Vortex counting in vortex worldsheet theory

Toshiaki Fujimori (National Taiwan University)

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and work in progress

Taro Kimura (CEA Saclay)
Muneto Nitta (Keio University)
Keisuke Ohashi (Osaka City University)

1. Introduction

Similarity between 2d and 4d gauge theories

generation of mass gap , asymptotically free, anomaly ･･･

instantons : non-perturbative effects

4d theory	2d thoery
• Yang-Mills instanton	• vortex

➡ **2d models are “toy modesl” of 4d gauge theories**

Lessons on 4d instantons from 2d vortices

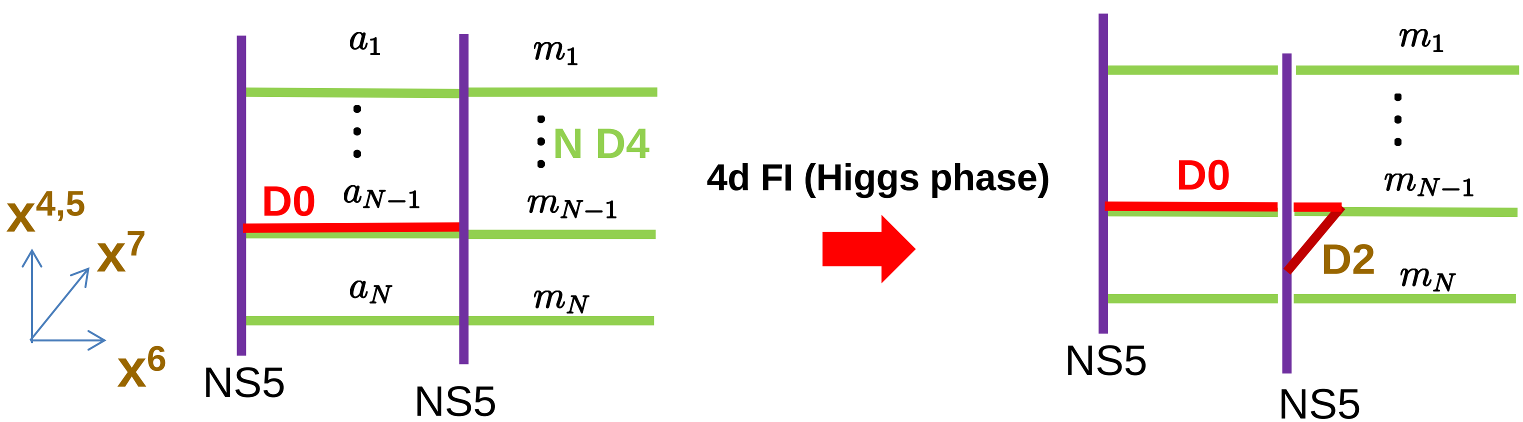
relation between 4d instanton and 2d vortex counting

- supersymmetry ··· smilarities → exact relations

4d U(N) gauge theory + N fund. hypers	2d U(k) gauge theory one ad. + N find. chirals

D4: 4d theory

D2: 2d theory



- root of Higgs branch

$$\vec{a} = \vec{m}$$

D0: YM instantons

= vortex on vortex string

relation between 4d instanton and 2d vortex counting

$$Z_{4d}(\vec{a} = \vec{m} - \epsilon_1 \vec{k}, \tau) \;=\; A\, Z_{2d}(\vec{m}, \vec{k}, \tau)$$

- ϵ_i : **Omega deformation parameters**
- $\vec{k}$: **magnetic flux of vortex strings** $\quad \frac{1}{2\pi}\int F = \vec{H}\cdot\vec{k}$

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- $\epsilon_2 \to 0$ **limit** $\quad \widetilde{\mathcal{W}}_{\vec{k}} - \widetilde{\mathcal{W}}_{\vec{k}'} = \lim_{\epsilon_2 \to 0}\left(-\frac{\epsilon_2}{2\pi} \log \frac{Z_{4d,\, \vec{a}=\vec{m}-\epsilon_1\vec{k}}}{Z_{4d,\, \vec{a}=\vec{m}-\epsilon_1\vec{k}'}} \right)$

[Chen-Dorey-Hollowood-Lee, 2011]

- $\epsilon_{1,2} \to 0$ **limit** $\quad \lim_{\epsilon_1 \to 0}\left(\widetilde{\mathcal{W}}_{\vec{k}} - \widetilde{\mathcal{W}}_{\vec{k}'}\right) \;=\; -\frac{1}{2\pi}(\vec{k} - \vec{k}')\cdot \frac{\partial}{\partial \vec{a}}\mathcal{F}$

- **exact correspondence of 2d kink and 4d monopole masses** [Dorey, 1998]

2. Vortex partition function

2d Omega deformation dimensional reduction from 4d

$$ds^2 = |dz - iz(\epsilon dw + \bar{\epsilon} d\bar{w})|^2 + |dw|^2 \longleftarrow \text{two torus}$$

- background connections for R- and flavor symemtries

$$V_i dx^i = -\frac{1}{2}(\epsilon dw + \bar{\epsilon} d\bar{w}) \Rightarrow \text{Killing spinors} \quad (\nabla_i - iV_i)\,\epsilon = 0$$

$$A_i^a dx^i = m^a dw + \bar{m}^a d\bar{w} \Rightarrow \text{twisted mass}$$

- The form of the SUSY transformations are the same as 4d N=1

vortex partition function

$$\mathcal{Q}^2 = \epsilon\left(J - \frac{1}{2}R\right) + m_a F_a, \quad \mathcal{Q}^2 V = \mathcal{Q} I = 0$$

$$Z = \int [\mathcal{D}\varphi] \, \exp\left(-\mathcal{Q}V + I\right)$$ **··· invariant under deformations of V**

Q-exact part

$$V = \sum_{fermions} \Psi \overline{\mathcal{Q}\Psi} + V_{FI}$$

$$\mathcal{Q}V = S - \tau \int F$$

$\epsilon \to 0$ ➡ **N=(2,2) action without topological term**

Q-closed operator

$$\widetilde{W} = i\tau\Sigma$$ **constant $\mathcal{T}$ ··· topological term**

$\mathcal{T}$ **···background twisted superfield** $\lim_{|z|\to\infty} \tau = 0$ $\quad \lim_{\epsilon\to 0} = \tau_0 = const.$

$$I = \int dz^2 d\theta d\bar{\theta}\, \widetilde{W} = \frac{2\pi i \tau_0}{\epsilon}\sigma + \mathcal{O}(\epsilon)$$

localization = WKB method

$QV = 0$ **saddle points**

$$\mathcal{D}_{\bar{z}}\phi_a = 0, \quad 2iF_{z\bar{z}} + g^2(|\phi_a|^2 - r) = 0$$

$$F_{\xi\bar{\xi}} = F_{\xi z} = F_{\xi\bar{z}} = \mathcal{D}_\xi\phi_a = \mathcal{D}_\xi\bar{\phi}_a = 0$$

$$\xi^i\partial_i = \partial_w + i\epsilon(z\partial_z + \bar{z}\partial_{\bar{z}})$$

Killing vector

- **saddle points = BPS vortex configurations**

$$\phi_a = \sqrt{r}\, e^{-\frac{1}{2}\psi} z^k, \quad A_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}\psi, \quad A_\xi = -m_a - k\epsilon \longrightarrow \sigma = -m_a - k\epsilon$$

fluctuations

$$QV = \delta\Phi^\dagger\Delta_B\delta\Phi + \delta\Psi^\dagger\Delta_F\delta\Psi + \cdots$$

$$Z = \sum_{s.p.} \exp(I)\frac{\det(\Delta_F)}{\det(\Delta_B)} = \sum_{s.p.} \exp\left(-\frac{2\pi i\sigma\tau}{\epsilon}\right)\frac{1}{\det(-i\mathcal{D}_\xi)}\bigg|_{short}$$

- **short multiplets = solutions of linearized vortex equation**

- **solutions of linearized eq. ・・・ holomorphic polynomials**

$$\delta\phi_a = \sqrt{r} e^{-\frac{1}{2}\psi}\left[\Delta h_a(z) - \frac{1}{2}\Delta\psi h_a(z)\right], \qquad \Delta h_a(z) = \sum_{l=0}^{\infty} c_{a,l} z^l$$

- **each monomial ・・・ eigenmode** $-i\mathcal{D}_\xi = \sigma + m_a + l\epsilon$

$$\frac{1}{\det(-i\mathcal{D}_\xi)} = \prod_{a=1}^{N}\prod_{l=0}^{\infty}\frac{\Lambda_0}{\sigma + m_a + l\epsilon} \sim \left(\frac{\Lambda_0}{\epsilon}\right)^N \prod_{a=1}^{N}\Gamma\left(\frac{\sigma + m_a}{\epsilon}\right)$$

zeta function regularization

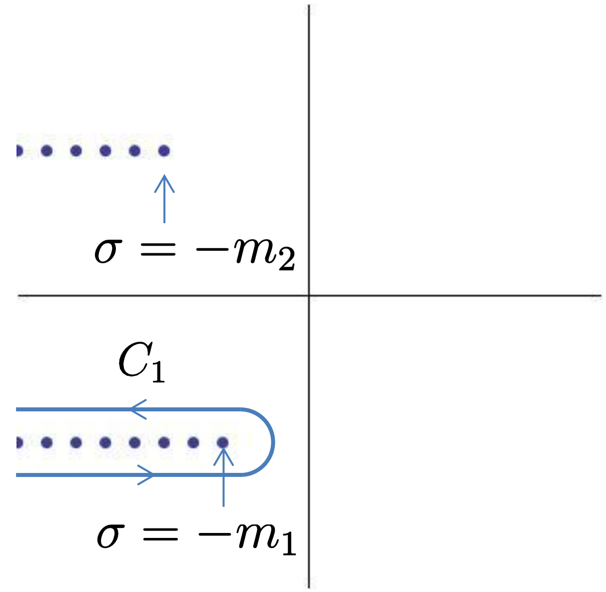
unphysical "gauge" modes $\Gamma\left(\frac{\sigma - \sigma_s}{\epsilon}\right) \qquad \sigma_s = -m_a - k\epsilon$

$$\Rightarrow \quad \exp(I)\,\frac{\det(\Delta_F)}{\det(\Delta_B)} = \left[\exp\left(-\frac{2\pi i\sigma\tau}{\epsilon}\right)\prod_{a=1}^{N}\Gamma\left(\frac{\sigma + m_a}{\epsilon}\right)\right]\Gamma\left(\frac{\sigma - \sigma_s}{\epsilon}\right)^{-1}$$

- **value at the saddle point** $\sigma \to \sigma_s = -m_a - k\epsilon$ ➡ **residue**

Integral representation of vortex partition function

$$Z_a = \int_{C_a} \frac{d\sigma}{2\pi i \epsilon} \exp\left(-\frac{2\pi i \sigma \tau}{\epsilon}\right) \prod_{b=1}^{N} \Gamma\left(\frac{\sigma + m_b}{\epsilon}\right)$$



- **expectation value of σ**

$$\langle \sigma \rangle_a = \lim_{\epsilon \to 0} \left(-\frac{\epsilon}{2\pi} \partial_\tau \log Z_a\right)$$

agree with known results **[D'Adda et al, Witten, ･･･]**

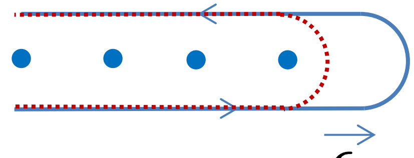
- **general gauge group and matter representation**

$$Z = \int \prod_{i=1}^{r} \left(\frac{d\sigma_i}{2\pi i \epsilon}\right) \exp\left(-\frac{2\pi i \vec{\sigma} \cdot \vec{\tau}}{\epsilon}\right) \left[\prod_{\vec{\alpha} \in G} \Gamma\left(\frac{\vec{\alpha} \cdot \vec{\sigma}}{\epsilon}\right)\right]^{-1} \prod_{a=1}^{N_F} \prod_{\vec{\rho}_a \in R_a} \Gamma\left(\frac{\vec{\rho}_a \cdot \vec{\sigma} + m_a}{\epsilon}\right)$$

differential equation for Z

$$Z_a = \int \frac{d\sigma}{2\pi i \epsilon} e^{-\frac{2\pi i \sigma \tau}{\epsilon}} \prod \Gamma\left(\frac{\sigma + m_b}{\epsilon}\right)$$

- **shift of the contour** $\sigma \to \sigma + \epsilon$



$$Z_a = \int \frac{d\sigma}{2\pi i \epsilon} e^{-\frac{2\pi i (\sigma+\epsilon)\tau}{\epsilon}} \prod_{b=1}^{N} \frac{\sigma + m_b}{\epsilon} \Gamma\left(\frac{\sigma + m_b}{\epsilon}\right)$$

$$= e^{-2\pi i \tau} \left(\prod_{b=1}^{N} \frac{\hat{\sigma} + m_b}{\epsilon}\right) Z_a \quad \Rightarrow \quad \left[\prod_{b=1}^{N} \frac{\hat{\sigma} + m_b}{\epsilon} - e^{2\pi i \tau}\right] Z_a = 0$$

differential operator $\hat{\sigma} \equiv -\frac{\epsilon}{2\pi i}\partial_\tau$

- **vortex partition functions** $Z_a \ (a = 1, 2, \cdots, N)$ **are linearly independent regular solutions**

$$\langle \sigma \rangle_a = \lim_{\epsilon \to 0} \left(-\frac{\epsilon}{2\pi}\partial_\tau \log Z_a\right)$$

regular in the limit

differential equation (general case)

$$G = U(1)^n \times G'$$

$$Z = \int \prod_{i=1}^{r} \frac{d\sigma_i}{2\pi i \epsilon} \exp\left(-\frac{2\pi i \vec{\sigma}\cdot\vec{\tau}}{\epsilon}\right) \cdots$$

- **physical τ_i are defined only for U(1) parts**

➡ **for all the Cartan parts** $\hat{\sigma}_i = -\frac{\epsilon}{2\pi i}\frac{\partial}{\partial \tau_i} \quad (i = 1, \cdots, r)$

r = rank G

- **r independent differential equations**

$$\left[\Delta(\vec{\hat{\sigma}} + \epsilon\vec{\lambda})P_\lambda^+(\vec{\hat{\sigma}}) - \epsilon^{\vec{\rho}_t\cdot\vec{\lambda}} \exp\left\{2\pi i\vec{\lambda}\cdot(\vec{\tau} + \vec{\rho}_w)\right\} \Delta(\vec{\hat{\sigma}} - \epsilon\vec{\lambda})P_\lambda^-(\vec{\hat{\sigma}})\right] Z = 0$$

$$\Delta(\vec{\sigma}) = \prod_{\vec{\alpha}>0} \vec{\alpha}\cdot\vec{\sigma} \qquad P_\lambda^\pm(\vec{\sigma}) = \prod_{a=1}^{N_F} \prod_{\rho_a \in R_{a,\lambda}^\pm} \prod_{j=1}^{|\vec{\rho}_a\cdot\vec{\lambda}|} \left\{\vec{\rho}_a\cdot\vec{\sigma} + m_a + (j-1)\epsilon\right\}$$

3. Vortex counting on vortex worldsheet

4d $\mathcal{N}=2$ U(N) gauge theory

$N \leq N_{\mathrm{F}} \leq 2N$

N fundamental hypermutiplets $\vec{m}$

- ϵ_1, ϵ_2 : 4d omega deformation parameters
- root of Higgs branch $\vec{a} = \vec{m} - \epsilon_1 \vec{k}$
- magnetic flux $\frac{1}{2\pi}\int F = \vec{H}\cdot\vec{k}$, $k = |\vec{k}|$: number of vortex strings

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effective worldsheet theory of k-vortex strings

2d $\mathcal{N}=(2,2)$ U(k) gauge theory effective vortex worldsheet theory

[Hanany-Tong, 2003]

2d $\mathcal{N}=(2,2)$ U(k) gauge theory

one adjoint + N fundamental

- Higgs branch ･･･ moduli space of vortices
- $\vec{k}$: label of vacua (in mass and omega deformed theory)
- ϵ_2 : 2d omega deformation parameter
- ϵ_1 : mass for adjoint chiral ･･･ positions of vortex strings

vortex on vortex worldsheet = Yang-Mills instanton

$$Z_{4d,\,instanton} \Longleftrightarrow Z_{2d,\,vortex}$$

vortex partition function in vortex worldsheet effective theory

integral representation

adjoint — fundamental

$$Z(\tau_1,\cdots,\tau_k)=\int\prod_{i=1}^{k}\left[\frac{d\sigma_i}{2\pi i\epsilon_2}e^{-\frac{2\pi i\sigma_i\tau}{\epsilon_2}}\right]\prod_{i\neq j}\frac{\Gamma\left(\frac{\sigma_i-\sigma_j-\epsilon_1}{\epsilon_2}\right)}{\Gamma\left(\frac{\sigma_i-\sigma_j}{\epsilon_2}\right)}\prod_{i=1}^{k}\prod_{a=1}^{N}\Gamma\left(\frac{\sigma_i+m_a}{\epsilon_2}\right)$$

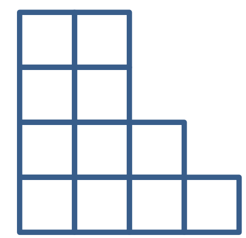
gauge

- **each pole is labeled by N Young tableaux** $Y_a=(\lambda_a^1,\lambda_a^2,\cdots,\lambda_a^{k_a})$

$$\sigma_i=-m_a+(j-k_a)\epsilon_1-\lambda_a^p\epsilon_2$$

$Y_a=(4,3,2,2)$

$k_a=4$ **(height)**



- **perturbative part (vacuum = empty Young tableaux)**

$$Z_{2d,pert}=\exp\left[\frac{\pi i\tau}{\epsilon_2}\left\{2\vec{k}\cdot\vec{m}-(\vec{k}\cdot\vec{k}-k)\epsilon_1\right\}\right]\prod_{a=1}^{N}\prod_{j=1}^{k_a}\prod_{b=1}^{N}\Gamma\left(\frac{-m_{ab}-(j-k_{ab})\epsilon_1}{\epsilon_2}\right)$$

4d perturbative part

$$Z_{4d,\,pert} = \exp\left[\frac{\pi i \vec{a}\cdot\vec{a}\tau}{\epsilon_1\epsilon_2}\right]\prod_{a=1}^{N}\prod_{b=1}^{N}\frac{\Gamma_2(a_a - m_b|\epsilon_1,\epsilon_2)}{\Gamma_2(a_{ab}|\epsilon_1,\epsilon_2)}$$

double gamma function $\Gamma_2(x|\epsilon_1,\epsilon_2) = \Gamma(x/\epsilon_1)\Gamma_2(x+\epsilon_1|\epsilon_1,\epsilon_2)$

$$\vec{a} = \vec{m} - \epsilon_1\vec{k}$$

root of Higgs branch

$$Z_{4d,pert} = \exp\left[\frac{\pi i\tau}{\epsilon_2}(\vec{m}-\vec{k}\epsilon_1)^2\right]\prod_{a=1}^{N}\prod_{b=1}^{N}\prod_{j=1}^{k_b}\Gamma\left(\frac{m_{ab}-(j-k_a)\epsilon_1}{\epsilon_2}\right)$$

$$= A\ Z_{2d,pert}$$

independent of choice of vacuum $\vec{k}$

instanton (vortex) part

$$Z = Z_{pert} \sum_{\vec{Y}} e^{2\pi i |\vec{Y}| \tau} Z_{\vec{Y}}$$

differential equations

$$Z_{2d}(\tau_1, \cdots, \tau_k) = \sum_{\vec{Y}} \exp\left(- \sum_{i=1}^{k} \frac{2\pi i \sigma_i \tau_i}{\epsilon_2} \right) Z_{\vec{Y}}$$

$$\left[\prod_{a=1}^{N} (m_a + \hat{\sigma}_n) \right] \left[\prod_{i \neq n} (\hat{\sigma}_n - \hat{\sigma}_i - \epsilon_1)(\hat{\sigma}_n - \hat{\sigma}_i + \epsilon_2) \right] Z_{2d} =$$

$$\epsilon_2^N e^{2\pi i \tau_n} \left[\prod_{i \neq n} (\hat{\sigma}_n - \hat{\sigma}_i + \epsilon_1)(\hat{\sigma}_n - \hat{\sigma}_i - \epsilon_2) \right] Z_{2d}$$

↓

recursion relations

$|\vec{Y}'| = |\vec{Y}| + 1$ ··· vortex (instanton) number

$$Z_{\vec{Y}'} = e^{2\pi i \tau_i} \left[\prod_{j \neq i} \frac{(\sigma_i - \sigma_j - \epsilon_2)(\sigma_i - \sigma_j - M)}{(\sigma_i - \sigma_j)(\sigma_i - \sigma_j - \epsilon_1 - \epsilon_2)} \right] \left[\prod_{a=1}^{N} \frac{\epsilon_2}{\sigma_i + m_a - \epsilon_2} \right] Z_{\vec{Y}}$$

4d instanton part

[Kanno-Matsuo-Zhang, 2013]

➡ **Simplifies at** $\vec{a} = \vec{m} - \epsilon_1 \vec{k}$

$$Z_{4d,\vec{Y}} = \int \prod_{l=1}^{|\vec{Y}|} \left[\frac{d\Phi_l}{2\pi i} \frac{\epsilon_+}{\epsilon_1 \epsilon_2} \prod_{m<l} \frac{\Phi_{lm}^2 (\Phi_{lm}^2 - \epsilon_+^2)}{(\Phi_{lm}^2 - \epsilon_1^2)(\Phi_{lm}^2 - \epsilon_2^2)} \prod_{a=1}^{N} \frac{\Phi_l - m_a}{(\Phi_l - a_a)(a_a + \epsilon - \Phi_l)} \right]$$

- **poles ··· positions of boxes** (i, j) **in Young tableaux**

$$\Phi_l = a_a + (j-1)\epsilon_1 + (i-1)\epsilon_2 \;\Rightarrow\; m_a + (j - k_a - 1)\epsilon_1 + (i-1)\epsilon_2$$

- **residue at** $\Phi_l = m_a$ **vanishes**

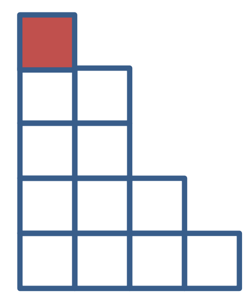
box at $(i, j) = (k_a + 1, 1)$

➡ **same recursion relation**

$$Z_{4d,full} = A\, Z_{2d,full}$$

$Y_a = (4, 3, 2, 2)$

$k_a = 4$ **(height)**



4. Summary

- **vortex partition function in general 2d** $\mathcal{N}=(2,2)$ **models**

- **integral representation** $Z_a = \int_{C_a} \frac{d\sigma}{2\pi i \epsilon} \exp\left(-\frac{2\pi i \sigma \tau}{\epsilon}\right) \prod_{b=1}^{N} \Gamma\left(\frac{\sigma + m_b}{\epsilon}\right)$

- **differential equation (recursion relation)** $\left[\prod_{b=1}^{N} \frac{\hat{\sigma} + m_b}{\epsilon} - e^{2\pi i \tau}\right] Z_a = 0$

- **relation between 2d vortex counting and 4d instanton counting**

$$Z_{4d}(\vec{a} = \vec{m} - \epsilon_1 \vec{k}, \tau) = A\, Z_{2d}(\vec{m}, \vec{k}, \tau)$$