Inflation and quantum gravity in a Born-Oppenheimer context

Phys.Lett. B726 (2013) 518-522

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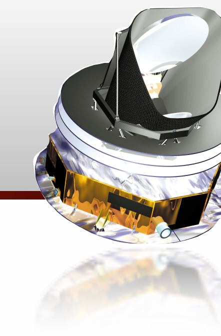
Taipei - 21/11/2013

Introduction



- Planck results are compatible with inflationary paradigm
- Tighter constraints on viable inflationary models
- Power loss in the lowest multipole interval
- Such a deviation from expected results may encode some quantum gravitational (QG) effect
- One would naively expect QG to affect more the dynamics of short wavelength modes of the spectrum of inflationary perturbations
- However long wavelength modes exit the horizon at earlier stages during inflation and re-enter later…
- …the high curvature/high energy effects may affect their evolution for a longer period of time w.r.t. short wavelength modes!

Wheeler deWitt approach

- General Relativity is invariant upon re-parametrization $(x^i, t) \to (\tilde{x}^i, \tilde{t})$

- The Hamiltonian of space-time d.o.f. is proportional to a linear combination of first class constraints

$$H_{GR} = \int d^4x \left(N^i \mathcal{H}_i + N\mathcal{H}\right)$$

- In particular the time reparametrization invariance is associated with the super-Hamiltonian constraint $\mathcal{H} = 0$

- On a homogeneous and isotropic space-time

$$ds^2 = -n(\tau)^2 d\tau^2 + a(\tau)^2 \delta_{ij} dx^i dx^j$$

- the super-Hamiltonian constraint is non trivial and at the quantum level plays the role of the time independent Schroedinger equation.

Wheeler deWitt approach

- The canonical quantization proceeds in a standard way

$$S = \int d^4x \sqrt{-g}\frac{\mathrm{M_P}^2}{12} R = V \int dt\; n \left[-\frac{\mathrm{M_P}^2}{2} \frac{a\dot{a}^2}{n^2} \right]$$

- **Note**: in order to get rid of the volume of 3-space and to keep the correct dimensionality after quantization we absorb $V^{1/3} a \to a \quad ([a] = M^{-1})$

$$\pi_a = -\mathrm{M_P}^2 a\dot{a}/n \to -i\frac{d}{da} \qquad\qquad \hbar = c = 1$$

- The **WdW equation** is the quantized Hamiltonian constraint

$$H\left(a, -i\frac{d}{da}\right)\Psi(a) = 0$$

with $H = \frac{\partial \int d^3x N\mathcal{H}}{\partial N} = \int d^3x \mathcal{H}$ and $N = n$

- $\Psi(a)$ is the **wave function of the Universe**

The Matter-Gravity System

- **Inflaton field** is added $\phi(\vec{x},\eta) = \phi_0(\eta) + \delta\phi(\vec{x},\eta)$
- The full "effective" action can be written as

$$z \equiv \frac{\phi_0'}{H}$$

$$S = \int d\eta \left\{ -\frac{{\mathrm{M_P}}^2}{2} a'^2 + \frac{a^2}{2}\left[\phi_0'^2 - V(\phi)a^2\right] + \sum_{i=1,2}\sum_{k\neq 0}^{\infty}\left[v_{i,k}'(\eta)^2 + \left(-k^2 + \frac{z''}{z}\right) v_{i,k}(\eta)^2 \right] \right\}$$

- The inhomogeneous parts of the field and the metric are described by the Sasaki-Mukhanov variable $v(\vec{x},\eta) = a(\eta)\delta\phi(\vec{x},\eta)$ in the uniform curvature gauge
- **Approx**: $\pi_a = -{\mathrm{M_P}}^2 a\dot{a}/n \rightarrow -i\frac{d}{da}$
- The WdW equation for the matter-gravity system is:

$$\left\{ \frac{1}{2{\mathrm{M_P}}^2}\frac{\partial^2}{\partial a^2} - \frac{1}{2a^2}\frac{\partial^2}{\partial \phi_0^2} + Va^4 + \sum_{k\neq 0}^{\infty}\left[-\frac{1}{2}\frac{\partial^2}{\partial v_k^2} + \frac{\omega_k^2}{2} v_k^2 \right] \right\} \Psi\left(a, \phi_0, \{v_k\}\right) = 0$$

Born-Oppenheimer Approx.

- **BO approximation** has been widely applied to Quantum Chemistry for calculating the spectra of complex atoms and molecules
(M. Born and J.R. Oppenheimer, Ann. Physik 84, 457 - 1927)

- It is applied when the Quantum System can be divided in "fast" (light) + "slow" (heavy) degrees of freedom (such as electrons and nuclei in Q.Chem.)

- In the matter-gravity system gravity is the heavy d.o.f. associated to the Planck mass
(T. Banks, 1985; R. Brout, 1987; R. Brout and G. Venturi, 1989)

- One can make the ansatz $\Psi\left(a,\phi_0,\{v_k\}\right)=\psi(a)\chi\left(a,\phi_0,\{v_k\}\right)=\psi(a)\prod_{k=0}^{\infty}\chi_k\left(a,v_k\right)$

$\phi_0 \equiv v_0$

- The **BO decomposition** of the system is:

where

$\psi = e^{-i\int^a \mathcal{A} da'}\tilde{\psi},\ \chi = e^{i\int^a \mathcal{A} da'}\tilde{\chi},\ \mathcal{A} = -i\langle\chi|\frac{\partial}{\partial a}|\chi\rangle$

$\langle\hat{O}\rangle = \langle\tilde{\chi}|\hat{O}|\tilde{\chi}\rangle \qquad \langle\chi_k|\chi_k\rangle = \int dv_k \chi_k^* \chi_k$

$$\left[\frac{1}{2{\rm M_P}^2}\frac{\partial^2}{\partial a^2}+\langle\hat{H}^{(M)}\rangle\right]\tilde{\psi}=-\frac{1}{2{\rm M_P}^2}\langle\frac{\partial^2}{\partial a^2}\rangle\tilde{\psi}$$

Equation for homogeneous gravity

$$\tilde{\psi}^*\tilde{\psi}\left[\hat{H}^{(M)}-\langle\hat{H}^{(M)}\rangle\right]\tilde{\chi}+\frac{1}{{\rm M_P}^2}\left(\tilde{\psi}^*\frac{\partial}{\partial a}\tilde{\psi}\right)\frac{\partial}{\partial a}\tilde{\chi}=\frac{1}{2{\rm M_P}^2}\tilde{\psi}^*\tilde{\psi}\left[\langle\frac{\partial^2}{\partial a^2}\rangle-\frac{\partial^2}{\partial a^2}\right]\tilde{\chi}$$

for each k-mode

Equation for matter (homogeneous inflation + cosm. perturbations)

Semiclassical limit and time emergence

- The equation for gravity is $\left[\frac{1}{2{M_P}^2}\frac{\partial^2}{\partial a^2}+\langle\hat{H}^{(M)}\rangle\right]\tilde{\psi}=-\frac{1}{2{M_P}^2}\langle\frac{\partial^2}{\partial a^2}\rangle\tilde{\psi}$
- On neglecting the r.h.s. and taking the semiclassical limit

$$\tilde{\psi}(a)=(|\pi_a|)^{-1/2}\exp\left(i\int da\ \pi_a\right)\Rightarrow\quad \pi_a(a)\simeq\pm\sqrt{2{M_P}^2\langle\hat{H}^{(M)}\rangle}$$

- One recovers the Friedmann equation $-\frac{{M_P}^2}{2}a'^2+\sum_k\langle\hat{H}_k^{(M)}\rangle=-\frac{1}{2{M_P}^2}\sum_k\langle\tilde{\chi}_k|\partial_a^2|\tilde{\chi}_k\rangle$
- A time parameter related to the scale factor also appears $\eta\leftrightarrow a(\eta)\Rightarrow\frac{\partial}{\partial a}=\frac{1}{a'}\frac{\partial}{\partial\eta}$
- The equation for matter is Schwinger-Tomonaga namely the time dependent Schroedinger equation (**TDSE**) plus correction

$$i\partial_\eta|\chi_k\rangle_s-\hat{H}_k^{(M)}|\chi_k\rangle_s=\epsilon\left[\hat{\Omega}_k-\langle\hat{\Omega}_k\rangle_s\right]|\chi_k\rangle_s \qquad \epsilon\equiv\frac{1}{2{M_P}^2}$$

where: $|\chi_k\rangle_s\equiv e^{-i\int^\eta\langle\tilde{\chi}_k|\hat{H}_k^{(M)}|\tilde{\chi}_k\rangle d\eta'}|\tilde{\chi}_k\rangle \qquad \langle\hat{O}\rangle_s\equiv\ {}_s\langle\chi_k|\hat{O}|\chi_k\rangle_s$

- Q.G. corrections encoded in the operator $\hat{\Omega}_k=\frac{1}{a'^2}\frac{d^2}{d\eta^2}+\left[2i\frac{\langle\hat{H}_k^{(M)}\rangle_s}{a'^2}-2\frac{a''}{a'^3}\right]\frac{d}{d\eta}$

Scalar fields evolution

- Each k-mode evolves independently. In particular we assume that homogeneous parts are classical and determine the overall background evolution!

- Inhomogeneities evolve according to the standard dynamics by neglecting the contributions of order $\mathcal{O}\left({\mathrm{M_P}}^{-2}\right)$

- The solutions of the **TDSE** with Hamiltonian $\hat{H}_k^{(M)} = \frac{\hat{\pi}_k^2}{2} + \frac{\omega_k^2}{2}\hat{v}_k^2$

  can be generated by the invariant operators $\hat{I},\ \hat{I}^\dagger$: $\hat{I}|\mathrm{vac}\rangle = 0$

Invariant definition

$$i\frac{d}{d\eta}\hat{I} + \left[\hat{I}, \hat{H}\right] = 0$$

Linear invariant operator

$$\hat{I} = \frac{e^{i\Theta}}{\sqrt{2}}\left[\left(\frac{1}{\rho} - i\rho'\right)\hat{v} + i\rho\hat{\pi}\right] \qquad \Theta = \int^\eta \frac{d\eta'}{\rho^2}$$

- ρ satisfies the **Pinney equation** $\rho'' + \omega^2\rho = \frac{1}{\rho^3}$

- The properly normalized Bunch-Davies vacuum is given by:

$$\langle v|\mathrm{vac}\rangle = \frac{1}{(\pi\rho^2)^{1/4}}\exp\left[\frac{i}{2}\int^\eta \frac{d\eta'}{\rho^2} - \frac{v^2}{2}\left(\frac{1}{\rho^2} - i\frac{\rho'}{\rho}\right)\right]$$

Two Point Function

- The spectrum of scalar fluctuations is related to observable quantities

$$p(\eta) \equiv {}_s\langle 0|\hat{v}^2|0\rangle_s = \langle \hat{v}^2\rangle_0 \qquad\qquad |0\rangle_s \leftrightarrow |\mathrm{vac}\rangle$$

- When QG effects are taken into account the vacuum satisfies the non linear eq.

$$0 = i\frac{d}{d\eta}|0\rangle_s - \hat{H}|0\rangle_s - \left[\left(2i\langle\hat{H}\rangle_0 g\left(\eta\right) + g'\left(\eta\right)\right)\left(\frac{d}{d\eta} - \langle\frac{d}{d\eta}\rangle_0\right) + g(\eta)\left(\frac{d^2}{d\eta^2} - \langle\frac{d^2}{d\eta^2}\rangle_0\right)\right]|0\rangle_s$$

- We translate the eq. for the vacuum $|0\rangle_s$ into an eq. for $p(\eta)$

$$g(\eta) = \frac{1}{2\mathrm{M_P}^2 a'^2}$$

$$\frac{d\langle\hat{v}^2\rangle_0}{d\eta} = \langle\{\hat{v},\hat{\pi}\}\rangle_0 - iR(\hat{v}^2)$$

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$$\frac{d\langle\{\hat{v},\hat{\pi}\}\rangle_0}{d\eta} = -i\langle\left[\{\hat{v},\hat{\pi}\}\,,\hat{H}\right]\rangle_0 - iR\left(\{\hat{v},\hat{\pi}\}\right)$$

$$\left[\{\hat{v},\hat{\pi}\}\,,\hat{H}\right] = 2i\left(\hat{\pi}^2 - \omega^2\hat{v}^2\right)$$

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$$\frac{d^3\langle\hat{v}^2\rangle_0}{d\eta^3} = \frac{d\langle\hat{\pi}^2\rangle_0}{d\eta} - 4\omega\omega'\langle\hat{v}^2\rangle_0 - 2\omega^2\frac{d\langle\hat{v}^2\rangle_0}{d\eta} - i\frac{dR\left(\{\hat{v},\hat{\pi}\}\right)}{d\eta} - i\frac{d^2R(\hat{v}^2)}{d\eta^2}$$

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$$\frac{d\langle\hat{\pi}^2\rangle_0}{d\eta} = i\omega^2\langle\left[\hat{v}^2,\hat{H}\right]\rangle_0 - iR(\hat{\pi}^2)$$

$$\langle\left[\hat{v}^2,\hat{H}\right]\rangle_0 = i\frac{d\langle\hat{v}^2\rangle_0}{d\eta} - R(\hat{v}^2)$$

$$\frac{d^2\langle\hat{v}^2\rangle_0}{d\eta^2} = 2\left(\langle\hat{\pi}^2\rangle_0 - \omega^2\langle\hat{v}^2\rangle_0\right) - iR(\{\hat{v},\hat{\pi}\}) - i\frac{dR(\hat{v}^2)}{d\eta}$$

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where

Purely imaginary function of eta

$$R(\hat{O}) = -\langle\hat{O}\rangle_0\left(\left(2ig\langle\hat{H}\rangle_0 + g'\right)\langle\partial_\eta\rangle_0 + g\langle\partial_\eta^2\rangle_0 - c.c.\right) + \left(2ig\langle\hat{H}\rangle_0 + g'\right)\langle\hat{O}\partial_\eta\rangle_0 + g\langle\hat{O}\partial_\eta^2\rangle_0 - c.c.$$

Solutions for p(eta)

- Finding the exact solution for $p(\eta)$ is a hopeless task
- A **perturbative approach** is needed and sufficient given the present precision of cosmological data:
  - Neglect QG effects $(\mathrm{M_P} \to \infty, R(\hat{O}) \to 0)$ and find the zero order vacuum solution
  - Evaluate QG corrections for the zero order vacuum
  - Express results up to the ${\mathrm{M_P}}^{-2}$ order

First method

- The approximate solution is $p \simeq p^{(0)} + {\mathrm{M_P}}^{-2} p^{(1)} = \dfrac{\rho^2}{2} + {\mathrm{M_P}}^{-2} p^{(1)}$
- On using $|\mathrm{vac}\rangle$ the QG corrections can be expressed through

$$R(\hat{O}) = {\mathrm{M_P}}^{-2} F_{R;\hat{O}}(\rho, \rho', \eta) + \mathcal{O}\left({\mathrm{M_P}}^{-4}\right) = {\mathrm{M_P}}^{-2} \tilde{F}_{R;\hat{O}}(p, p', \eta) + \mathcal{O}\left({\mathrm{M_P}}^{-4}\right)$$

- One finally has the non linear equation

$$0 = \frac{d^3 p}{d\eta^3} + 4\omega^2 \frac{dp}{d\eta} + 2\frac{d\omega^2}{d\eta} p - \frac{1}{{\mathrm{M_P}}^2} \frac{d^3}{d\eta^3} \frac{\left(p'^2 + 4\omega^2 p^2 - 1\right)}{4a'^2} + \frac{1}{{\mathrm{M_P}}^2} \frac{d^2}{d\eta^2} \frac{p'\left(p'^2 + 4\omega^2 p^2 + 1\right)}{4pa'^2}$$
$$+ \frac{1}{{\mathrm{M_P}}^2} \frac{d}{d\eta} \left\{ \frac{1}{8a'^2 p^2} \left[\left(1 - 4\omega^2 p^2\right)^2 + 2p'^2 \left(1 + 4\omega^2 p^2\right) + p'^4 \right] \right\} - \frac{1}{{\mathrm{M_P}}^2} \frac{\omega\omega' \left(p'^2 + 4\omega^2 p^2 - 1\right)}{a'^2}$$

Solutions for p(eta)

Second method

- One can explicitly use the exact (or approx.) solution for $\rho(\eta)$ and write

$$R(\hat{O}) = \mathrm{M_P}^{-2} f_{R;\hat{O}}(\eta) + \mathcal{O}\left(\mathrm{M_P}^{-4}\right)$$

- This method is exactly equivalent but is preferred if the analytical expression for $\rho(\eta)$ can be obtained
- The first method can be used for numerical results

- In any case one needs to specify
  - the **background** (homogeneous) evolution
  - the **initial conditions**

- Initial conditions: **BD vacuum** seems appropriate (and generally adopted for cosmological perturbations)
  - BD at the **unperturbed level** means (**type I**)

$$p^{(0)}\left(-\infty\right) = \rho\left(-\infty\right)^2/2 = 1/\left(2k\right) \Rightarrow \rho\left(-\infty\right) = 1/\sqrt{k}$$

  - BD at the **perturbed level** means (**type II**) $\quad p(-\infty) = 1/\left(2k\right)$

Application to De Sitter

- Background evolution: $a(\eta) = -\dfrac{1}{H\eta}$, $\omega = \sqrt{k^2 - \dfrac{2}{\eta^2}}$
- The unperturbed BD vacuum solution corresponds to $\rho = \sqrt{\dfrac{1}{k} + \dfrac{1}{k^3\eta^2}}$
- The full equation for $p(\eta)$ is very compact and can be exactly solved

$$\frac{d^3p}{d\eta^3} + 4\left(k^2 - \frac{2}{\eta^2}\right)\frac{dp}{d\eta} + \frac{8}{\eta^3}p + \frac{4H^2}{\mathrm{M_P}^2 k^4 \eta^3} = 0$$

- It's solution is $p = \dfrac{1}{2k^4\eta^2}\Big\{c_+\left(1 + k^2\eta^2\right) + \cos(2k\eta)\left[2c_0 k\eta - c_-\left(k^2\eta^2 - 1\right)\right]$

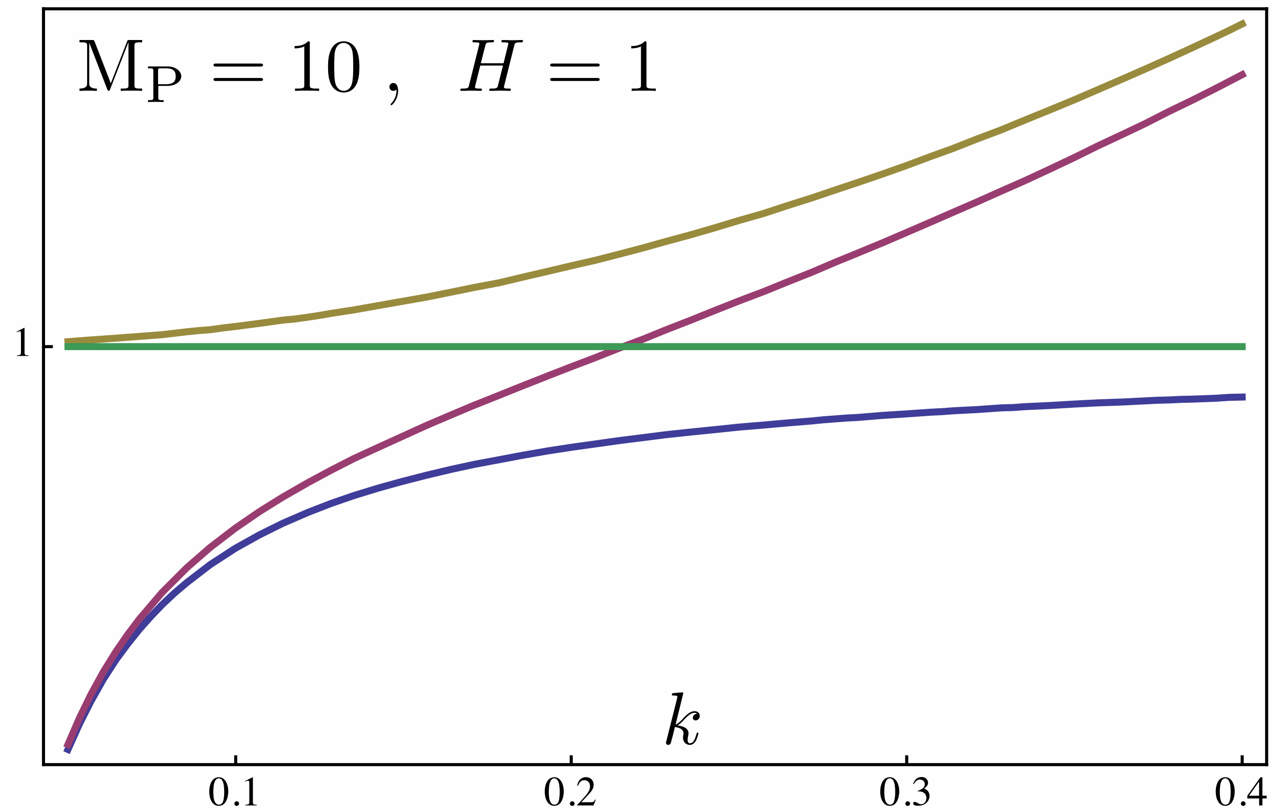
$$+ \sin(2k\eta)\left[c_0\left(k^2\eta^2 - 1\right) + 2c_- k\eta\right] - \frac{H^2}{\mathrm{M_P}^2}\eta^2\Big\}$$

- Imposing the **initial conditions of type I**: $c_- = c_0 = 0$, $c_+ = k$

$$\mathcal{P}_v = \frac{k^3}{2\pi^2}p = \frac{a^2H^2}{4\pi^2}\left(1 + \frac{k^2}{a^2H^2} - \frac{1}{a^2 k \mathrm{M_P}^2}\right) \overset{-k\eta\to 0}{=} \frac{a^2H^2}{4\pi^2}\left(1 - \frac{1}{a^2\mathrm{M_P}^2 k}\right)$$

Application to De Sitter

- **Analytical (qualitative) results for modes outside the Horizon: k/aH<<1**



Exact DS

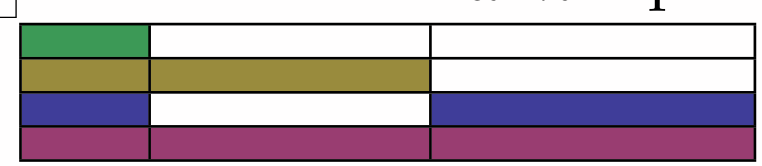
DS plus QG corrections

DS in long wavelength limit (FLAT)

DS plus QG corrections in long wavelength limit

Terms which contribute

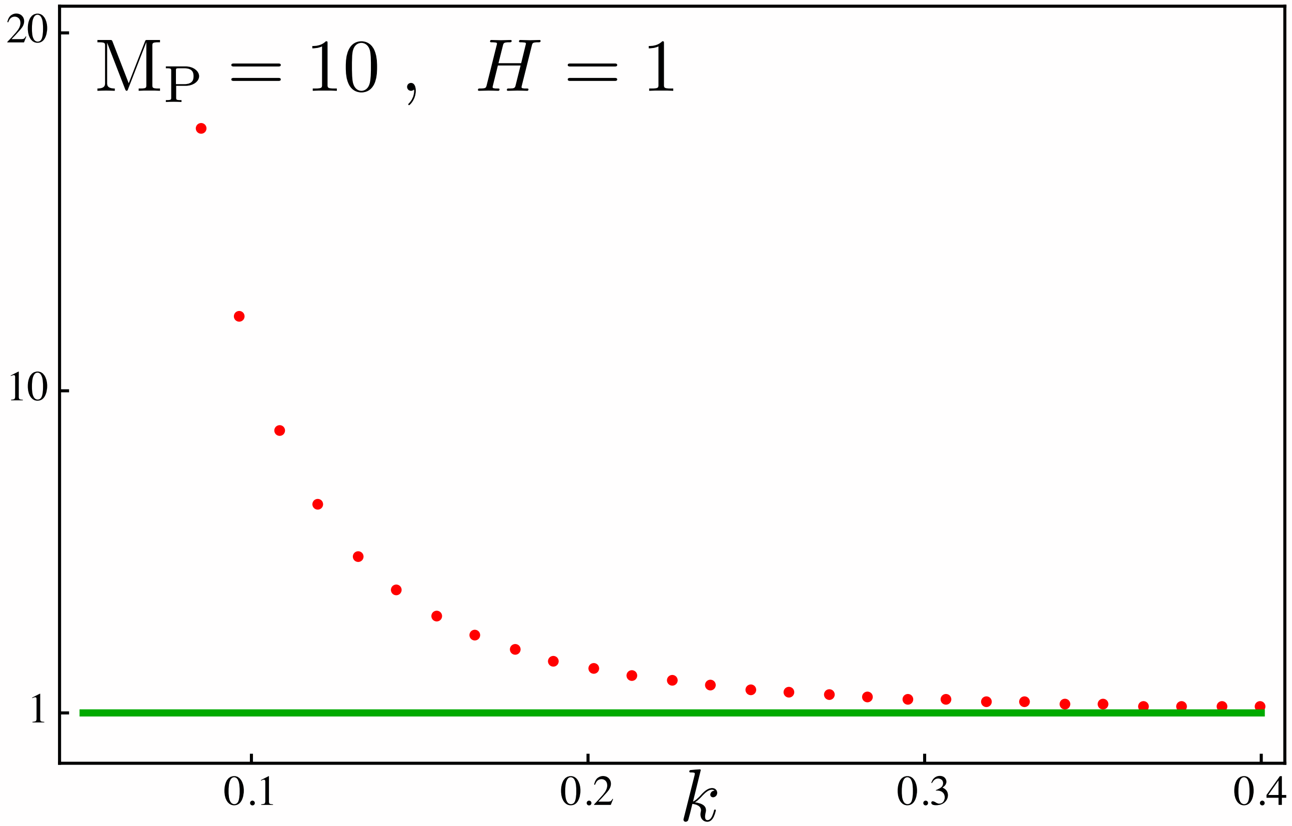
$$1 + \frac{k^2}{a^2 H^2} - \frac{1}{a^2 k {M_P}^2}$$



- For k large the QG corrections are negligible! (**no trans-Planckian problem!**)
- For k small we observe a **power-loss** w.r.t. standard de Sitter flat spectrum
- The QG correction evolve in time as the sub-leading term

Application to De Sitter

- **Numerical (qualitative) results for modes outside the Horizon: k/aH<<1**



- Numerical results (red) of the full equation for p
- Initial conditions of **type II**
- Deviation at small k
- Power enhancement w.r.t. the pure de Sitter case

DS plus QG corrections
DS in long wavelength limit (FLAT)

- There's a mismatch between the two estimates but different initial conditions!

- Such a mismatch is originated by slightly different way to fix initial conditions:
  - Analytical case: BD vacuum is the unperturbed initial state (type I)
  - Numerical case: BD vacuum is the perturbed initial state (type II)

  - The above difference is in the **small k region**!

Application to De Sitter

- Consider the exact analytical spectrum we obtained:

$$\mathcal{P}_v = \frac{k^3}{2\pi^2} p = \frac{a^2 H^2}{4\pi^2} \left(1 + \frac{k^2}{a^2 H^2} - \frac{1}{a^2 k {\mathrm{M_P}}^2} \right)$$

Long WL behav. — **Short WL behav.** — **Quantum Grav. Correction**

- The following relation must hold for QG correction to be observable

$$\frac{k^2}{a^2 H^2} \ll \frac{1}{a^2 {\mathrm{M_P}}^2 k} \ll 1 \quad \Rightarrow \quad \frac{1}{a^2 {\mathrm{M_P}}^2} \ll k \ll \left(\frac{H^2}{{\mathrm{M_P}}^2} \right)^{1/3}$$

- **NOTE**: if QG effects are observable in Long WL regime they dominate over Short WL term forever! BD vacuum is modified by QG corrections!

- One can make a different choice of integration constants, consistent with out approx.:

$$c_- = c_0 = 0 \ , \quad c_+ = k + \left(H^2 / {\mathrm{M_P}}^2 \right) c(k)$$

- Choose $c(k)$ to eliminate the pathological behavior at k/aH>>1

Application to De Sitter

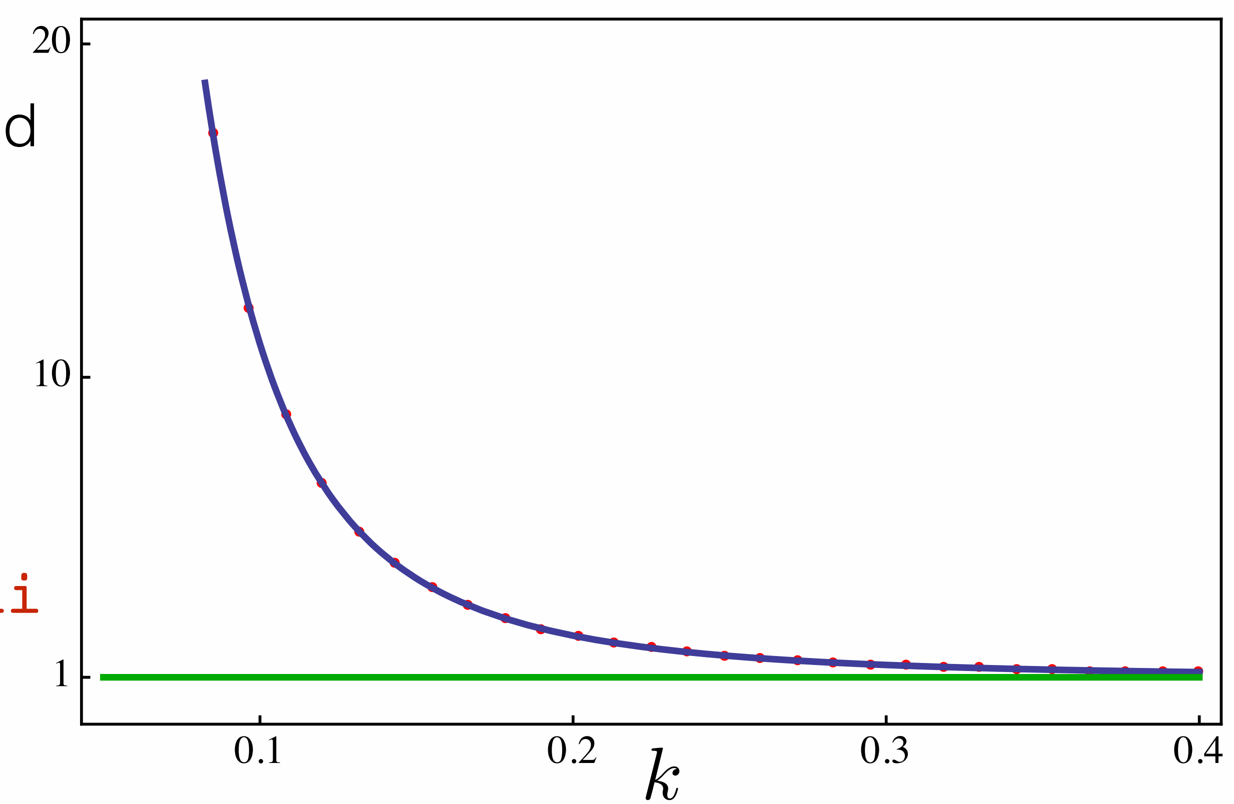
- These modified initial conditions and the above requirement may be equivalent to fixing **type II** initial conditions!

$$\tilde{\mathcal{P}}_v = \frac{a^2H^2}{4\pi^2}\left(1+\frac{k^2}{a^2H^2}-\frac{1}{a^2k\mathrm{M_P}^2}+\frac{c(k)}{k}\frac{H^2}{\mathrm{M_P}^2}+\frac{c(k)k^2}{a^2k\mathrm{M_P}^2}\right)$$

- Take $c(k)=1/k^2$ and the final result is:

$$\tilde{\mathcal{P}}_v = \frac{a^2H^2}{4\pi^2}\left(1+\frac{k^2}{a^2H^2}+\frac{1}{k^3}\frac{H^2}{\mathrm{M_P}^2}\right) \overset{-k\eta\to 0}{=} \frac{a^2H^2}{4\pi^2}\left(1+\frac{1}{k^3}\frac{H^2}{\mathrm{M_P}^2}\right).$$

- The modified spectrum is different from the previous analytical estimate
- The modified spectrum is exactly that obtained in the numerical solution with type II initial conditions!
- **Similar results:**
- D.Bini, G.Esposito, C.Kiefer, M.Kraemer and F.Pessina (2013)
- G.L.Alberghi, R.Casadio, A.Tronconi (2006)
- G. Calcagni (2012)



Comparison with observations?

- The spectrum is evaluated around some **pivot scale** $k \sim k_* \simeq 0.002\ \mathrm{Mpc}^{-1}$
- First one need to **adjust dimensions** (a length scale is hidden)
- **For the type II case**

$$\frac{1}{k^3}\frac{H^2}{{\mathrm{M_P}}^2} \to \delta_{QG} = \left(\frac{k_0}{k}\right)^3 \frac{H^2}{{\mathrm{M_P}}^2} \quad \text{with} \quad (H/\mathrm{M_P})^2 \lesssim 10^{-6}$$

$k_0 \simeq 1.4 \cdot 10^{-4}\mathrm{Mpc}^{-1}$ (smaller observable mode) $\delta_{QG} \simeq 3.4 \cdot 10^{-10}$

$k_0 \sim k_*$ (comparable with the pivot scale) $\delta_{QG} \simeq 10^{-6}$

- **For the type I case**

$$\frac{1}{{\mathrm{M_P}}^2 a^2 k} \to \delta_{QG} = \frac{k_0^3}{{\mathrm{M_P}}^2 a^2 k} = \frac{(k_0/a_0)^2}{{\mathrm{M_P}}^2}\left(\frac{a_0}{a}\right)^2 \frac{k_0}{k}$$

$$\begin{cases} \dfrac{k_0}{a_0} \simeq H \\ \dfrac{k_0}{a_{\mathrm{today}}} \simeq 1.4 \cdot 10^{-4}\mathrm{Mpc}^{-1} \end{cases} \qquad \begin{cases} \delta_{QG} \simeq \left(\dfrac{a_0}{a}\right)^2 \cdot 10^{-7} \\ \left(\dfrac{a_0}{a}\right)^2 \ll 1 \Rightarrow \delta_{QG} \ll 10^{-7} \end{cases}$$

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$k_0 \sim k_*$ (comparable with the pivot scale) $\delta_{QG} \simeq 10^{-6}$

Tiny corrections
Maybe observable
by future experiments

- **For the type I case**

$$\frac{1}{{\mathrm{M_P}}^2 a^2 k} \to \delta_{QG} = \frac{k_0^2}{{\mathrm{M_P}}^2 a^2 k} = \frac{(k_0/a_0)^2}{{\mathrm{M_P}}^2}\left(\frac{a_0}{a}\right)^2 \frac{k_0}{k}$$

$$\begin{cases} \dfrac{k_0}{a_0} \simeq H \\ \dfrac{k_0}{a_{\mathrm{today}}} \simeq 1.4 \cdot 10^{-4}\mathrm{Mpc}^{-1} \end{cases} \qquad \begin{cases} \delta_{QG} \simeq \left(\dfrac{a_0}{a}\right)^2 \cdot 10^{-7} \\ \left(\dfrac{a_0}{a}\right)^2 \ll 1 \Rightarrow \delta_{QG} \ll 10^{-7} \end{cases}$$

Conclusions

- We calculated the quantum gravitational corrections to the spectrum of cosmological perturbations through the canonical quantization of the full matter gravity system
- On adopting a BO decomposition we could decouple the dynamics of the homogeneous d.o.f. and that of scalar perturbations
- We were able to obtain exact equations governing the dynamics of perturbations and that of the two-point function
- These equations can be solved, depending on the background evolution, via an analytical or a numerical approach
- We studied two different prescriptions for initial conditions (type I and type II)
- The machinery was applied to the simplified but still important de Sitter case
- Initial conditions crucially determine the evolution
- **Small k-modes are more affected** by QG effects as expected
- **Power enhancement or power loss** can be obtained depending on initial conditions
- Still QG corrections are **small compared to the precision of present experiments**
- The method **can be easily generalized to other cases** (and that is what we are doing now): gravitational waves, power law inflation, slow-roll inflation

Thank You