

Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory in Ω -background

Hiroaki Nakajima

(National Taiwan University)

in collaboration with

Katsushi Ito (Tokyo Tech) and **Shin Sasaki** (Kitasato Univ.)

based on JHEP 10 (2013) 080 [arXiv:1307.7565]

Nov. 20, 2013 @ PASCOS, NTU

1. Introduction

Ω -background deformation [Moore-Nekrasov-Shatashvili]

Ω -background for $\mathcal{N} = 2$ theory (\leftarrow 6D $\mathcal{N} = (1, 0)$ theory)
= 6D curved background for deformation + SU(2) R-symmetry Wilson line to preserve SUSY

- 6D Ω -background metric (on $\mathbb{R}^4 \times \mathbf{T}^2$)

$$ds^2 = 2d\bar{z}dz + (dx^\mu + \Omega^{\mu\nu} x_\nu d\bar{z} + \bar{\Omega}^{\mu\nu} x_\nu dz)^2.$$

Here constant antisymmetric matrix $\Omega^{\mu\nu}$ and $\bar{\Omega}^{\mu\nu}$ commute with each other.

Hence they can be taken of the form

$$\Omega^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\epsilon}_2 \\ 0 & 0 & \bar{\epsilon}_2 & 0 \end{pmatrix}.$$

- $\bar{Q}_0 \rightarrow \bar{Q}$, $\bar{Q}^2 = (\text{gauge transf.}) + (\text{U(1)}^2 \text{ rotation by } \epsilon_1, \epsilon_2)$
- $S_0 = \bar{Q}_0 \Xi_0 \rightarrow S = \bar{Q} \Xi = \bar{Q}(\Xi_0 + \dots)$

equivariant localization \implies exact computation of path integral

- Nekrasov partition function

Generalization of Ω -background for $\mathcal{N} = 4$ super Yang-Mills theory
[Ito-H.N.-Saka-Sasaki], [Ito-H.N.-Sasaki 2012]

generalized Ω -background

= 10D curved background + SU(4) R-symmetry Wilson line

In 10D, SU(4) = SO(6) R-symmetry is the subgroup of local Lorentz symmetry (No R-symmetry in 10D SYM).

\Rightarrow contribution to spin (and Affine) connection (\sim torsion)

\Rightarrow deformation of parallel (or Killing) spinor equation

$$\nabla_{\mathcal{M}} \zeta = 0 \quad \longrightarrow \quad \widehat{\nabla}_{\mathcal{M}} \zeta = \left(\nabla_{\mathcal{M}} + \frac{1}{4} \mathcal{A}_{\mathcal{M}, NP} \Gamma^{NP} \right) \zeta = 0.$$

Classification of parallel spinors \rightarrow supersymmetry in the theory

contents

1. Introduction (done.)
2. $\mathcal{N} = 4$ SYM in Ω -background and topological twist
3. Off-shell scalar supersymmetry
 - (a) half twist
 - (b) Vafa-Witten twist
 - (c) Marcus twist (GL twist)
4. Summary

2. Ω -background and SUSY condition

Ω -background for $\mathcal{N} = 2$ theory

= 6D curved background + SU(2) R-symmetry Wilson line

generalized Ω -background

= 10D curved background + SU(4) R-symmetry Wilson line
(con)torsion

- 10D metric for generalized Ω -background (on $\mathbb{R}^4 \times \mathbb{T}^6$)

$$ds_{10D}^2 = (dx^\mu + \Omega_a^\mu dx^a)^2 + dx^a dx^a, \quad \Omega_a^\mu = \Omega^\mu{}_{\nu a} x^\nu.$$

Here $\Omega_{\mu\nu a} = -\Omega_{\nu\mu a}$ are constant and commute with each other.

(μ, ν, \dots : 4D indices, a, b, \dots : 6D indices)

commuting $\Omega_{\mu\nu a} \longrightarrow \Omega_{\mu\nu a} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_{1a} & 0 & 0 \\ -\epsilon_{1a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{2a} \\ 0 & 0 & \epsilon_{2a} & 0 \end{pmatrix}.$

R-symmetry Wilson line: $\mathcal{A}_{bc} = \mathcal{A}_{a,bc} dx^a$, $\mathcal{A}_{a,bc}$: constant

Action

$$S_{\Omega} = \int d^4x \frac{1}{\kappa g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_{\mu} \varphi_a - F_{\mu\nu} \Omega_a^{\nu})^2 \right. \\ \left. + \frac{1}{4} \left[[\varphi_a, \varphi_b] + i \Omega_a^{\mu} D_{\mu} \varphi_b - i \Omega_b^{\mu} D_{\mu} \varphi_a - i \Omega_a^{\mu} \Omega_b^{\nu} F_{\mu\nu} \right. \right. \\ \left. \left. - i (\mathcal{A}_{a,b}{}^c - \mathcal{A}_{b,a}{}^c) \varphi_c \right]^2 + (\text{fermions}) \right].$$

- parallel spinor condition and topological twist

$$\partial_\mu \zeta = 0, \quad \frac{1}{4} (\Omega_{\mu\nu a} \Gamma^{\mu\nu} + \mathcal{A}_{a,bc} \Gamma^{bc}) \zeta = 0.$$

\implies 4D and 6D rotation should be canceled.

$$\Omega_{\mu\nu a} \in U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R.$$

In order to cancel 4D rotation, we restrict the contorsion s.t.

$$\mathcal{A}_{a,bc} \in U(1)_{L'} \times U(1)_{R'} \subset SU(2)_{L'} \times SU(2)_{R'} \subset SU(4).$$

cancellation of $U(1)$ charges \implies topological twist

SUSY condition can be solved for each topological twist.
(We assume generic ϵ_1, ϵ_2 .)

- half twist $(\text{SU}(2)_R \sim \text{SU}(2)_{R'}) \Rightarrow \mathcal{N} = 2^*$ theory
parameters : $\epsilon_1, \epsilon_2, \bar{\epsilon}_1, \bar{\epsilon}_2, m, \bar{m}$ (m, \bar{m} : $\mathcal{N} = 2^*$ mass)
one scalar and one tensor supercharges are preserved.
- Vafa-Witten twist $(\text{SU}(2)_R \sim \text{diag}(\text{SU}(2)_{L'} \times \text{SU}(2)_{R'}))$
parameters : $\epsilon_1, \epsilon_2, \bar{\epsilon}_1, \bar{\epsilon}_2, \hat{\epsilon}_1, \hat{\epsilon}_2, \check{\epsilon}_1, \check{\epsilon}_2$
two scalar and two tensor supercharges are preserved.
- Marcus twist $(\text{SU}(2)_L \sim \text{SU}(2)_{L'}, \text{SU}(2)_R \sim \text{SU}(2)_{R'})$
parameters : $\epsilon_1, \epsilon_2, \bar{\epsilon}_1, \bar{\epsilon}_2$ (special case of half twist)
two scalar and two tensor supercharges are preserved.

3. Off-shell scalar supersymmetry

(a) half twist ($\mathcal{N} = 2^*$ theory)

$$\begin{aligned}
 A_\mu &\rightarrow A_\mu, & \varphi_a &\rightarrow (\varphi, \bar{\varphi}, \varphi^{\dot{\alpha}\hat{A}}), \\
 \Lambda_\alpha^{\hat{A}} &\rightarrow (\Lambda_\mu, \Lambda_\alpha^{\hat{A}}), & \bar{\Lambda}_{\hat{A}}^{\dot{\alpha}} &\rightarrow (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \bar{\Lambda}_{\hat{A}}^{\dot{\alpha}})
 \end{aligned}$$

$\bar{\Lambda}_{\mu\nu}^-$ and $\Lambda_\alpha^{\hat{A}}$ need auxiliary fields for off-shell closure.

auxiliary fields

$$\begin{aligned}
 \bar{Q}\bar{\Lambda}_{\mu\nu} &= -2F_{\mu\nu}^- - i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}\dot{\alpha}}[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}] + 2D_{\mu\nu}^-, \\
 \bar{Q}\Lambda_\alpha^{\hat{A}} &= \sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}}D_\mu\varphi^{\dot{\alpha}\hat{A}} + 2K_\alpha^{\hat{A}}.
 \end{aligned}$$

Algebra of scalar supercharges

$$\bar{Q}^2 = 2\sqrt{2} \left(\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1, \epsilon_2) + \delta_{\text{flavor}}(m) \right),$$

$\delta_{\text{gauge}}(\varphi)$: gauge transformation by φ .

$\delta_{\text{Lorentz}}(\epsilon_1, \epsilon_2)$: $U(1)_L \times U(1)_{R+R'}$ rotation by parameter ϵ_1, ϵ_2 .

$\delta_{\text{flavor}}(m)$: $U(1)_{L'}$ rotation by parameter m .

The action S is written as \bar{Q} -exact form up to the topological term:

$$S = \bar{Q}\Xi + \int d^4x \frac{1}{\kappa g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right].$$

(b) Vafa-Witten twist [Vafa-Witten], [Dijkgraaf-Moore], etc.

$$\begin{aligned}
 A_\mu &\rightarrow A_\mu, & \varphi_a &\rightarrow (\varphi, \bar{\varphi}, \hat{\varphi}, \hat{\varphi}_{\mu\nu}^-), \\
 \Lambda_\alpha^A &\rightarrow (\Lambda_\mu, \hat{\Lambda}_\mu), & \bar{\Lambda}_A^{\dot{\alpha}} &\rightarrow (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \hat{\Lambda}, \hat{\Lambda}_{\mu\nu}^-)
 \end{aligned}$$

$\Lambda_\mu, \hat{\Lambda}_\mu, \bar{\Lambda}_{\mu\nu}^-$ and $\hat{\Lambda}_{\mu\nu}^-$ need auxiliary fields for off-shell closure.

\implies auxiliary fields: $K_\mu, D_{\mu\nu}^-$

Algebra of scalar supercharges

$$\begin{aligned}
 \bar{Q}^2 &= 2\sqrt{2}(\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1, \epsilon_2)), \\
 \hat{Q}^2 &= -2\sqrt{2}(\delta_{\text{gauge}}(\bar{\varphi}) + \delta_{\text{Lorentz}}(\bar{\epsilon}_1, \bar{\epsilon}_2)), \\
 \{\bar{Q}, \hat{Q}\} &= 2\sqrt{2}(\delta_{\text{gauge}}(\hat{\varphi}) + \delta_{\text{Lorentz}}(\hat{\epsilon}_1, \hat{\epsilon}_2)).
 \end{aligned}$$

The action is written in the exact form with respect to the two scalar supercharges simultaneously:

$$S = \bar{Q}\hat{Q}\mathcal{F} + \int d^4x \frac{1}{\kappa g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right].$$

Here

$$\begin{aligned} \mathcal{F} = \int d^4x \frac{1}{\kappa g^2} \text{Tr} & \left[-\frac{1}{2\sqrt{2}} \hat{\varphi}^{\mu\nu} F_{\mu\nu}^- + \frac{1}{8} \bar{\Lambda}^{\mu\nu} \hat{\Lambda}_{\mu\nu} + \frac{1}{8} \Lambda^\mu \Lambda_\mu - \frac{1}{8} \bar{\Lambda} \hat{\Lambda} + \frac{i}{24\sqrt{2}} \hat{\varphi}^{\mu\nu} [\hat{\varphi}_\mu^\lambda, \hat{\varphi}_{\lambda\nu}] \right. \\ & + \frac{1}{16\sqrt{2}} \hat{\varphi}^{\mu\nu} (\hat{\Omega}^{\rho, \mu\sigma} D_\rho \hat{\varphi}_\nu^\sigma - \hat{\Omega}^{\rho, \nu\sigma} D_\rho \hat{\varphi}_\mu^\sigma - i \hat{\Omega}_{\mu\nu, \rho\sigma} \hat{\varphi}^{\rho\sigma} + i \hat{\Omega}_{\rho\sigma, \mu\nu} \hat{\varphi}^{\mu\nu}) \\ & \left. + \frac{3}{2\sqrt{2}} \hat{\Omega}^{[\rho, \mu\nu]} \left(A_{[\mu} F_{\nu\rho]} - \frac{i}{3} A_{[\mu} A_\nu A_{\rho]} \right) \right], \end{aligned}$$

$$\hat{\Omega}^{\rho, \mu\nu} = \hat{\Omega}^{\rho\sigma, \mu\nu} x_\sigma, \quad \hat{\Omega}_{12,12} = -\hat{\Omega}_{12,34} = -\frac{1}{\sqrt{2}} \check{\epsilon}_1, \quad \hat{\Omega}_{34,12} = -\hat{\Omega}_{34,34} = \frac{1}{\sqrt{2}} \check{\epsilon}_2.$$

(c) Marcus twist (GL twist) [Marcus], [Kapustin-Witten]

$$\begin{aligned} A_\mu &\rightarrow A_\mu, & \varphi_a &\rightarrow (\varphi, \bar{\varphi}, \varphi_\mu), \\ \Lambda_\alpha^A &\rightarrow (\Lambda_\mu, \Lambda, \Lambda_{\mu\nu}^+), & \bar{\Lambda}_A^{\dot{\alpha}} &\rightarrow (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \bar{\Lambda}_\mu) \end{aligned}$$

Λ , $\bar{\Lambda}$, $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$ need auxiliary fields for off-shell closure.

\implies auxiliary fields: K , $K_{\mu\nu}^+$, $D_{\mu\nu}^-$

On-shell algebra of scalar supercharges

$$\begin{aligned} Q^2 = \bar{Q}^2 &= 2\sqrt{2} \left(\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1, \epsilon_2) \right), \\ \{Q, \bar{Q}\} &= 0. \end{aligned}$$

The first equation holds **off-shell** but the second does **not** hold off-shell on the fields $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$.

linear combination of scalar supercharges [Kapustin-Witten]

$$Q = uQ + v\bar{Q}, \quad u, v \in \mathbb{C},$$

$$Q^2 = 2\sqrt{2}(u^2 + v^2) \left(\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\epsilon_1, \epsilon_2) \right).$$

The above algebra holds **off-shell**.

Q -exactness of the action

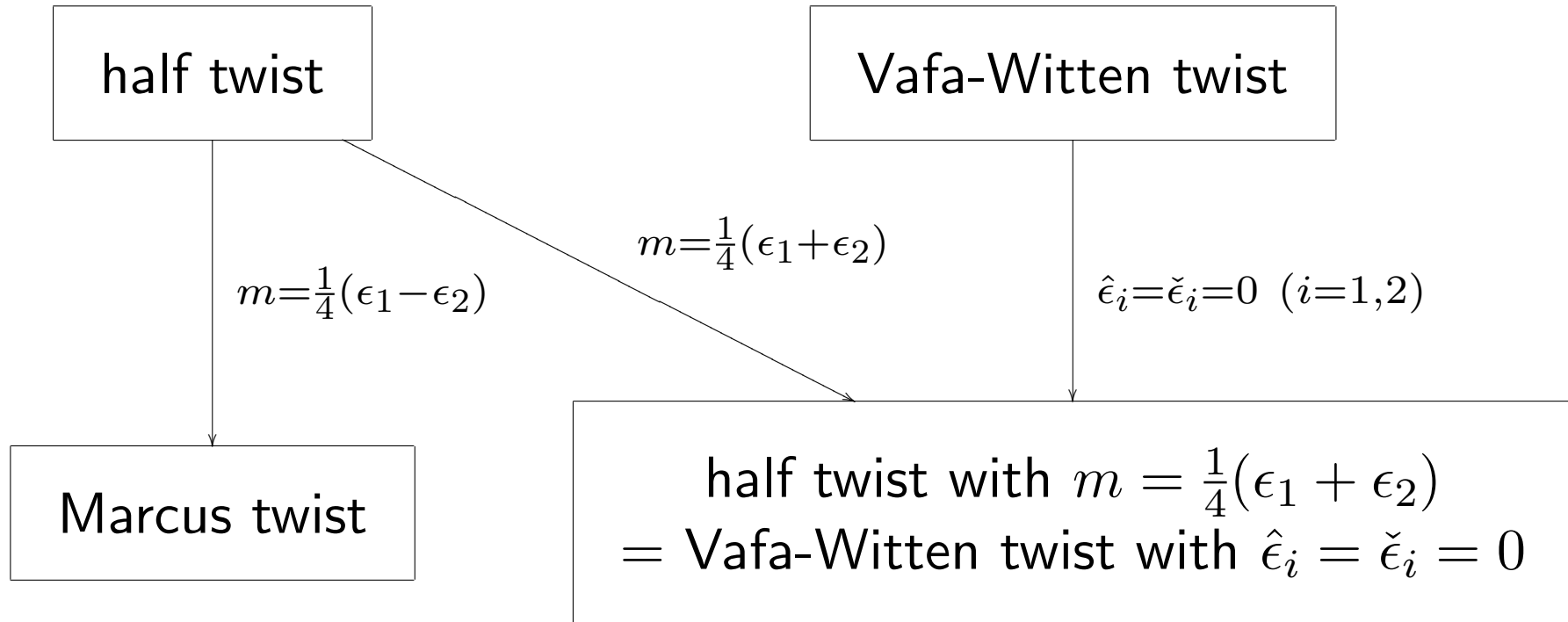
- $u^2 + v^2 \neq 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + \int d^4x \frac{1}{\kappa g^2} \text{Tr} \left[\frac{u^2 - v^2}{4(u^2 + v^2)} F_{\mu\nu} \tilde{F}^{\mu\nu} \right].$$

- $u^2 + v^2 = 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + (\mathcal{Q}\text{-closed term, not } \mathcal{Q}\text{-exact}).$$

Relation among three topological twists



cf. [Pestun], [Pestun-Okuda], etc.

4. Summary

summary

1. We have constructed off-shell scalar supercharge(s) in Ω -deformed $\mathcal{N} = 4$ super Yang-Mills.
2. The action is written as the exact form with respect to the supercharge(s) up to the topological term except the case of Marcus twist with $u^2 + v^2 = 0$.

future work

- Nekrasov-Shatashvili limit, enhancement of (off-shell) SUSY
- extension to more complicated backgrounds
[Festuccia-Seiberg], [Dumitrescu-Festuccia-Seiberg],
[Hama-Hosomichi], [Klare-Zaffaroni] etc.
- auxiliary field formalism in 10D [Berkovits], etc.
- embedding to superstring/SUGRA
R-R 3-form (in instanton ($D(-1)$) effective action)
[Ito-H.N.-Sasaki], [Ito-H.N.-Saka-Sasaki],
[Antoniadis-Florakis-Hohenegger-Narain-Zein Assi], etc.
(cf. [Hellerman-Orlando-Reffert], [Reffert], [Nakayama-Ooguri])