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Quantum Entanglement
and
Holographic Spacetime

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Based on

(i) arXiv:1208.3469 [JHEP 1210 (2012) 193]

with Masahiro Nozaki (YITP, Kyoto)

and Shinsei Ryu (Illinois, Urbana–Champaign)

(ii) arXiv:1311.nnnn

with Ali Mollabashi (IPM, Iran), Masahiro Nozaki and Shinsei Ryu

Contents

① Introduction

String Theory ⇒ a unified theory of quantum gravity

It has been still difficult to compute quantum corrections in cosmological spacetimes like big bang, de-Sitter etc.

However, a generalization of AdS/CFT (or holography) may be able to resolve this problem:

``Quantum Gravity = Quantum Many-body Systems''

For this, we need to understand the basic mechanism of AdS/CFT. ⇒ A key concept is **quantum entanglement**.

What is the quantum entanglement ?

In quantum mechanics,
a physical state =a vector in Hilbert space.

Consider a spin of an electron, any state is described by a linear combination:

$$\left|\Psi\right\rangle = a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Consider the following states in **two spin systems**:

(i) A direct product state (unentangled state)

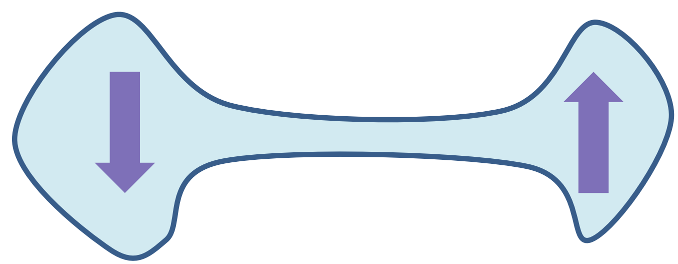
$$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right].$$

Independent

(ii) An entangled state (EPR pair)

$$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right] / \sqrt{2}.$$

One determines the other !



∃ Non-local correlation

A measure of quantum entanglement is known as the **entanglement entropy** defined as follows.

Divide a quantum system into two subsystems A and B.

$$H_{tot} = H_A \otimes H_B \quad .$$

Define the reduced density matrix ρ_A by

$$\rho_A = \mathrm{Tr}_B \left| \Psi \right\rangle \left\langle \Psi \right| \; .$$

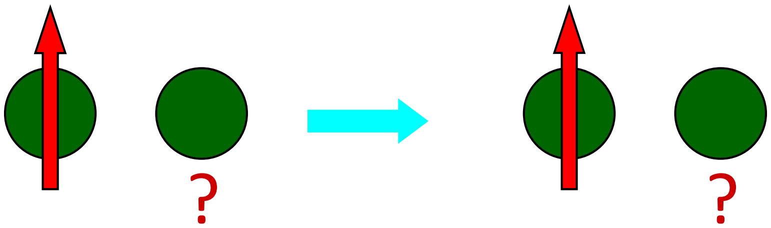
The entanglement entropy S_A is now defined by

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A \, . \qquad \text{(von-Neumann entropy)}$$

The Simplest Example: two spins (2 qubits)

(i) $$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\otimes\left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$$

$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right].$$



Not Entangled

$$\mathrm{S}_{\mathrm{A}} = 0$$

(ii) $$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right] /\sqrt{2}$$

$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right].$$

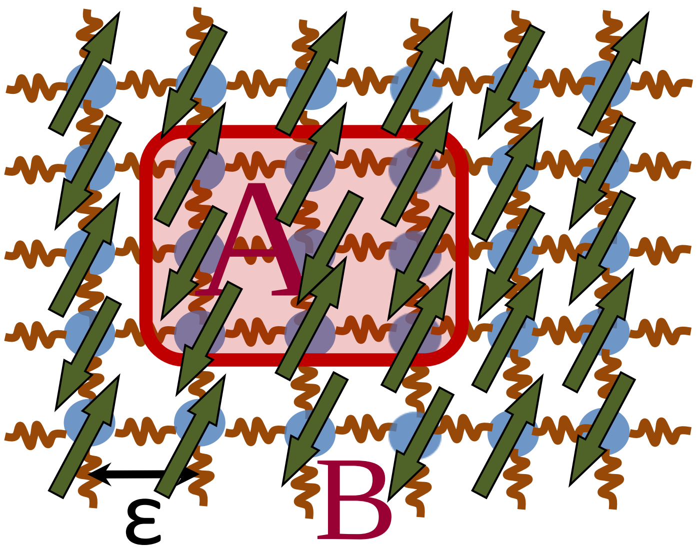
Entangled

$$\mathrm{S}_{\mathrm{A}} = \log 2$$

EE in Quantum Many-body Systems and QFTs

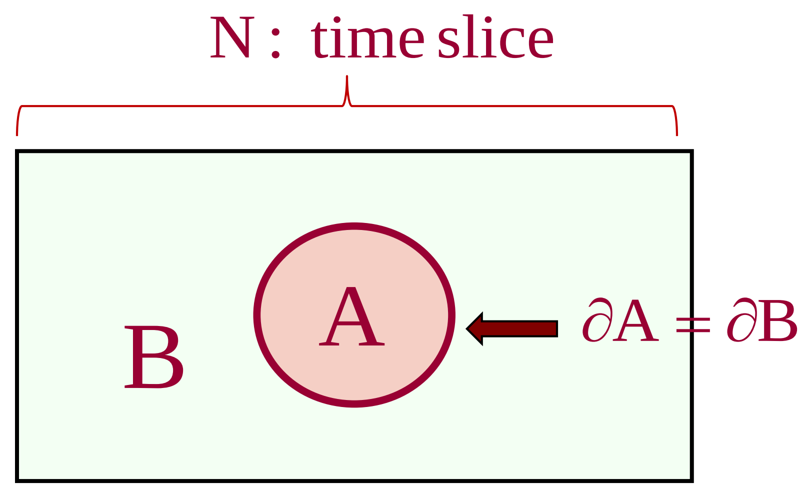
We can define the EE geometrically:

Quantum Many-body Systems



Continuum Limit ε→0

Quantum Field Theories (QFTs)



$$H_{tot} = H_A \otimes H_B \quad .$$

In this talk, we will explain a deep connection between **quantum entanglement** and **spacetime geometries in gravity**.

The Bekenstein-Hawking formula of black hole entropy:

Statistical Mechanics

$$S_{BH} = \frac{k_B \cdot c^3}{4 G_N \cdot \hbar} \cdot \mathrm{Area(Horizon)}.$$

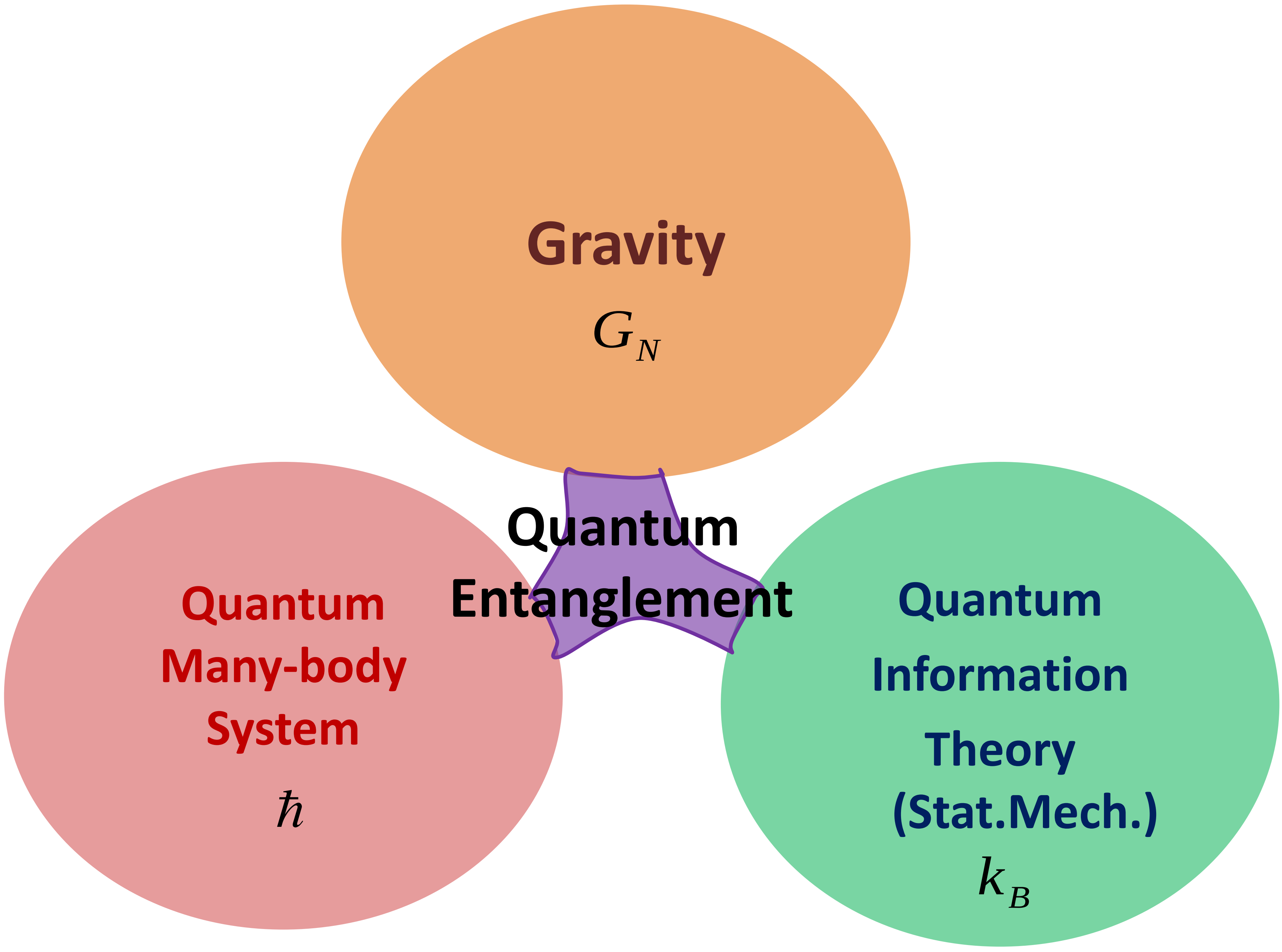
Gravity (General Relativity)

Quantum Mechanics

suggests a deep connection between

quantum entanglement and space time geometry .

[Entropy] **[Area]**


Gravity
G_N
Quantum Entanglement
Quantum Many-body System
$\hbar$
Quantum Information Theory (Stat.Mech.)
k_B

② Holographic Entanglement Entropy

(2-1) AdS/CFT (the best example of holography)

[Maldacena 97]

AdS/CFT

Quantum Gravity (String theory) on d+2 dim. AdS spacetime (anti de-Sitter space)	=	Conformal Field Theory (CFT) on d+1 dim. Minkowski spacetime
⬇ Classical limit		⬇ Large N limit Strong coupling limit
General relativity with Λ＜0 (Geometrical)		**Strongly interacting quantum many-body systems**

Basic Principle

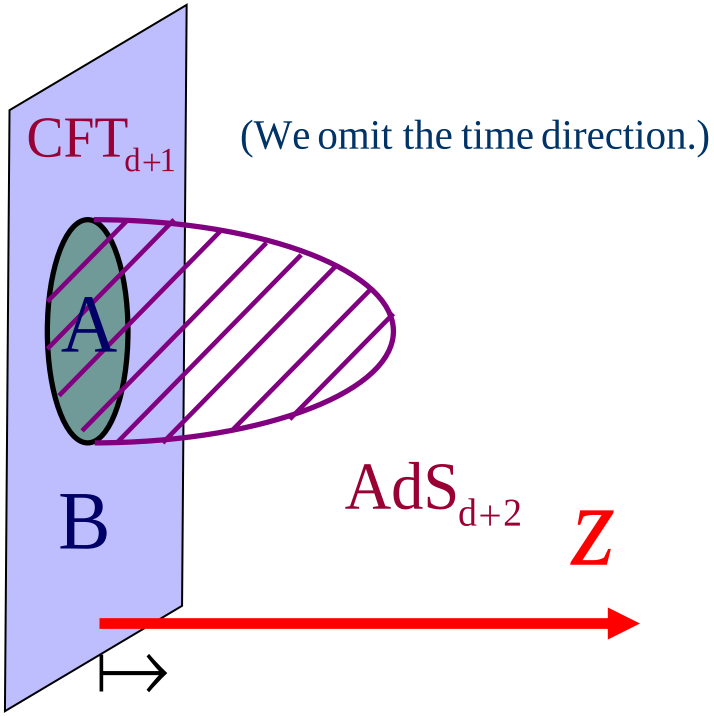
（Bulk-Boundary relation）:

$$Z_{Gravity} = Z_{CFT}$$

(2-2) Holographic Entanglement Entropy Formula

[Ryu-TT 06; Proved by Lewkowycz-Maldacena 13]

$$S_A = \underset{\substack{\partial\gamma_A = \partial A \\ \gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right]$$



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial\gamma_A \ \text{ and } \ A \sim \gamma_A \ .$$

homologous

$z > \varepsilon$ (UV cut off)

Note: In time-dependent spacetimes, we need to take extremal surfaces.

[Hubeny-Rangamani-TT 07]

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

Verification of HEE

- Confirmations of basic properties:

Area law, Strong subadditivity (SSA), Conformal anomaly,….

- Direct Derivation of HEE from AdS/CFT:

(i) Pure AdS, A = a round sphere [Casini-Huerta-Myers 11]

(ii) Euclidean AdS/CFT [Lewkowycz-Maldacena 13, Faulkner 13, cf. Fursaev 06]

(iii) Disjoint Subsystems [Headrick 10, Faulkner 13, Hartman 13]

(iv) General time-dependent AdS/CFT → Not yet.

[But, ∃ confirmations of SSA: Allais-Tonni 11, Callan-He-Headrick 12, Wall 13]

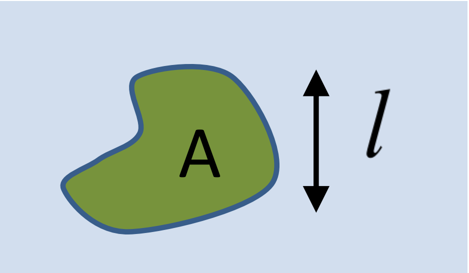
- Corrections to HEE beyond the supergravity limit:

[Higher derivatives: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11,…..]

[1/N effect: Barrella-Dong-Hartnoll-Martin 13,…]

[Higher spin gravity: de Boer-Jottar 13, Ammon-Castro-Iqbal 13]

General Behavior of HEE [Ryu-TT 06]



$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[p_1\left(\frac{l}{\varepsilon}\right)^{d-1} + p_3\left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots + \begin{cases} p_{d-1}\left(\frac{l}{\varepsilon}\right) + p_d & (\text{if } d+1=\text{odd}) \\ p_{d-2}\left(\frac{l}{\varepsilon}\right)^2 + q\log\left(\frac{l}{\varepsilon}\right) & (\text{if } d+1=\text{even}) \end{cases} \right],$$

Area law divergence

where $p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$

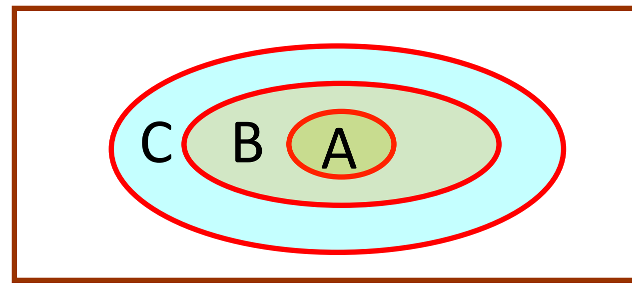
..... $q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!!$.

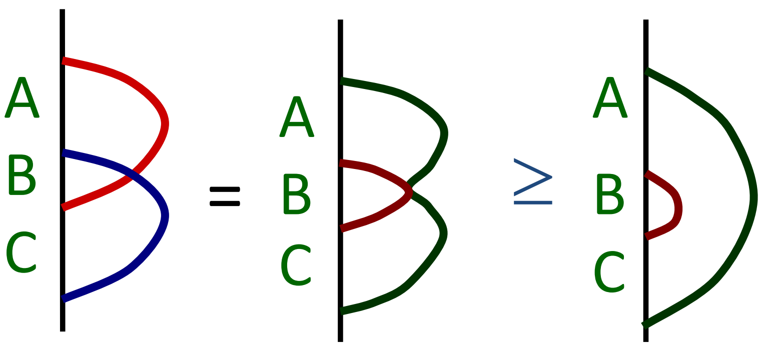
A universal quantity (**F**) which characterizes **odd dim. CFT**.
⇒A proof of c-theorem in 3 dim. (F-theorem). [Casini-Huerta 12, Liu-Mezei 12, Myers-Singh 12, …]

Agrees with **conformal anomaly** (central charge) in **even dim. CFT**
[Calabrese-Cardy 04, Solodukhin 08, Hung-Myers-Smolkin 11 …]

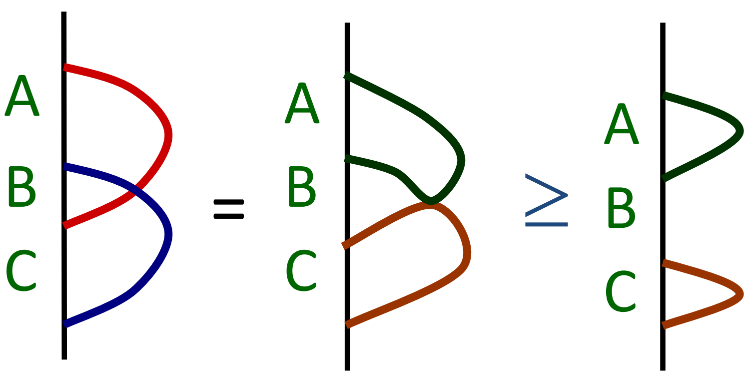
Holographic Strong Subadditivity [Headrick-TT 07]

We can easily derive the strong subadditivity, which is the most important inequality satisfied by EE. [Lieb-Ruskai 73]





$$\Rightarrow \; S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B$$



$$\Rightarrow \; S_{A+B} + S_{B+C} \geq S_A + S_C$$

Note: the HEE formula can be regarded as a generalization of Bekenstein-Hawking formula of black hole entropy:

$$S_{BH} = \frac{\text{Area of BH}}{4G_N}.$$

A Killing horizon (time independent black holes)

⇔ All components of extrinsic curvature are vanishing.

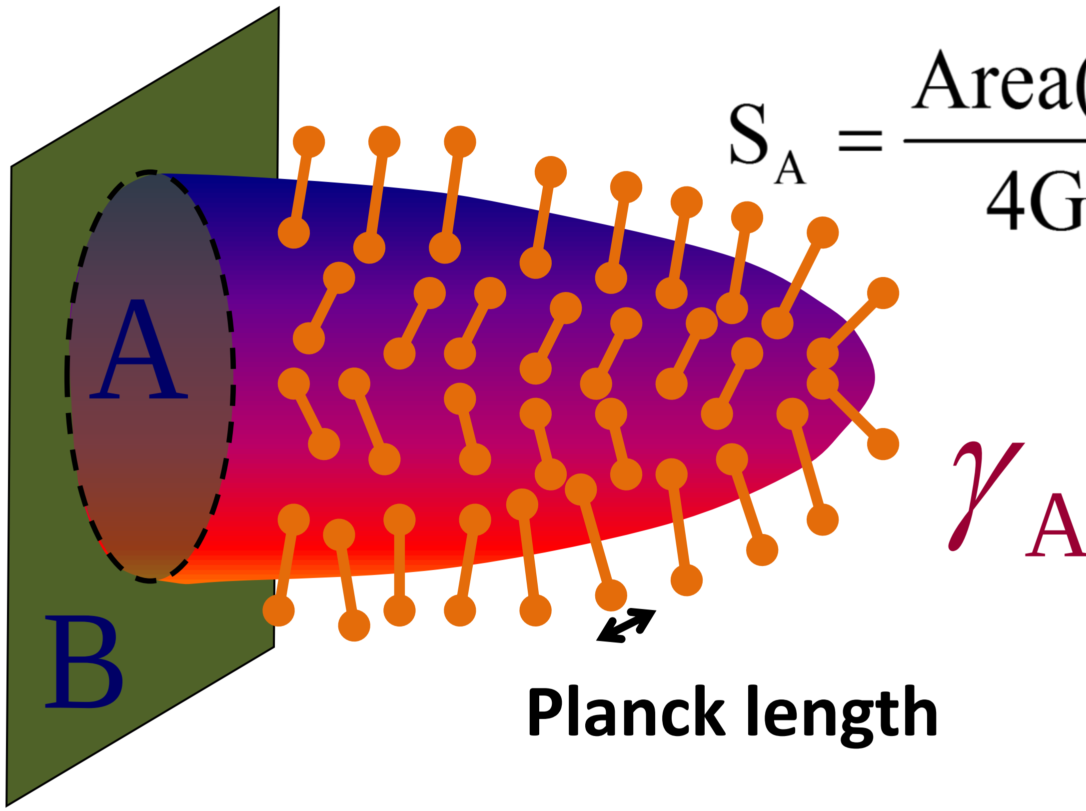
$$\cap$$

A minimal surface (or extremal surface)

⇔Traces of extrinsic curvature are vanishing.

The HEE suggests that

A spacetime in gravity
= Collections of bits of quantum entanglement



$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^2}.$$

A framework for this is the **entanglement renormalization**.

③ Entanglement Renormalization and AdS/CFT

(3-1) Tensor Network (TN) [See e.g. the review Cirac-Verstraete 09]

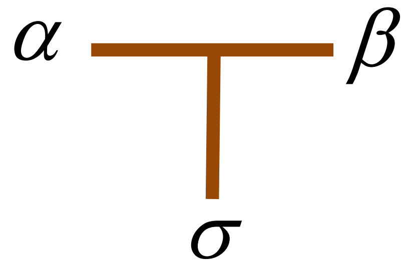
Recent remarkable progresses in numerical algorithms for quantum lattice models:

⇒ **Tensor network states**
= Efficient variational ansatz for the ground state wave functions in quantum many-body systems.

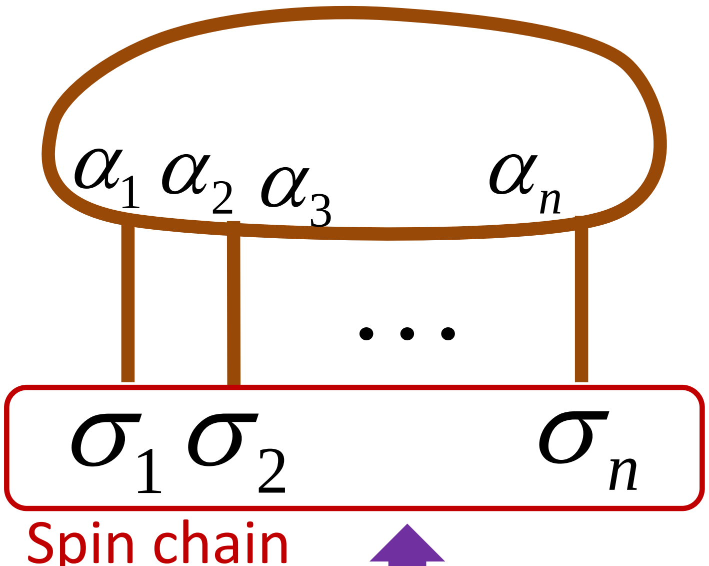
⇒ An ansatz is good if it respects the quantum entanglement of the true ground state.

Ex. Matrix Product State (MPS)

[DMRG: White 92,...,
Rommer-Ostlund 95,..]



$$\Longleftrightarrow \quad M_{\alpha\beta}(\sigma)$$



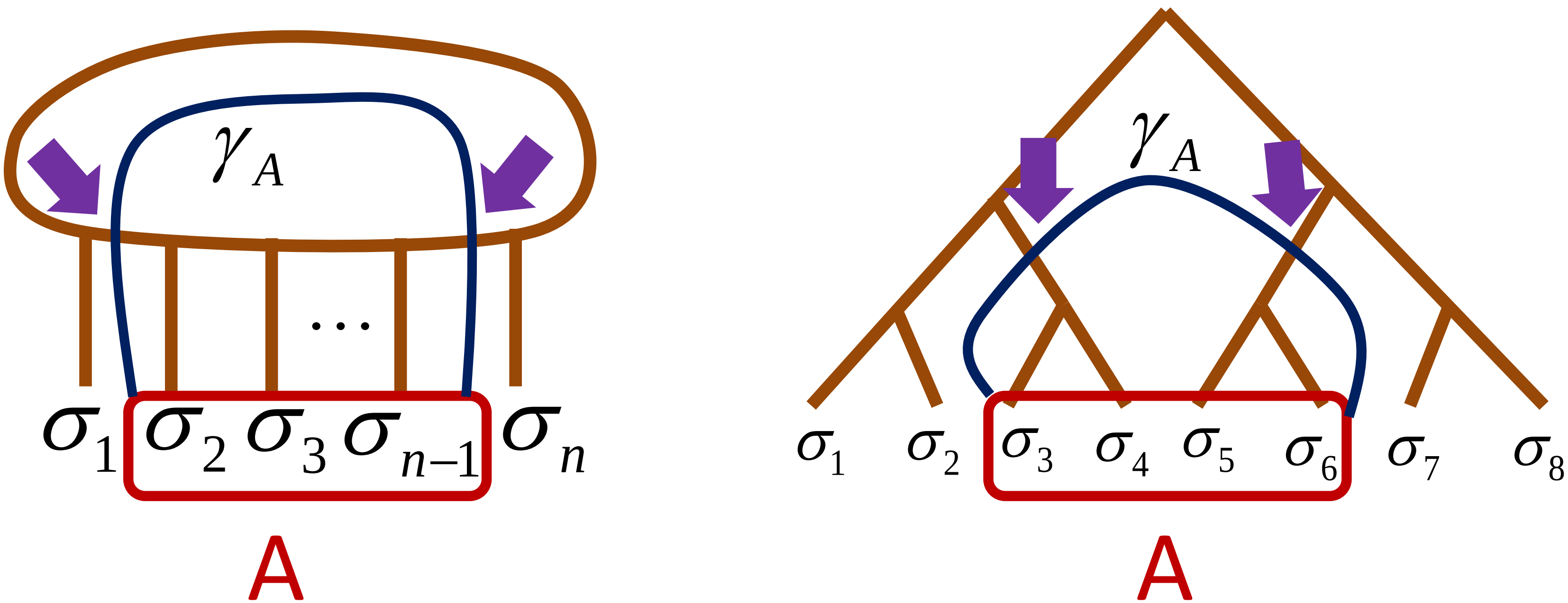
$$\alpha_i = 1, 2, \ldots, \chi,$$

$$\sigma_i = \uparrow \text{ or } \downarrow .$$

$$\left|\Psi\right\rangle = \sum_{\sigma_1,\sigma_2,\cdots,\sigma_n} \mathrm{Tr}[M(\sigma_1)M(\sigma_2)\cdots M(\sigma_n)]\underbrace{\left|\sigma_1,\sigma_2,\cdots,\sigma_n\right\rangle}_{n\ Spins}$$

MPS and TTN are not good near quantum critical points (CFTs) because entanglement entropies are too small:

$$S_A \leq 2\log\chi \quad (<< \log L \sim S_A^{CFT}).$$



In general,

$$S_A \sim N_{\mathrm{int}} \cdot \log\chi,$$
$$N_{\mathrm{int}} \equiv \min[\#\,\text{Intersections of } \gamma_A].$$

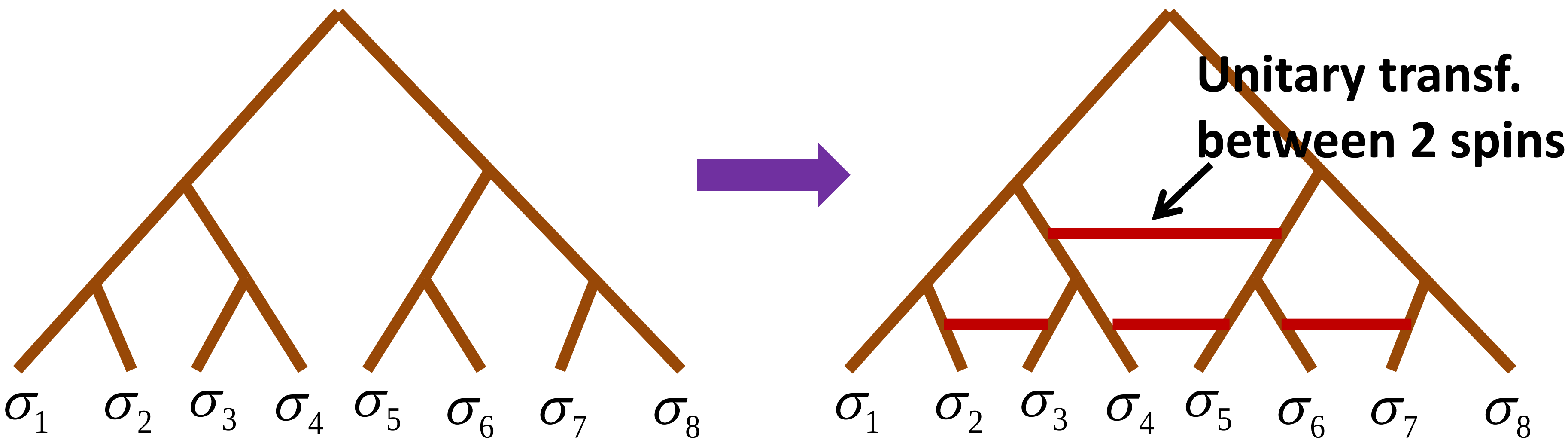
(3-2) AdS/CFT and (c)MERA

<u>MERA</u> (Multiscale Entanglement Renormalization Ansatz):

⇒ An efficient variational ansatz for CFT ground states.

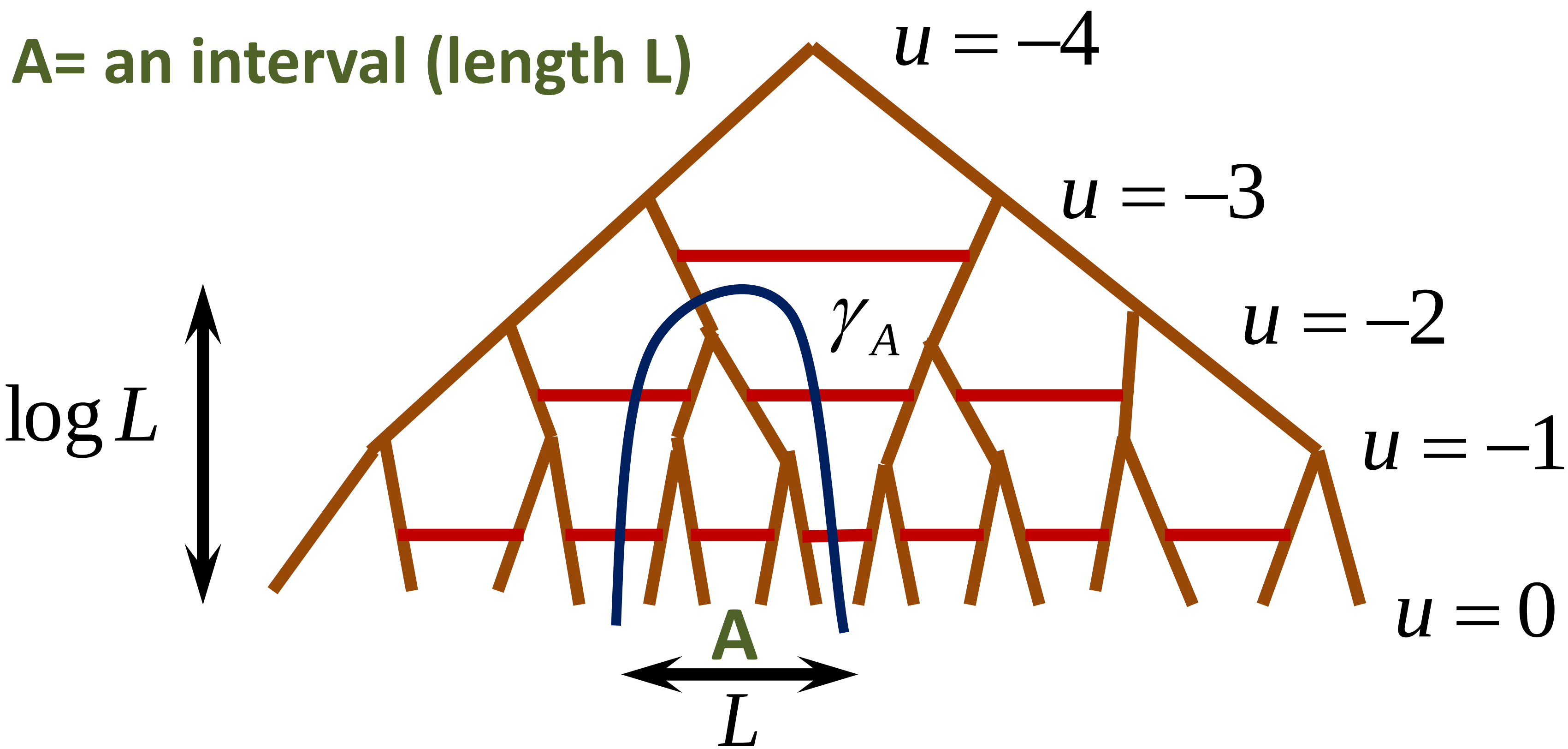
[Vidal 05 (for a review see 0912.1651)]

To increase entanglement in a CFT, we add (dis)entanglers.



Calculations of EE in 1+1 dim. MERA

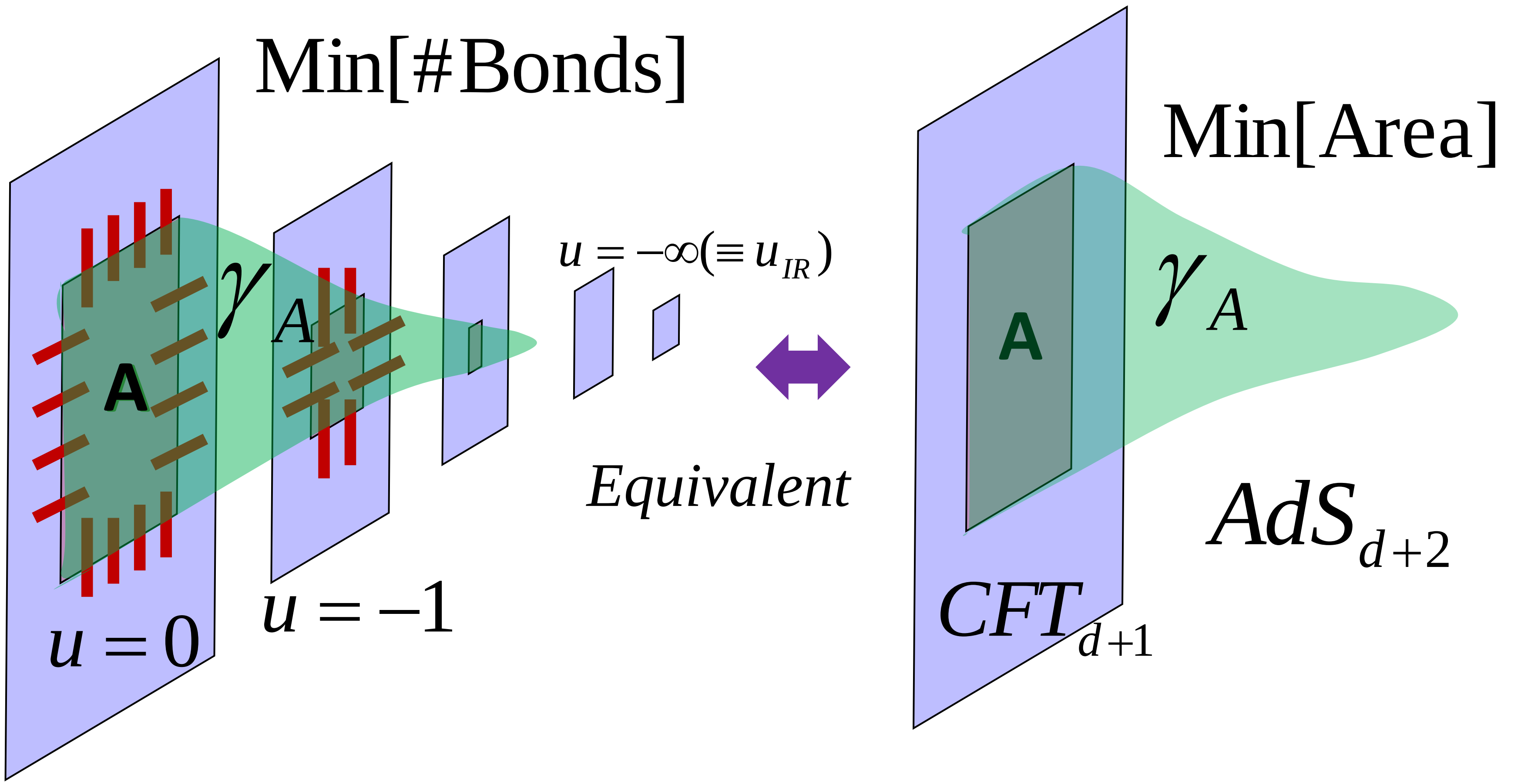
A= an interval (length L)



$$S_A \propto \mathrm{Min}[\#\mathrm{Bonds}] \propto \log L$$

$\Rightarrow$ agrees with 2d CFTs.

A conjectued relation to AdS/CFT [Swingle 09]



$$\text{Metric} = ds^2 + \frac{e^{2u}}{\varepsilon^2}(-dt^2 + d\vec{x}^2) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2},$$

$$where \quad z = \varepsilon \cdot e^{-u} \quad .$$

To make relate to AdS/CFT directly, we want to consider the MERA for quantum field theories.

Continuous MERA (cMERA) [Haegeman-Osborne-Verschelde-Verstraete 11]

$$\underbrace{|\Psi(u)\rangle}_{\substack{\text{True ground state}\\ \text{(highly entangled)}}} = P \cdot \exp\left(-i\int_{-\infty}^{u} ds[K(s)+L]\right) \cdot \underbrace{|\Omega\rangle}_{\substack{\text{IR state}\\ \text{(no entanglement)}}},$$

$$\Rightarrow \text{Real space renormalization flow}: \ \text{length scale} \sim \varepsilon \cdot e^{-u}.$$

K(u) : disentangler, L: scale transformation

Conjecture

$$d+1 \ \ \text{dim. cMERA} = \text{gravity on AdS}_{\text{d+2}} \qquad z = \varepsilon \cdot e^{-u}.$$

(3-3) Emergent Metric from cMERA [Nozaki-Ryu-TT 12]

We focus on gravity duals of translational invariant static states, which are not conformal in general.

Our conjecture: metric in the extra direction

$$g_{uu}du^2 = N \cdot \left(1 - \left| \langle \Psi(u) | e^{iLdu} | \Psi(u+du) \rangle \right|^2 \right).$$

$N^{-1} \equiv \int dx^d \cdot \int_0^{\Lambda e^u} dk^d =$ The total volume of phase space at energy scale u.

$(\Lambda = 1/\varepsilon : \text{cut off})$

Bures metric (Fisher information metric)

The **Bures distance** between two states is defined by

$$D(\psi_1,\psi_2) = 1 - \left|\langle \psi_1 | \psi_2 \rangle\right|^2.$$

When the state depends on the parameters {ξi},

Bures metric (Fisher information metric) is defined as

$$D\left[\psi(\xi),\psi(\xi + d\xi)\right] = g_{ij}\, d\xi^i d\xi^j.$$

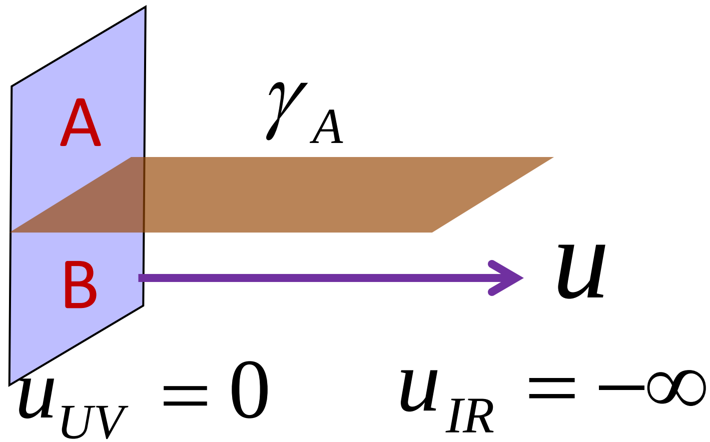
⇒ Reparameterization invariant (in our case: u→u')

The operation e^{iLdu} removes the coarse-graining procedure to extract the density of disentanglers.

⇒ Our metric guu = the density of disentanglers

= the metric guu in the gravity dual

$$S_A \sim \int_{u_{IR}}^{0} du \sqrt{g_{uu}} \cdot e^{(d-1)u}$$



(3-4) Emergent Metric in a (d+1) dim. Free Scalar Theory

Hamiltonian: $H = \frac{1}{2}\int dk^d [\pi(k)\pi(-k) + (k^2 + m^2)\phi(k)\phi(-k)].$

Ground state $|\Psi\rangle$ **:** $\quad a_k|\Psi\rangle = 0.$

The IR state $|\Omega\rangle$ and the disentanglers $\hat{K}(u)$ are defined :

$$a_x = \sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x), \quad (M \equiv \sqrt{\Lambda^2 + m^2}),$$

$$a_x|\Omega\rangle = 0, \quad |\Omega\rangle = \prod_x |0\rangle_x \quad \Rightarrow \quad S_{\mathrm{A}}(\Omega) = 0 \text{ for any A.}$$

$$\hat{K}(u) = \frac{i}{2}\int dk^d \left[\chi(u)\Gamma\left(ke^{-u}/\Lambda\right)a_k^+ a_{-k}^+ + (h.c.)\right], \quad \left(\Gamma(\mathrm{x}) \equiv \theta(1 - |\,\mathrm{x}\,|)\right).$$

$$\text{Metric}: \ g_{uu} = \chi(u)^2$$

$$ds^2_{Gravity} = g_{uu} du^2 + \frac{e^{2u}}{\varepsilon^2} \cdot d\vec{x}^2 - g_{tt} dt^2$$

(i) Massless scalar (E=k)

$$g_{uu} = \chi(u)^2 = \frac{1}{4} \qquad \Rightarrow \text{ the pure } AdS$$

(ii) Lifshitz scalar (E=k^ν)

$$g_{uu} = \frac{\nu^2}{4} \qquad \Rightarrow \text{ the Lifshitz geometry}$$

(iii) Massive scalar

$$g_{uu} = \frac{e^{4u}}{4(e^{2u} + m^2/\Lambda^2)^2}.$$

$$\Rightarrow ds^2 = \frac{dz^2}{z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right)(d\vec{x}^2 - dt^2).$$

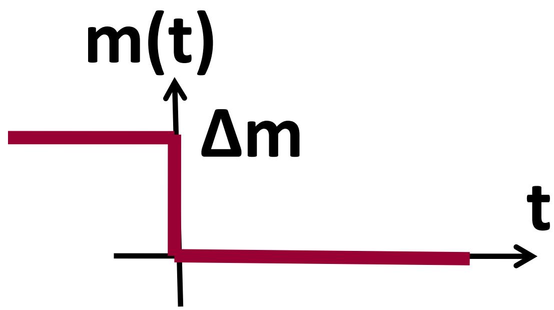
Capped off in the IR z<1/m

④ Finite Temperature CFT and AdS Black Holes

(4-1) Excited States in MERA

Quantum quench

= Excited state induced by a sudden change of Hamiltonian

e.g. Mass shift: 

⇒ an excited state in CFT

$$\left|\Psi(t=0)\right\rangle \approx e^{-\frac{\beta H}{4}} \cdot \left|B\right\rangle.$$

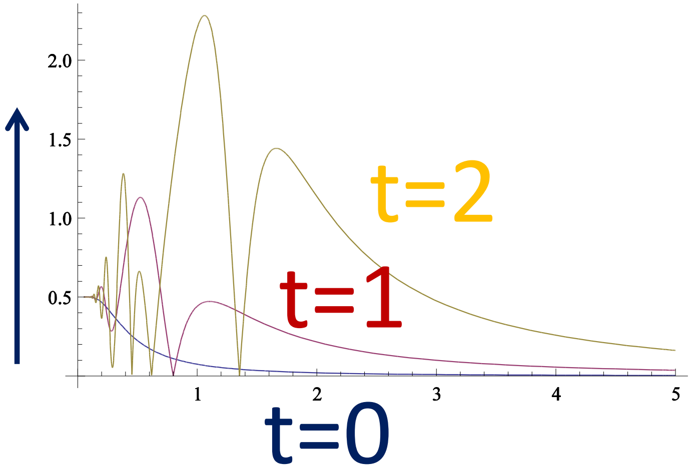
[Calabrese-Cardy 05,… Ho-Li-Lin-Ning 13, Ugajin 13]

Regularization factor
Δm~1/β (~ effective temp.)

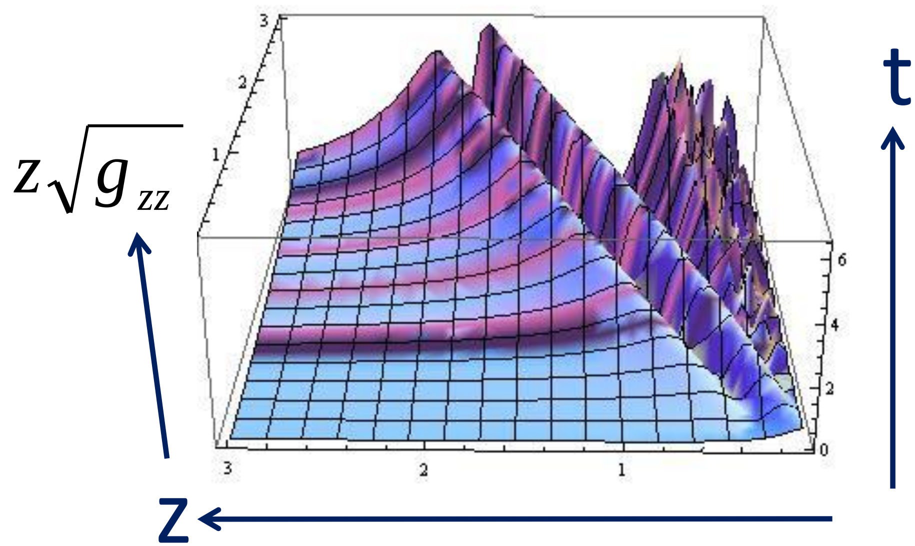
Boundary state

Metric from cMERA for 2d Quantum Quench

$$z\sqrt{g_{zz}} = g(u)$$



Light cone: looks like a gravitational wave.

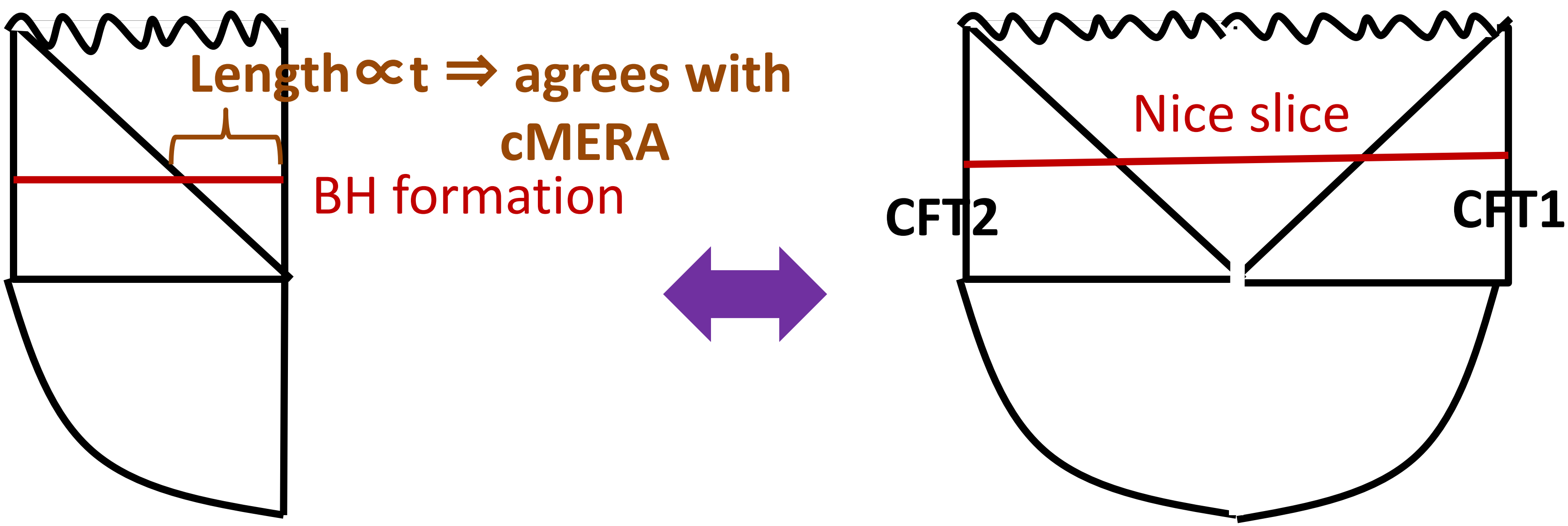


We can also (analytically) confirm the linear growth: SA∝t because g(u)∝t at late time. This is also true in higher dim.

This is consistent with the known CFT (2d) [Calabrese-Cardy 05]. and with the holographic result (any d). [Hartman-Maldacena 13]

Comparison with AdS BH

The gravity dual of a quantum quench was analytically constructed as a half of eternal AdS BH. [Hartman-Maldacena 13]



Gravity dual of quantum quench

Eternal AdS BH dual to finite temp. CFT

(4-2) cMERA for Finite Temperature CFT

Indeed, in our free scalar model, we find this relation:

$$\left|\Psi(t)\right\rangle_{QQ} = N \cdot \exp\left(-\frac{1}{2}\int dk e^{-\beta k/2} e^{-2ikt} a_k^+ a_{-k}^+ \right)\left|0\right\rangle.$$

Z2 projection: CFT1=CFT2

$$\left|\Psi(t)\right\rangle_{\text{Finite } T} = N' \cdot e^{-iHt} \cdot \sum_n e^{-\beta E_n/2} \left|n\right\rangle_1 \left|\tilde{n}\right\rangle_2$$

$$= N' \cdot \exp\left(-\int dk e^{-\beta k/2} e^{-2ikt} a_k^+ \tilde{a}_{-k}^+ \right)\left|0\right\rangle_1 \left|\tilde{0}\right\rangle_2.$$

$\Rightarrow$ A construction of cMERA for finite temp. CFT !

guu is the same as that for quantum quench.

⑤ Conclusions

- HEE suggests that spacetimes in gravity consist of bits of quantum entanglement.

  ⇒ A good framework for this is MERA.

  MERA can be a basic mechanism of the AdS/CFT.

- EE plays a role of quantum order parameter in any QFTs.

  (~ generalization of Wilson loop or central charge…)

- We proposed a holographic metric for cMERA and constructed cMERA both for QQs and finite T CFTs

  ⇒ This naturally describes the inside of black hole horizon.

  ∃ suggestion of ``charged BH horizon = Fermi surfaces'' .

Future Problems

- How to calculate gtt ? (Boosting the subsystem ?)

- Large N limit in cMERA ?
  (large N limit ⇔locality⇒saturation of entropy bound ?)

- Free field theories ⇒ Higher spin gauge theory?

- Einstein eq. from cMERA in the strong coupling limit ?
  [cf. Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharya-TT 13]

- Holography and MERA for more general spacetimes
  (e.g. A flat space shows a volume law
  ⇒ Non-local QFTs [Li-TT 11, Shiba-TT 13])