AGT relation in the light asymptotic limit

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Reference

・JHEP 1310, 152 (2013)
arXiv:1307.8174[hep-th] with K. Hosomichi (YITP)

・JHEP 1209, 033 (2012)
arXiv:1206.6359[hep-th] with K. Hosomichi

AGT relation

[09 Alday-Gaiotto-Tachikawa]

Partition function
of 4d $\mathcal{N}=2$ thy
on S^4

↕ equivalent

Correlators in
2d CFT (LFT, Toda)
whose coupling b=1

explained as
:2 different limits in
compactification of
6d (2,0) thy

AGT relation

Localization method

: deform the action by susy exact term

- one way to compute path integral exactly
- using nilpotent sym. of the system

  ～up to bosonic sym.

  (gauge sym. , Poincare, ……)

  here: susy

- only calculate susy closed observables

ex.) partition function $Z = \int \mathrm{d}a_E e^{-S_{\mathrm{cl}}(a_E)} Z_{1-\mathrm{loop}}(a_E) |Z_{\mathrm{inst}}|^2$

a_E : parameter of saddle pt

$e^{-S_{\mathrm{cl}}}$: classical part

$Z_{1-\mathrm{loop}}$: 1-loop det.

Z_{inst} : Nekrasov partition function

AGT relation

Partition function of 4d $\mathcal{N}=2$ thy on S^4

deform

Partition function of 4d $\mathcal{N}=2$ thy on ellipsoid S_b^4

Ellipsoid: $\frac{x_0^2}{r^2}+\frac{x_1^2+x_2^2}{\ell^2}+\frac{x_3^2+x_4^2}{\tilde{\ell}^2}=1 \quad \left(b=\sqrt{\frac{\ell}{\tilde{\ell}}}\right)$

On the poles of these mfds, the system become Ω-deformed $\mathbb{R}^4$

With Ω-deformation parameters

$$\epsilon_1=\epsilon_2$$

With Ω-deformation parameters

$$\frac{\epsilon_1}{\epsilon_2}=b^2$$

AGT relation

Partition function of 4d $\mathcal{N}=2$ thy on S^4

equivalent

Correlators in 2d CFT (LFT, Toda) whose coupling b=1

deform

Partition function of 4d $\mathcal{N}=2$ thy on ellipsoid S^4_b

equivalent

Correlators in 2d CFT (LFT, Toda) w/ general coupling b

Ellipsoid: $\frac{x_0^2}{r^2}+\frac{x_1^2+x_2^2}{\ell^2}+\frac{x_3^2+x_4^2}{\tilde{\ell}^2}=1 \quad \left(b=\sqrt{\frac{\ell}{\tilde{\ell}}}\right)$

Now...

we can take b to any value.

Can't help taking limit!

We take $b \to 0$ limit.

LFT side: Semiclassical limit [Zamolodchikov '84]

Gauge thy side: Nekrasov-Shatashvili limit

AGT relation

We want to "use" AGT relation as a tool to study 4d class $\mathcal{S}$ and 6d (2,0) thy.

To make a "dictionary" of AGT relation

ex.) Many sym.s of LFT correlators $< \prod_i V_{\alpha_i}(z_i) >$

- $b \leftrightarrow \frac{1}{b}$ ～Quantum sym.
- $V_\alpha(z) = R(\alpha) V_{Q-\alpha}(z)$

  ～Reflection sym. of primary op.s
- Permutation sym. of $V_{\alpha_i}(z_i)$ s

How are these sym.s respected in 4d gauge thy?

sym. of thy?   only BPS observables?   structure of mfd?

AGT relation

Here we use easiest situation in AGT relation to take the limit where the dictionary can be filled by explicit calculations.

4d SU(2) $N_f = 4$SQCD
w/ 4 (anti)fund.s on S_b^4

⇕

LFT on sphere w/ 4 punctures

and take **light asymptotic limit** (LA limit)

Light asymptotic limit

[84 Zamolodchikov]

A particular example of classical limit $b \to 0$ in LFT
path integral ⇒ finite dim. integral

break the quantum sym. $b \leftrightarrow \frac{1}{b}$

On gauge thy:
In $b \to 0$, instanton counting become easy.

Liouville Field Theory

Action: $S_{\mathrm{L}} = \frac{1}{4\pi}\int \sqrt{g}\mathrm{d}^2\sigma \left(g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\right)$

where b: Liouville coupling, $Q = b + \frac{1}{b}$

R: scalar curvature〜here: sphere → 0

・2d CFT which realizes Virasoro algebra whose central charge is $c = 1 + 6Q^2$

・Prim op.s are $V_\alpha = e^{2\alpha\phi}$, whose conformal dim. are $\alpha(Q-\alpha)$

・Their correlators are

$$\left\langle \prod_i V_{\alpha_i}(\sigma_i) \right\rangle = \int D\phi \exp\Big(-S_{\mathrm{L}} + \sum_i 2\alpha_i\phi(\sigma_i)\Big)$$

LFT in LA limit

Semiclassical limit $b \to 0$ w/ rescaling $b\phi \equiv \varphi$, μ ,

$$S_{\rm L} \to \frac{1}{\pi b^2} \int {\rm d}^2 z \Big(\partial\varphi\bar{\partial}\varphi + \tilde{\mu}\pi e^{2\varphi} \Big)$$

Only OVERALL FACTOR ⇒ semiclassical limit

Furthermore,

LA limit: Inserted momenta $\alpha_i = b\eta_i \to 0$ $(\eta_i : \text{fixed})$

⇒ can be ignored

⇒ path integral in correlators localizes onto the solutions of EoM

$$\partial\bar{\partial}\varphi = \pi\mu b^2 e^{2\varphi}$$

LFT in LA limit

Solutions of EoM are labeled by $g \in H_3^+$

3D HYPERBOLOID

Path integral become an integration over H_3^+;

$$\left\langle \prod_i V_{\alpha_i}(z_i) \right\rangle \Big|_{\mathrm{LA}} = \int_{H_3^+} \mathrm{d}g \prod_i \Phi_{z_i}^{\eta_i}(g) \equiv \left\langle \prod_i \Phi_{z_i}^{\eta_i} \right\rangle_{H_3^+}$$

where $\Phi_z^\eta(g) \equiv \left[(z \; 1)\, g \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \right]^{-2\eta}$ are wavefunctions:

$$\delta(g - g') = -\frac{1}{2\pi^3} \int \mathrm{d}\eta \int \mathrm{d}^2 z \; (2\eta - 1)^2 \Phi_z^\eta(g) \Phi_z^{1-\eta}(g')$$

LFT in LA limit

4-pt. function is inserted complete set,

$$\left\langle \Phi^{\eta_1}_{z_1}\Phi^{\eta_2}_{z_2}\Phi^{\eta_3}_{z_3}\Phi^{\eta_4}_{z_4}\right\rangle_{H_3^+} = -\int \frac{\mathrm{d}\eta_0 \mathrm{d}^2 z_0}{2\pi^3}(2\eta_0-1)^2 \left\langle \Phi^{\eta_1}_{z_1}\Phi^{\eta_2}_{z_2}\Phi^{\eta_0}_{z_0}\right\rangle_{H_3^+}\left\langle \Phi^{1-\eta_0}_{z_0}\Phi^{\eta_3}_{z_3}\Phi^{\eta_4}_{z_4}\right\rangle_{H_3^+}$$

$$= \frac{1}{4}|z_{12}|^{-2\eta_{1+2}}|z_{13}|^{2\eta_{-1-2-3+4}}|z_{14}|^{4\eta_2+2\eta_{3-4}}|z_{23}|^{2\eta_{1+2}}|z_{24}|^{-4\eta_2}|z_{34}|^{-2\eta_{3+4}}$$

$$\times\int \mathrm{d}\eta_0 \frac{\Gamma(\eta_{0+1+2}-1)\Gamma(\eta_{-0+1+2})\Gamma(\eta_{0+1-2})\Gamma(\eta_{0-1+2})}{\Gamma(2\eta_0)\Gamma(2\eta_0-1)\Gamma(2\eta_1)\Gamma(2\eta_2)}$$

$$\times\frac{\Gamma(\eta_{0+3+4}-1)\Gamma(\eta_{-0+3+4})\Gamma(\eta_{0+3-4})\Gamma(\eta_{0-3+4})}{\Gamma(2\eta_3)\Gamma(2\eta_4)}$$

$$\times|q|^{2\eta_0}F(\eta_{0-1+2},\eta_{0+3-4};2\eta_0;q)F(\eta_{0-1+2},\eta_{0+3-4};2\eta_0;\bar{q})$$

IN AGT RELATION,

where $q \equiv \frac{z_{43}z_{21}}{z_{42}z_{31}}$: cross ratio

only η_0 related parts appear
in the partition function of 4d gauge thy

Gauge theory set up

4 pt.s function on sphere corresponds to
SU(2) $N_f = 4$ SQCD
w/ 4 (anti)fund.s on S_b^4: $\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1$

4d		LFT	
COMPLEX GAUGE COUPLING	AGT	CROSS RATIO	$q = \exp\left(-\frac{8\pi^2}{g^2} + i\theta\right)$
FLAVOR MASS $\mu_{i=1,\dots,4}$	AGT	INSERTED MOMENTA	$\mu_1 = p_1 + p_2$ $\mu_2 = -p_1 + p_2$ $\mu_3 = -p_3 + p_4$ $\mu_4 = -p_3 - p_4$ $(\alpha_i = b\eta_i = \frac{Q}{2} + ip_i)$
	LA	2 of 4: MASSLESS 2 of 4: HEAVY	~permutation sym.?

1-loop Part

$$\left(\Upsilon(x) = \prod_{m,n\geq 0}(mb+\frac{n}{b}+x)(mb+\frac{n}{b}+Q-x)\right)$$

$$Z_{\text{1-loop}} = \frac{\Upsilon(2ia)\Upsilon(-2ia)}{\prod_{i=1}^{4}\Upsilon(\frac{Q}{2}+ia+i\mu_i)\Upsilon(\frac{Q}{2}-ia+i\mu_i)}$$

$$= \frac{\Upsilon(2b\eta_0-Q)\Upsilon(2b\eta_0)}{\Upsilon(b\eta_{0+1+2}-Q)\Upsilon(b\eta_{0+1-2})\Upsilon(b\eta_{0-1+2})\Upsilon(b\eta_{-0+1+2})\Upsilon(b\eta_{0+3+4}-Q)\Upsilon(b\eta_{0+3-4})\Upsilon(b\eta_{0-3+4})\Upsilon(b\eta_{-0+3+4})}$$

Only considering zeroes,
Υfunctions in LA limit are represented by

$$\begin{cases}\Upsilon(bx-Q) \xrightarrow{\text{LA}} b^{\frac{3}{2}-x}\dfrac{\sqrt{2\pi}}{\Gamma(x-1)}\\ \Upsilon(bx) \xrightarrow{\text{LA}} b^{\frac{1}{2}-x}\dfrac{\sqrt{2\pi}}{\Gamma(x)}\end{cases}$$

Then,

$$Z_{\text{1-loop}} \xrightarrow{\text{LA}} \frac{b^{-4+2\eta_{1+2+3+4}}}{(2\pi)^3}\frac{\Gamma(\eta_{0+1+2}-1)\Gamma(\eta_{0+1-2})\Gamma(\eta_{0-1+2})\Gamma(\eta_{-0+1+2})\Gamma(\eta_{0+3+4}-1)\Gamma(\eta_{0+3-4})\Gamma(\eta_{0-3+4})\Gamma(\eta_{-0+3+4})}{\Gamma(2\eta_0)\Gamma(2\eta_0-1)}$$

$$\left\langle \Phi_{z_1}^{\eta_1}\Phi_{z_2}^{\eta_2}\Phi_{z_3}^{\eta_3}\Phi_{z_4}^{\eta_4}\right\rangle_{\eta_0}$$

$$= \int d\eta_0 \frac{\Gamma(\eta_{0+1+2}-1)\Gamma(\eta_{-0+1+2})\Gamma(\eta_{0+1-2})\Gamma(\eta_{0-1+2})}{\Gamma(2\eta_0)\Gamma(2\eta_0-1)\Gamma(2\eta_1)\Gamma(2\eta_2)}$$

$$\times\frac{\Gamma(\eta_{0+3+4}-1)\Gamma(\eta_{-0+3+4})\Gamma(\eta_{0+3-4})\Gamma(\eta_{0-3+4})}{\Gamma(2\eta_3)\Gamma(2\eta_4)}$$

$$\times|q|^{2\eta_0}F(\eta_{0-1+2},\eta_{0+3-4};2\eta_0;q)F(\eta_{0-1+2},\eta_{0+3-4};2\eta_0;\bar{q})$$

reproduce!

Instanton Part

[09 Mironov, Morozov]

Take LA limit to Nekrasov's formula...

LA: only partiton
$\vec{Y} = (\ \square\square\square\cdots\square\ , 0)$ can contributes to Z_{inst}

$$Z_{\text{inst}} \overset{\text{LA}}{\Rightarrow} \sum_{k\geq 0} \frac{q^k}{k!} \prod_{m=0}^{k-1} \frac{(\eta_{0-1+2}+m)(\eta_{0+3-4}+m)}{2\eta_0+m} = F(\eta_{0-1+2}, \eta_{0+3-4}; 2\eta_0; q)$$

$$\left\langle \Phi_{z_1}^{\eta_1} \Phi_{z_2}^{\eta_2} \Phi_{z_3}^{\eta_3} \Phi_{z_4}^{\eta_4} \right\rangle_{\eta_0}$$
$$= \int d\eta_0 \frac{\Gamma(\eta_{0+1+2}-1)\Gamma(\eta_{-0+1+2})\Gamma(\eta_{0+1-2})\Gamma(\eta_{0-1+2})}{\Gamma(2\eta_0)\Gamma(2\eta_0-1)\Gamma(2\eta_1)\Gamma(2\eta_2)}$$
$$\times \frac{\Gamma(\eta_{0+3+4}-1)\Gamma(\eta_{-0+3+4})\Gamma(\eta_{0+3-4})\Gamma(\eta_{0-3+4})}{\Gamma(2\eta_3)\Gamma(2\eta_4)}$$
$$\times |q|^{2\eta_0} F(\eta_{0-1+2}, \eta_{0+3-4}; 2\eta_0; q) F(\eta_{0-1+2}, \eta_{0+3-4}; 2\eta_0; \bar{q})$$

reproduce!

Instanton Part

[09 Mironov, Morozov]

Take LA limit to Nekrasov's formula...

LA

only partiton
$\vec{Y} = (\square\square\square\cdots\square, 0)$ can contributes to Z_{inst}

Summation over 1direction

Vortex sum?
2d gauge thy?

Does squashing limit of ellipsoid reproduce this 2d mfd?