bar{\psi}_R^{(f)} \not{D} \psi_R^{(f)} \right) - \sum_f m_f \left(\bar{\psi}_R^{(f)} \psi_L^{(f)} + \bar{\psi}_L^{(f)} \psi_R^{(f)} \right)$$

$$\psi_L = P_L \psi\,, \quad \psi_R = P_R \psi\,, \quad P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$$

So neglecting fermion masses the Lagrangian has the larger symmetry

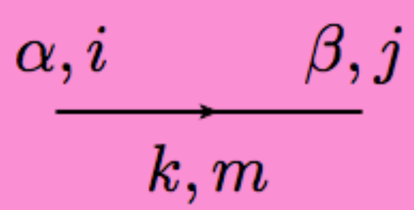
$$SU_L(N_f) \times SU_R(N_f) \times U_L(1) \times U_R(1)$$

Feynman rules: propagators

Obtain quark/gluon propagators from free piece of the Lagrangian

Quark propagator: replace i∂ → k and take the i × inverse

$$\mathcal{L}_{\text{q,free}} = \sum_f \bar{\psi}_i^{(f)} \left(i\not{\partial} - m_f \right) \delta_{ij} \psi_j^{(f)}$$



$$= \left(\frac{i}{\not{k} - m} \right)_{\alpha\beta} \delta_{ij}$$

Gluon propagator: replace i∂ → k and take the i × inverse ?

$$\mathcal{L}_{\text{g,free}} = \frac{1}{2} A^\mu \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) A^\nu$$

➡ inverse does not exist, since $\left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) \partial_\mu = \Box \partial_\nu - \Box \partial_\nu = 0$

How can one to define the propagator ?

Gauge fixing

Solution:
add to the Lagrangian a gauge fixing term which depends on an arbitrary parameter ξ

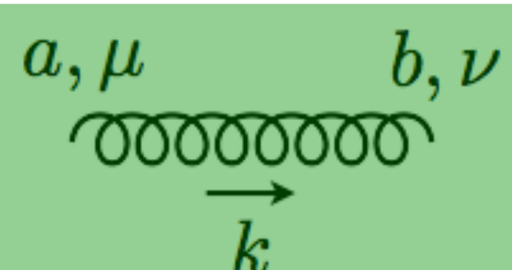
In covariant gauges:

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{\xi}\left(\partial^\mu A_\mu^A\right)^2$$

ξ=1 Feynman gauge
ξ=0 Landau gauge

Gluon propagator:

$$\frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab} =$$

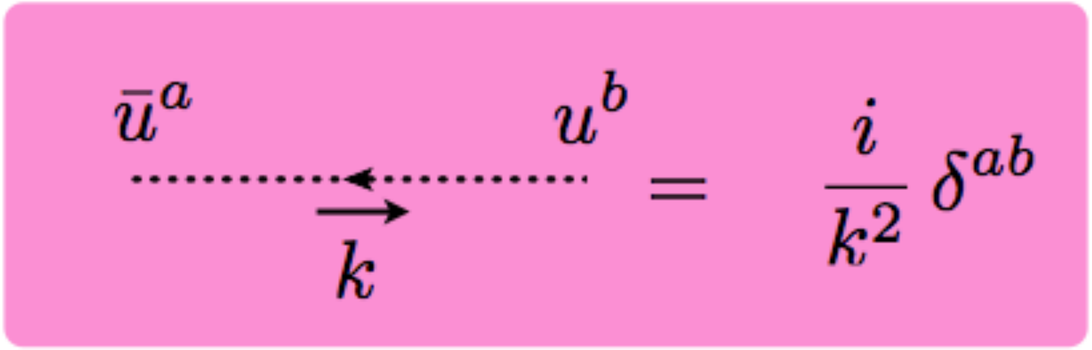


Gauge fixing explicitly breaks gauge invariance. However, in the end physical results are independent of the gauge choice. Powerful check of higher order calculations: verify that the ξ dependence fully cancels in the final result

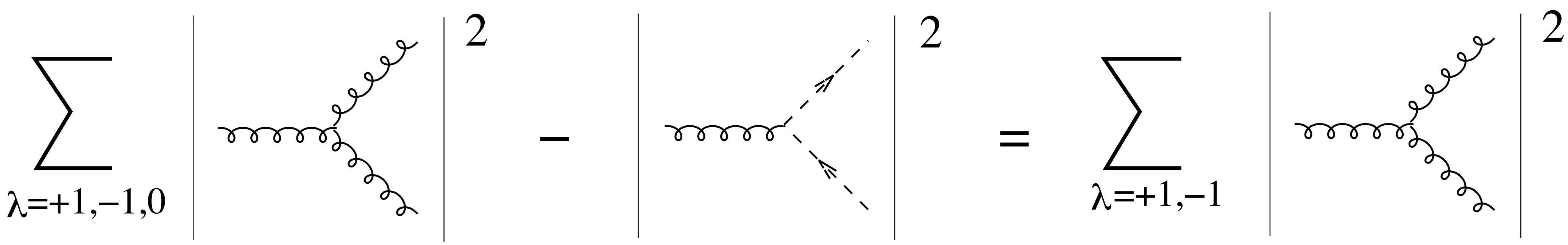
Ghosts

In covariant gauges gauge fixing term must be supplemented with ghost term to cancel unphysical longitudinal degrees of freedom which should not propagate

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \eta^{a\dagger} D^\mu_{ab} \eta^b$$



η: complex scalar field which obeys Fermi statistics



Axial gauges

Alternative: choose an axial gauge (introduce an arbitrary direction n)

$$\mathcal{L}_{\text{axial gauge}} = -\frac{1}{\xi}\left(n^\mu A_\mu^A\right)^2$$

The gluon propagator becomes

$$d_{\mu\nu} = \frac{i}{k^2}\left(-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} + \frac{(n^2+\xi k^2)k_\mu k_\nu}{(n\cdot k)^2}\right)\delta_{ab}$$

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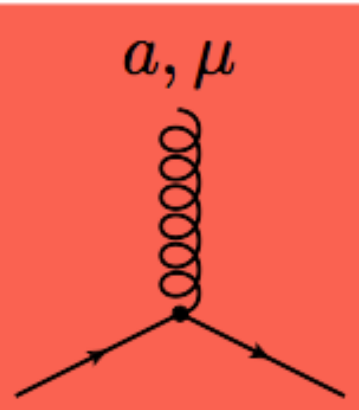
Light cone gauge: $n^2 = 0$ and $\xi = 0$

Axial gauges for $k^2 \to 0$

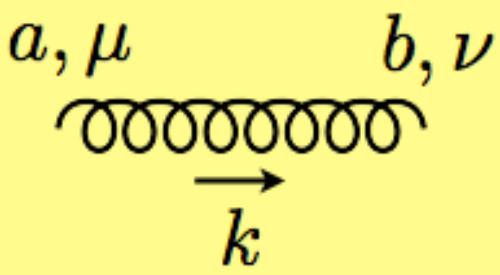
$$d_{\mu\nu} k^\mu = d_{\mu\nu} n^\mu = 0$$

i.e. only two physical polarizations propagate, that's why often the term physical gauge is used

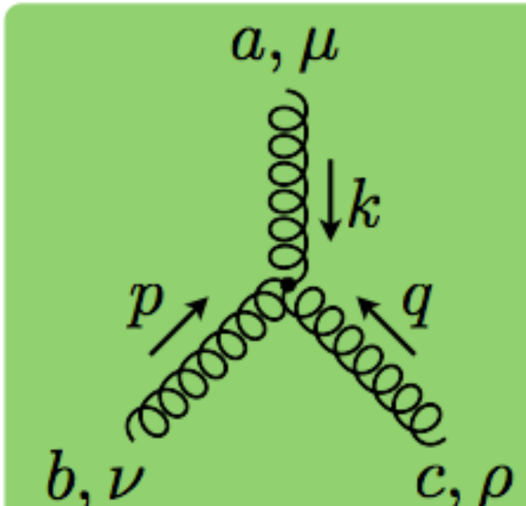
QCD Feynman rules: the vertices



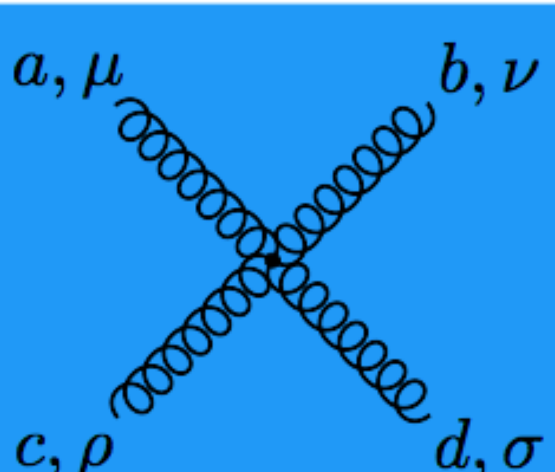
$$= \quad i g_s \gamma^\mu t^a$$



$$= \quad \left(\frac{-i g_{\mu\nu}}{k^2} \right) \delta^{ab}$$



$$= \quad g_s f^{abc} \Big[g^{\mu\nu} (k-p)^\rho + g^{\nu\rho} (p-q)^\mu + g^{\rho\mu} (q-k)^\nu \Big]$$

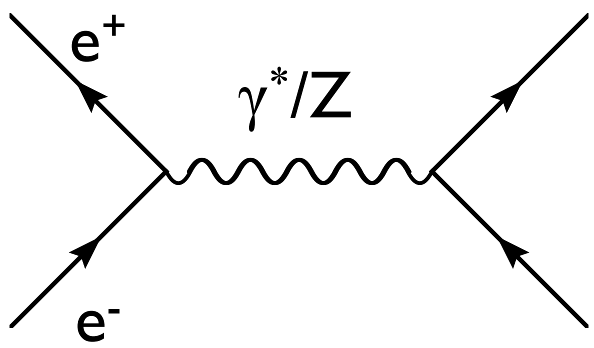


$$= \quad -i g_s^2 \Big[f^{abe} f^{cde} \left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ace} f^{bde} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) + f^{ade} f^{bce} \left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right) \Big]$$

Perturbative expansion of the R-ratio

The R-ratio is defined as

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



At lowest order in perturbation theory

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0(e^+e^- \to q\bar{q})$$

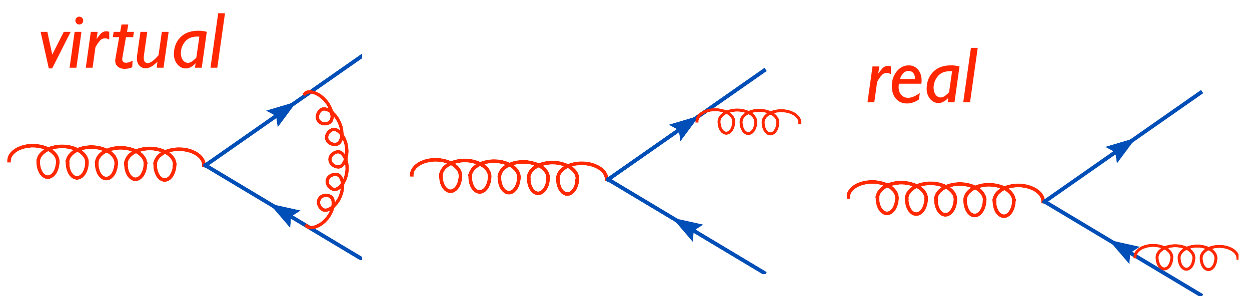
The PT treatment works since the scattering happens at large momentum transfer (short time), while hadronization happens at low momentum transfer, i.e. too late to change the original probability distribution

Since common factors cancel in numerator/denominator, to lowest order one finds

$$R_0 = \frac{\sigma_0(\gamma^* \to \text{hadrons})}{\sigma_0(\gamma^* \to \mu^+\mu^-)} = N_c \sum_f q_f^2$$

The R-ratio: perturbative expansion

First order correction



Real and virtual do not interfere since they have a different # of particles. The amplitude squared becomes

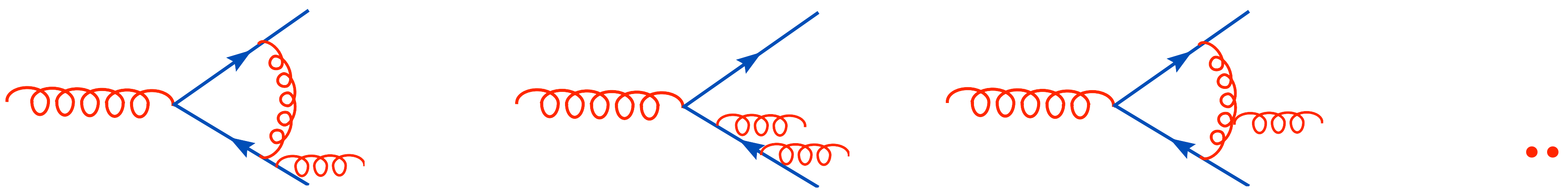
$$|A_1|^2 = |A_0|^2 + \alpha_s \left(|A_{1,r}|^2 + 2\mathrm{Re}\{A_0 A_{1,v}^*\} \right) + \mathcal{O}(\alpha_s^2) \qquad \alpha_s = \frac{g_s^2}{4\pi}$$

Integrating over phase space, the first order result reads

$$R_1 = R_0 \left(1 + \frac{\alpha_s}{\pi} \right)$$

R-ratio and UV divergences

To compute the second order correction one has to compute diagrams like these and many more



One gets

$$R_2 = R_0 \left(1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{M_{\rm UV}^2}{Q^2}\right)\right) \qquad b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

Ultra-violet divergences do not cancel. Result depends on UV cut-off.

Renormalization and running coupling

The divergence is dealt with by renormalization of the coupling constant

$$\alpha_s(\mu) = \alpha_s^{\text{bare}} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\text{bare}}\right)^2$$

R expressed in terms of the renormalized coupling is finite

$$R = R_0 \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{\mu^2}{Q^2}\right) + \mathcal{O}(\alpha_s^3(\mu))\right)$$

Renormalizability of the theory guarantees that *the same redefinition of the coupling removes all UV divergences from all physical quantities (massless case)*

Will not cover renomalization in these lectures, but it suffices to know that renormalization of S-matrix elements is achieved by replacing bare masses and bare coupling with renormalized ones

- the coupling ⇒ β function
- the masses ⇒ anomalous dimensions γ_m

The beta-function

$$\beta(\alpha_s^{\rm ren}) \equiv \mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2}$$

The renormalized coupling is

$$\alpha_s(\mu) = \alpha_s^{\rm bare} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\rm bare}\right)^2$$

So, one immediately gets

$$\beta = -b_0 \alpha_s^2(\mu) + \ldots$$

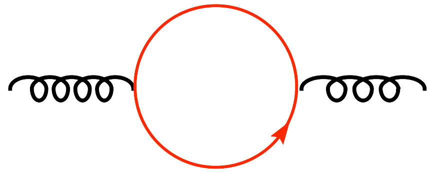
Integrating the differential equation one finds at lowest order

$$\frac{1}{\alpha_s(\mu)} = b_0 \ln \frac{\mu^2}{\mu_0^2} + \frac{1}{\alpha_s(\mu_0)} \qquad \Rightarrow \qquad \alpha_s(\mu) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$$

More on the beta-function

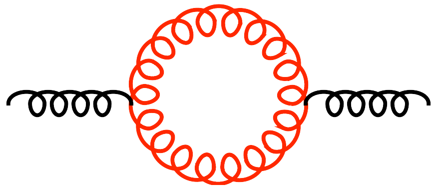
Roughly speaking:

(a) quark loop vacuum polarization diagram gives a negative contribution to $b_0 \sim n_f$



(a)

(b) gluon loop gives a positive contribution to $b_0 \sim N_c$



(b)

Since (b) > (a) $\Rightarrow$ $b_{0,QCD} > 0 \Rightarrow$ overall negative beta-function in QCD

While in QED (b) = 0 $\Rightarrow b_{0,QED} < 0$

$$\beta_{\mathrm{QED}} = \frac{1}{3\pi}\alpha^2 + \ldots$$

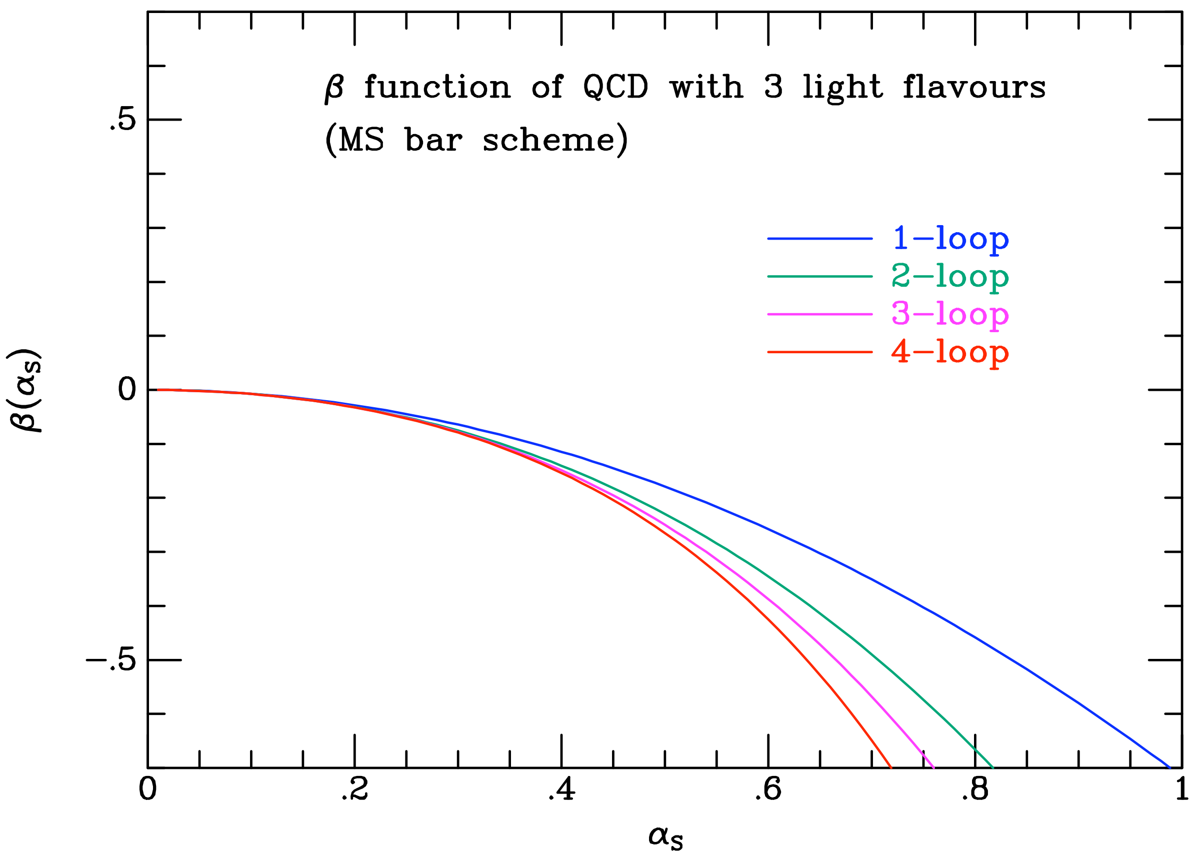
More on the beta-function

Perturbative expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu) \sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11N_c - 4n_f T_R}{12\pi}$$

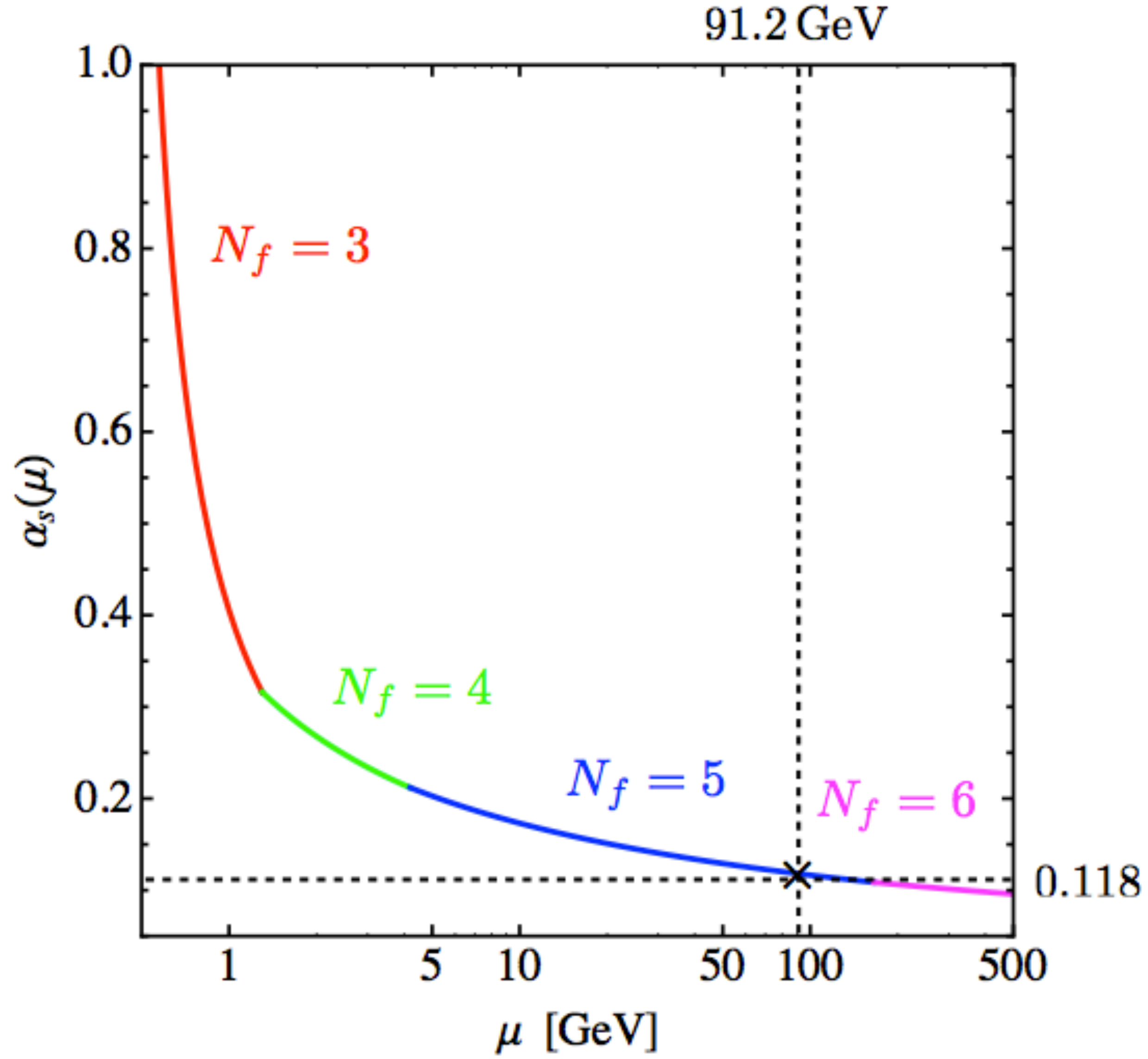
$$b_1 = \frac{17N_c^2 - 5N_c n_f - 3C_F n_f}{24\pi^2}$$



- n_f is the number of active flavours (depends on the scale)
- today, the beta-function known up to four loops, but only first two coefficients are independent of the renormalization scheme

Active flavours & running coupling

The active field content of a theory modifies the running of the couplings



Constrain New Physics by measuring the running at high scales?

Asymptotic freedom

Integrating the differential equation

$$\frac{\partial \alpha_s(Q)}{\partial t} = -b_0 \alpha_s^2(Q) + \mathcal{O}(\alpha_s^3) \qquad t = \ln\left(\frac{Q^2}{\mu^2}\right)$$

To lowest order one gets

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1 + b_0 \ln \frac{Q^2}{\mu^2} \alpha_s(\mu)}$$

So the coupling constant decreases logarithmically with increasing energy. The statement that the theory becomes free at high energy goes under the name of asymptotic freedom [N.B. the sign of b_0 is crucial], i.e. the non-abelian vacuum polarization has an anti-screening effect.

Renormalization Group Equation

Consider a dimensionless quantity A, function of a single scale Q. The dimensionless quantity should be independent of Q. However in quantum field theory this is not true, as renormalization introduces a second scale μ

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So, for any observable A one can write a renormalization group equation

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0$$

$$\alpha_s = \alpha_s(\mu^2) \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

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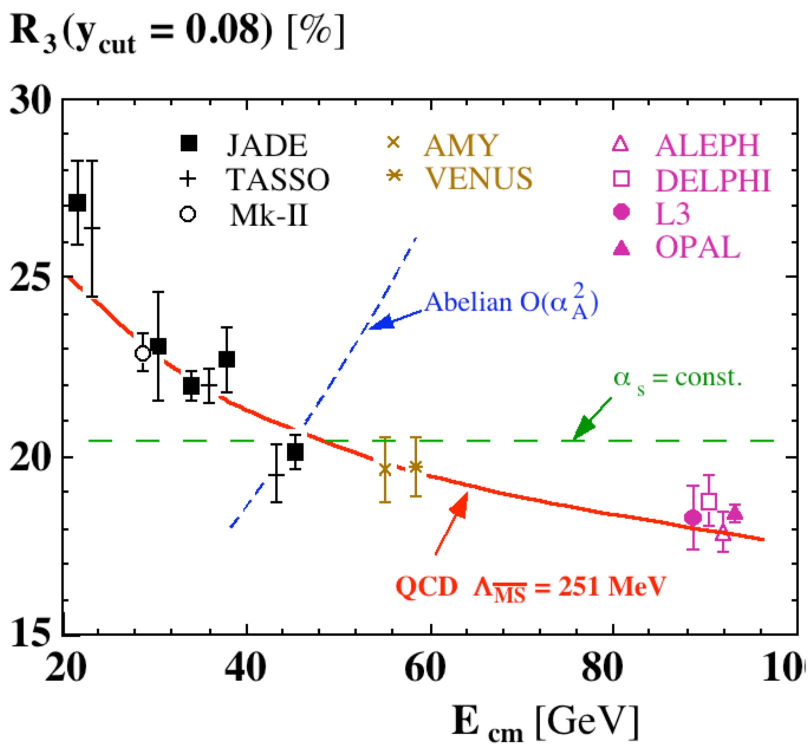
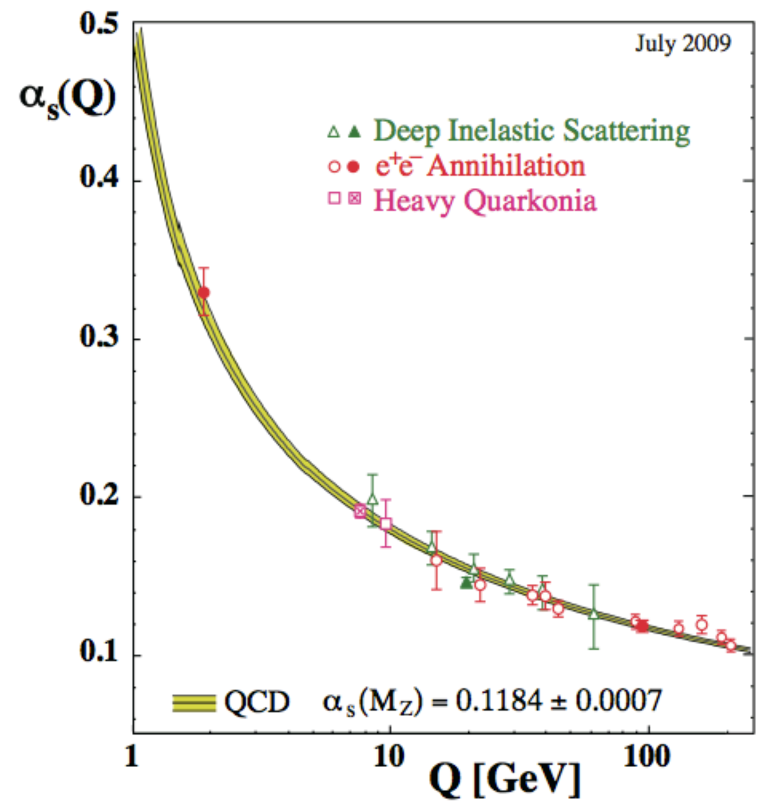
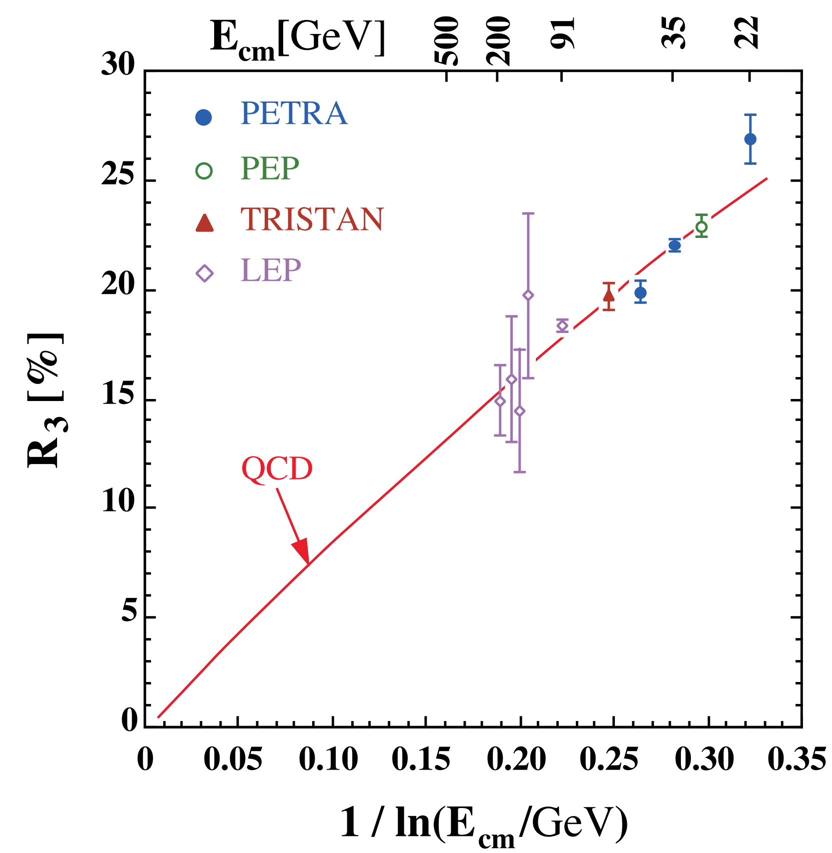
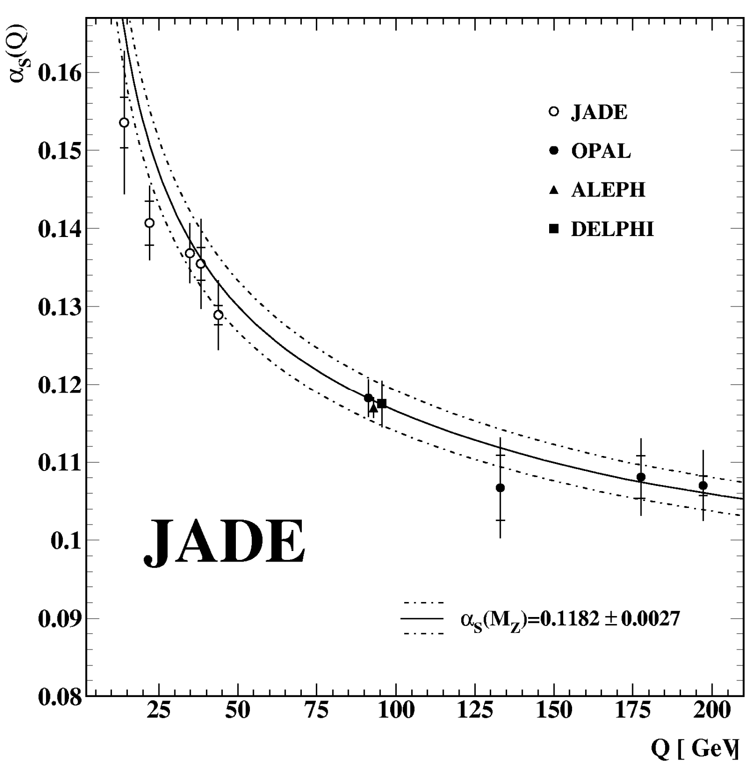
All scale dependence of A enters only through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q given the beta-function

Measurements of the running coupling



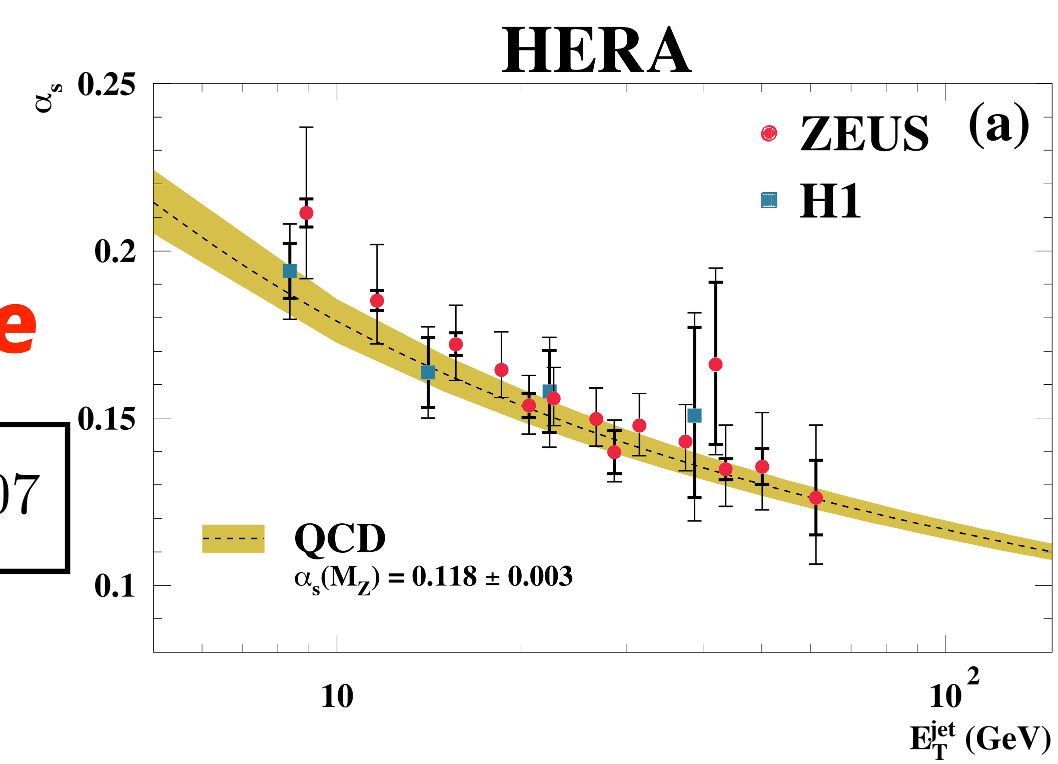






World average

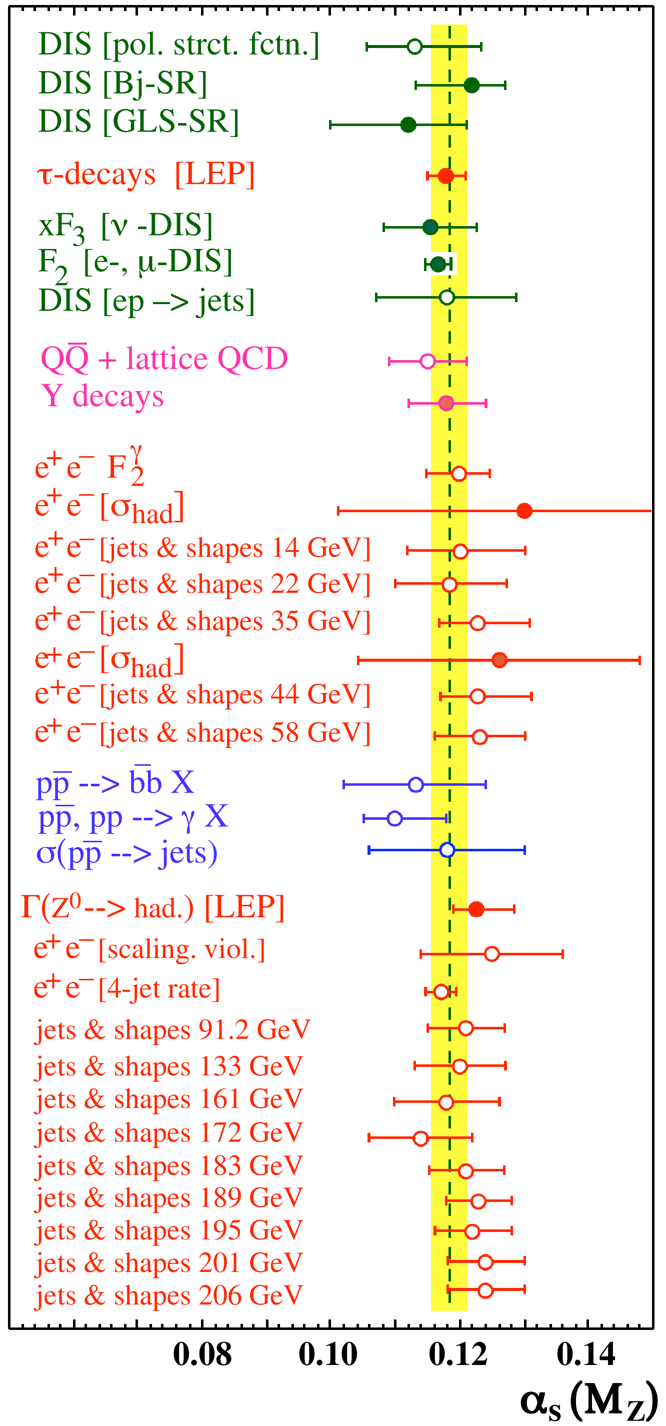
$$\alpha_s(M_{Z^0}) = 0.1184 \pm 0.007$$



Measurements of the running coupling

<u>Summarizing:</u>

- overall consistent picture: α_s from very different observables compatible
- α_s is not so small at current scales
- α_s decreases slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important



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Intermediate Recap

- QCD is in principle a simple theory based on a simple Lagrangian with gauge group is SU(3)
- Simple color algebra and Feynman rules are the necessary ingredients for perturbative calculations (see later)
- Today, we know three families of quarks, we briefly revisited the experiments which lead to their discovery
- There are UV divergences but they are dealt with by renormalization (coupling + masses)
- This is intimately related to the fact that the coupling runs $\Rightarrow$ beta-function
- The theory is asymptotically free and consistent with confinement

Next

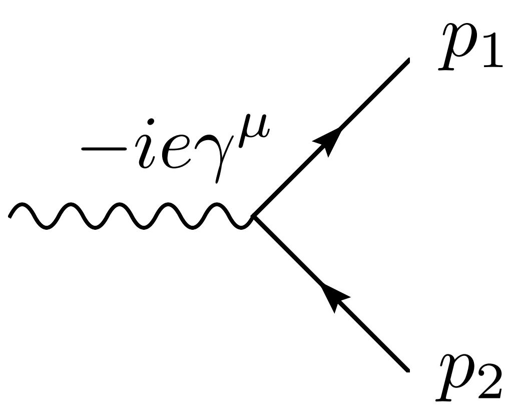
- Infrared and collinear divergences and IRsafety
- Parton model: incoherent sum of all partonic cross-sections
- Sum rules (momentum, charge, flavor conservation)
- Determination of parton densities (electron & neutrino scattering in DIS or Drell-Yan)
- Radiative corrections: failure of parton model
- Factorization of initial state divergences into scale dependent parton densities
- DGLAP evolution of parton densities $\Rightarrow$ measure gluon PDF

The soft approximation

Let's consider again the R-ratio. This is determined by $\gamma^* \to q\bar{q}$

At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$



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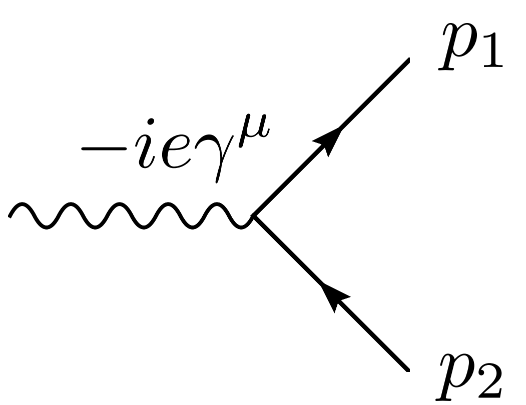
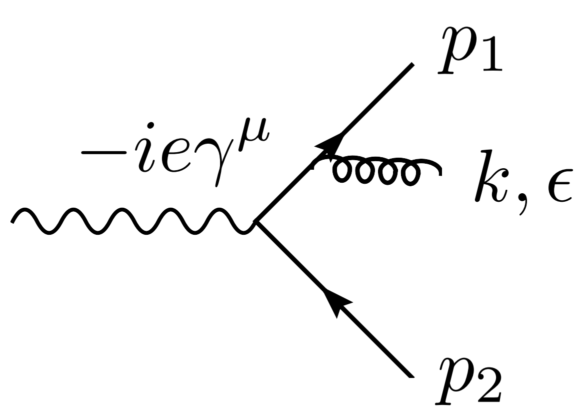


Emit one gluon:

$$\begin{aligned} M^\mu_{q\bar{q}g} &= \bar{u}(p_1)(-ig_s t^a \not\epsilon)\frac{i(\not p_1 + \not k)}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not p_2 - \not k)}{(p_2-k)^2}(-ig_s t^a \not\epsilon)v(p_2) \end{aligned}$$



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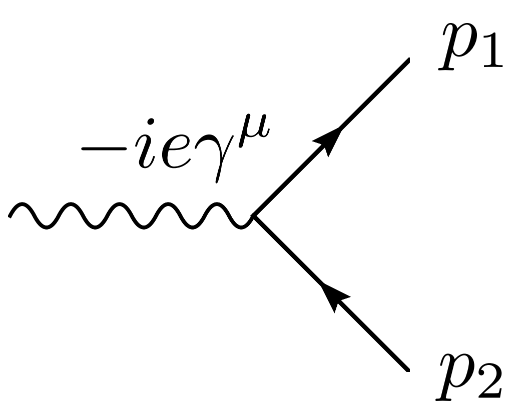

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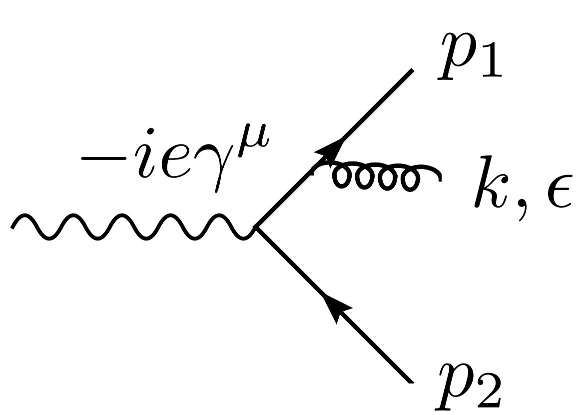

Consider the soft approximation: $k \ll p_1, p_2$

$$M^\mu_{q\bar{q}g} = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

$\Rightarrow$ factorization of soft part (crucial for resummed calculations)

Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a) v(p_2)\right) \left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2 \epsilon}{p_2 k}\right) \right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

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Including phase space

$$d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 \quad = \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$
$$= \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi}\frac{2\alpha_s C_F}{\pi}\frac{1}{\omega^2(1-\cos^2\theta)}$$

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The differential cross section is

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi}\frac{d\omega}{\omega}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

Soft & collinear divergences

Cross section for producing a $q\bar{q}$-pair and a gluon is infinite (IR divergent)

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

$\omega \to 0$: soft divergence

$\theta \to 0$: collinear divergence

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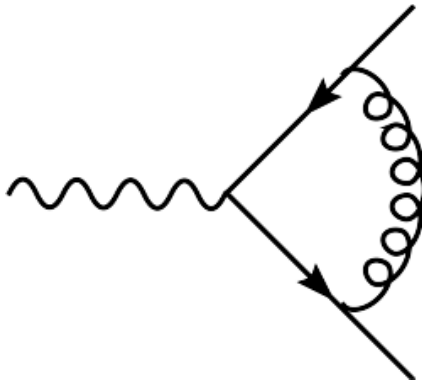
$\omega \to 0$: soft divergence

$\theta \to 0$: collinear divergence

But the full O(α_s) correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets a finite correction of α_s/π

Soft & collinear divergences

$\underline{\omega \to 0}$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\underline{\theta \to 0}$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

NB: the appearance of soft and collinear divergences discussed in the specific contect of $e^+e^- \to qq$ are a general property of QCD

Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get infinities, which can not be renormalized away (why not?)

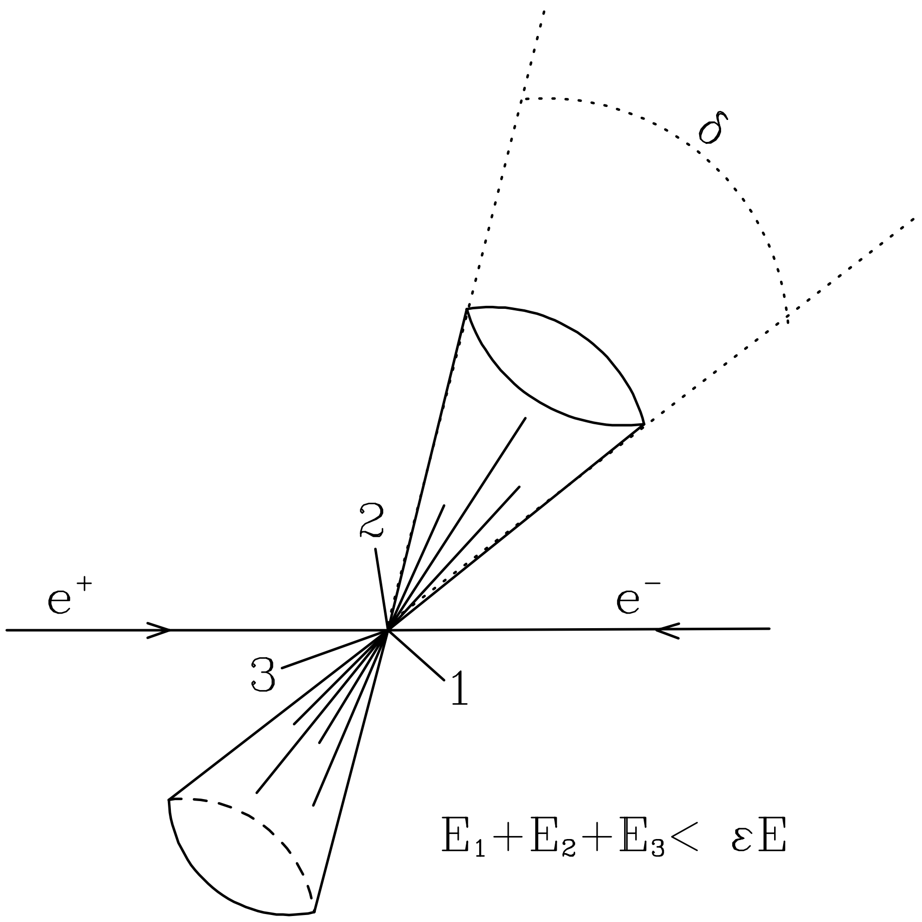
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters ε and δ:
a pair of Sterman-Weinberg jets are two cones of opening angle δ that contain all the energy of the event excluding at most a fraction ε



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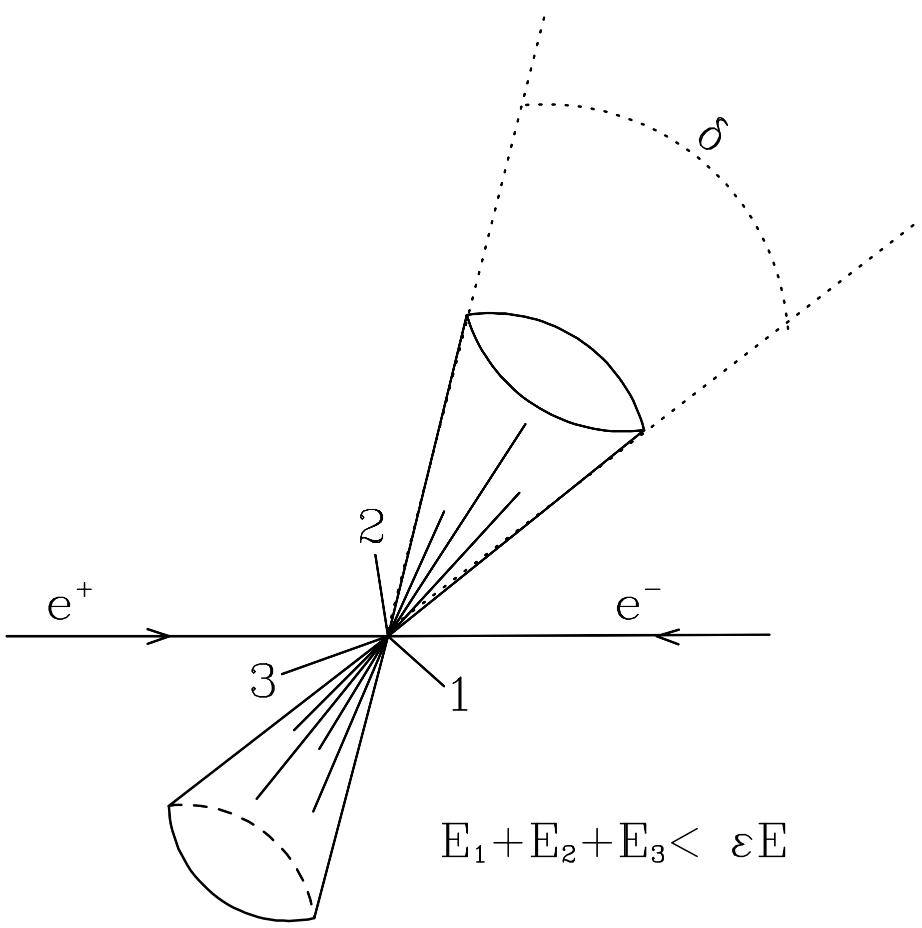

Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

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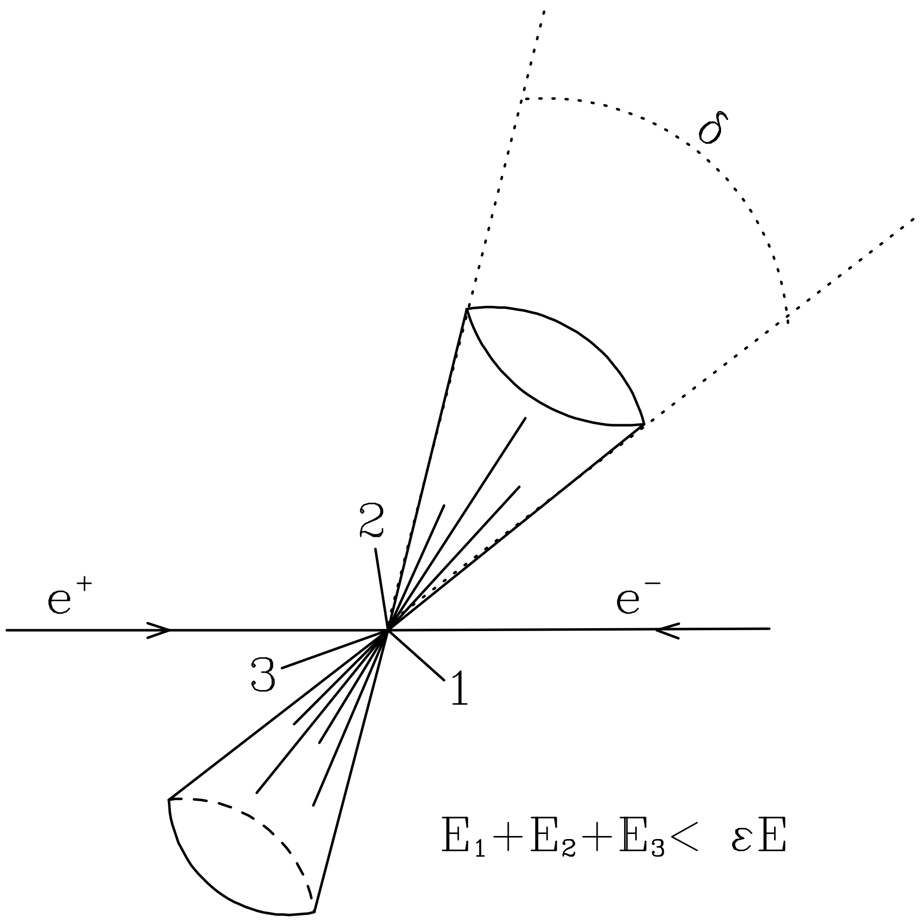

Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

Kinoshita-Lee-Nauenberg (KLN) theorem:
final-state infrared divergences cancel in measurable quantities (transition probabilities, cross-sections summed over indistinguishable states...)

Sterman-Weinberg jets

The Sterman-Weinberg jet cross-section up to O(α_s) is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln\epsilon \ln\delta^2\right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not α_s

α_s-expansion enhanced by a double log: left-over from real-virtual cancellation

- if more gluons are emitted, one gets for each gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm lnε
  - a collinear logarithm lnδ
- if ε and/or δ become too small the above result diverges
- if the logs are large, fixed order meaningless, one needs to resum large infrared and collinear logarithms to all orders in the coupling constant

Infrared safety: definition

An observable $\mathcal{O}$ is infrared and collinear safe if

$$\mathcal{O}_{n+1}(k_1, k_2, \ldots, k_i, k_j, \ldots k_n) \to \mathcal{O}_n(k_1, k_2, \ldots k_i + k_j, \ldots k_n)$$

whenever one of the k_i/k_j becomes soft or k_i and k_j are collinear

i.e. the observable is insensitive to emission of soft particles or to collinear splittings

Infrared safety: examples

Infrared safe ?

- energy of the hardest particle in the event
- multiplicity of gluons
- momentum flow into a cone in rapidity and angle
- cross-section for producing one gluon with $E > E_{min}$ and $\theta > \theta_{min}$
- jet cross-sections

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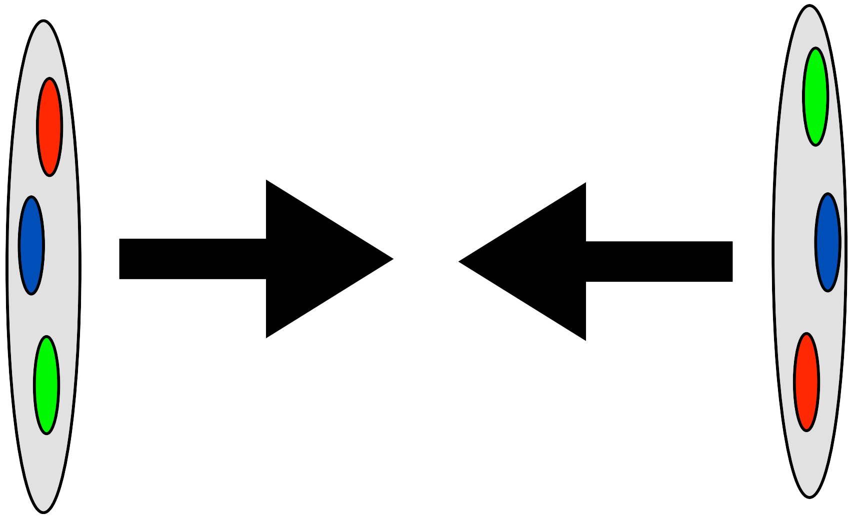
- cross-section for producing one gluon with $E > E_{min}$ and $\theta > \theta_{min}$ NO
- jet cross-sections DEPENDS

Partons in the initial state

- We talked a lot about final state QCD effects
- This is the only thing to worry about at e^+e^- colliders (LEP)
- Hera/Tevatron/LHC involve protons in the initial state
- Proton are made of QCD constituents

Next we will focus mainly on aspects related to initial state effects



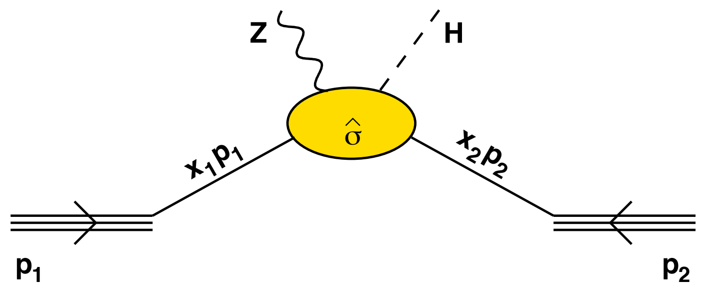
The parton model

Basic idea of the parton model: intuitive picture where in a high transverse momentum scattering partons behave as quasi free in the collision

$\Rightarrow$ cross section is the incoherent sum of all partonic cross-sections

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

NB: This formula is wrong/incomplete (see later)



$f_i^{(P_j)}(x_i)$: parton distribution function (PDF) is the probability to find parton i in hadron j with a fraction x_i of the longitudinal momentum (transverse momentum neglected), extracted from data

$\hat{\sigma}(x_1 x_2 s)$: partonic cross-section for a given scattering process, computed in perturbative QCD

Sum rules

Momentum sum rule: conservation of incoming total momentum

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

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Conservation of flavour: e.g. for a proton

$$\int_0^1 dx \left(f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left(f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

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In the proton: u, d valence quarks, all other quarks are called sea-quarks

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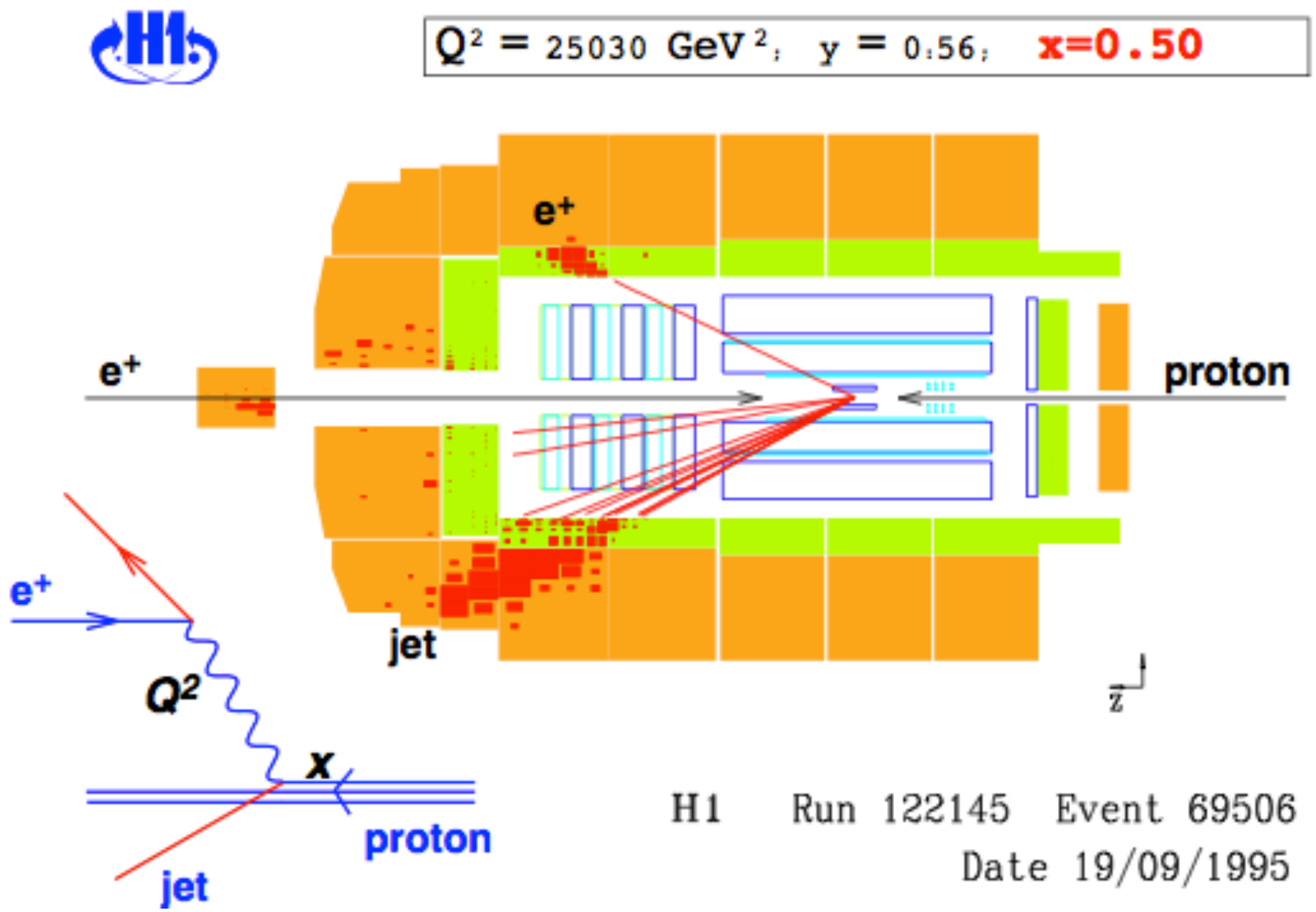
How can parton densities be extracted from data?

Deep inelastic scattering

Easier than processes with two incoming hadrons is the scattering of a lepton on a (anti)-proton



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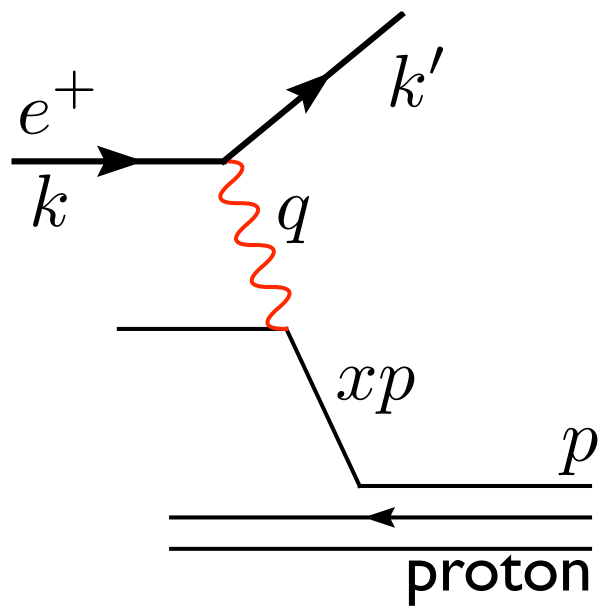

Kinematics:

$$Q^2 = -q^2 \quad s = (k+p)^2 \quad x_{Bj} = \frac{Q^2}{2p\cdot q} \quad y = \frac{p\cdot q}{k\cdot p}$$

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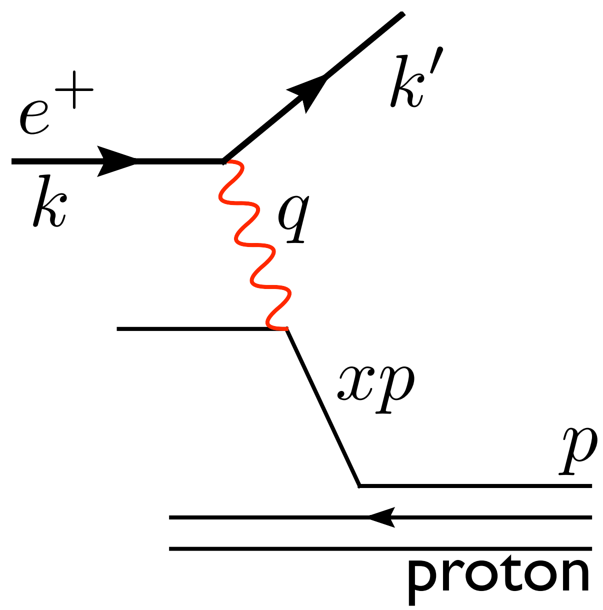

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$$\hat{p} = xp \quad \hat{s} = (k+\hat{p})^2 = 2k\cdot\hat{p} \quad \hat{y} = \frac{\hat{p}\cdot q}{k\cdot\hat{p}} = y \quad (\hat{p}+q)^2 = 2\hat{p}\cdot q - Q^2 = 0$$

$$\Rightarrow x = x_{Bj}$$

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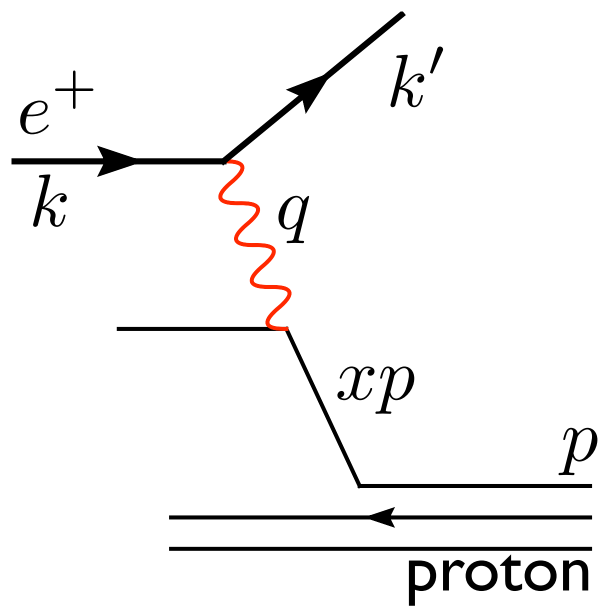

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Partonic cross section:

(just apply QED Feynman rules and add phase space)

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\,\pi\,\alpha_{em}\left(1 + (1-\hat{y})^2\right)$$

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Hadronic cross section:

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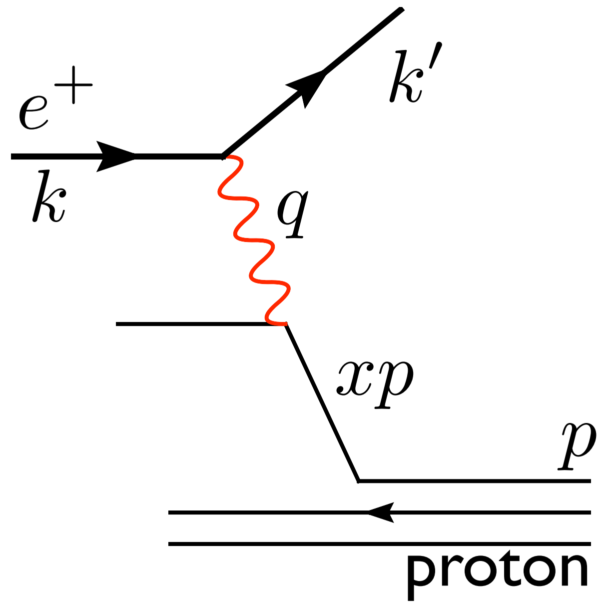
$$= \frac{2\pi\,\alpha_{em}^2 s x_{Bj}}{Q^4}\left(1 + (1-y)^2\right) \sum_l q_l^2 f_l^{(p)}(x_{Bj})$$



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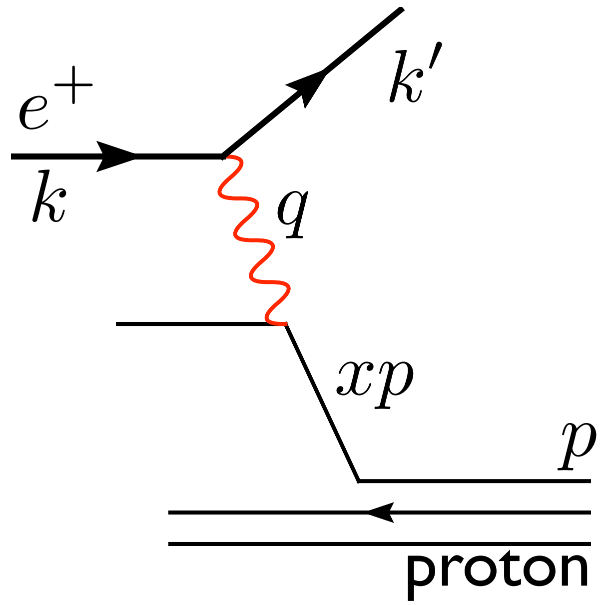

1. at fixed x_{Bj} and y the cross-section scales with s
2. the y-dependence of the cross-section is fully predicted and is typical of vector interaction with fermions ⇒Callan-Gross relation
3. can access (sums of) parton distribution functions
4. Bjorken scaling: pdfs depend on x and not on Q^2

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1+\left(1-y^2\right)F_2(x)\right. \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

F_2 is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Question: F_2 gives only a linear combination of u and d. How can they be extracted separately?

Isospin

Neutron is like a proton with u & d exchanged

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NB: experimentally get F_2^n from deuteron: $F_2^d(x) = F_2^p(x) + F_2^n(x)$

Sea quark distributions

Inside the proton there are fluctuations, and pairs of u$\bar{u}$,d$\bar{d}$,c$\bar{c}$,s$\bar{s}$ ... can be created

An infinite number of pairs can be created as long as they have very low momentum, because of the momentum sum rules.

We saw before that when we say that the proton is made of uud what we mean is

$$\int_0^1 dx\,(u_p(x) - \bar{u}_p(x)) = 2 \qquad \int_0^1 dx\,\left(d_p(x) - \bar{d}_p(x)\right) = 1$$

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How can one measure the difference?

Question: What interacts differently with particle and antiparticle?

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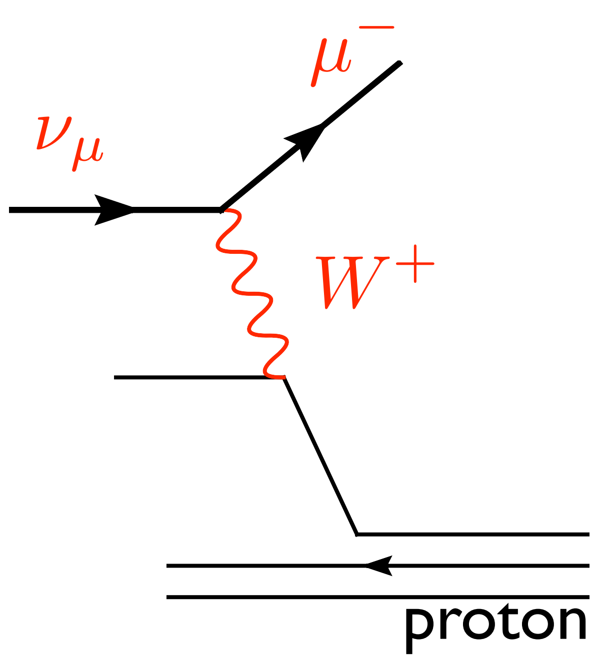

<u>Question:</u> What interacts differently with particle and antiparticle? W^+/W^- from neutrino scattering

Check of the momentum sum rule

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

u_v	0.267
d_v	0.111
u_s	0.066
d_s	0.053
s_s	0.033
c_c	0.016
total	0.546

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$\gamma/W^{+/-}$ don't interact with gluons

How can one measure gluon parton densities?

We need to discuss radiative effects first