

# Constraints on renormalization group flows

Based on published and unpublished work with  
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# Plan

- 1) Setup: RG flows ,nature of fixed points,trace anomalies ,Cardy's conjecture .
- 2)The two type of flows. The proof of the a-theorem for spontaneous breaking of conformal symmetry .
- 3)General proof of the a-theorem .
- 4)The relation between scale invariant and conformal invariant fixed points in  $d=4$ .
- 5)Conclusions and open problems.

# 1)Setup

Following Wilson ,a QFT is a flow between two massless theories: the UV and IR fixed points.

The fixed points could be just scale invariant or have the full conformal symmetry i.e. the energy momentum tensor obeys the exact equations at the quantum level:

$$\partial^m T_{mn} = 0 \qquad T^m_m = 0$$

the beta function being zero.

The implications of the equations obeyed by the energy momentum tensor in a conformal theory can be conveniently treated by coupling the conformal theory to an external, background metric:

$$g_{mn} = \eta_{mn} + h_{mn}$$

The conservation and tracelessness correspond respectively to the invariance of the effective action under reparametrizations and Weyl transformations ,i.e.:

$$g_{mn} \rightarrow e^{2\sigma(x)} g_{mn}$$

The generating functional  $W$  is invariant under Weyl transformations except trace anomalies reflecting violations of tracelessness in certain 3-point functions of the energy momentum tensors:

$$\delta_\sigma W = c \int d^4x \sqrt{g} \sigma C_{mnpq} C^{mnpq} - a \int d^4x \sqrt{g} \sigma E_4$$

The anomaly coefficients  $a$  and  $c$  characterize the Conformal Field Theory.

Following Zamolodchikov's theorem in  $d=2$  Cardy (1988) proposed that in  $d=4$  for any unitary theory flowing between two conformal

fixed points :

$$a_{IR} < a_{UV}$$

the “a-theorem”.

This implies the irreversibility of the flow .  
“a” measures the degrees of freedom with very special weights per degree of freedom:  
scalar=1 , massless fermion=11/4 ,  
gauge boson= 31 . For example in QCD the inequality is easily satisfied for a flow from free quarks and gluons in the UV to “pions” in the IR. In SUSY theories the inequality can be verified using the Intriligator-Wecht “a-maximization” procedure.

## 2) Two type of flows. The proof of the a-theorem for SBC

The massive flow between the fixed points can occur either by perturbing with a relevant operator or through a spontaneous breaking of conformal symmetry . In the second situation a dimensionful scalar operator gets a vacuum expectation value:

$$\langle 0|\mathcal{O}|0\rangle = v^\Delta$$

This phase has massless and massive states and due to Goldstone's theorem there is a massless

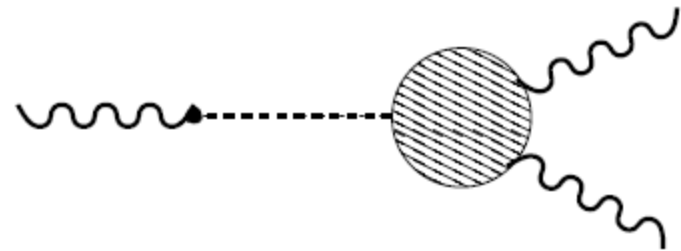
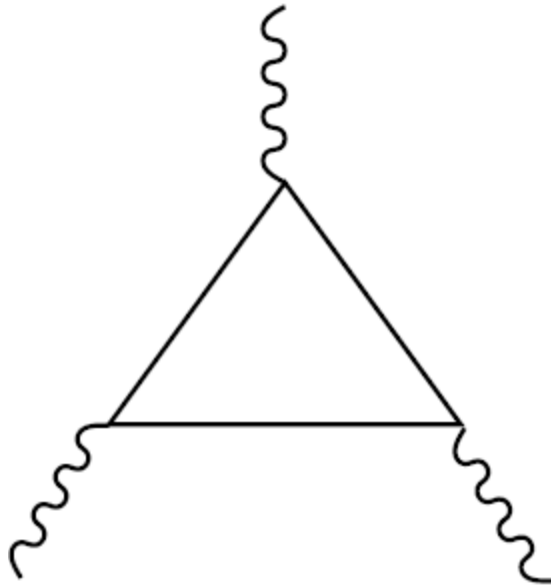
dilaton with a linear coupling to the energy momentum tensor :

$$\langle 0|T_{mn}|\tau; q\rangle = \frac{1}{3}f q_m q_n$$

In a SB conformal theory there is a matching of trace anomalies in the UV and IR analogous to the 't Hooft anomaly matching of chiral anomalies: the anomaly in the UV is reproduced by the anomaly in the IR and the dilaton with “strength” :

$$\Delta a \equiv a_{UV} - a_{IR}$$





The states circulating in the loop contribute the values of “a” in the UV and IR :the difference should be matched by the dilaton contribution fixing the dilaton couplings to two energy momentum tensors .

The transformation of the dilaton under the Weyl group :

$$g_{\mu\nu} \longrightarrow e^{2\sigma} g_{\mu\nu} , \quad \tau \longrightarrow \tau + \sigma .$$

and the anomaly fix a universal piece of the of the dilaton action , analogous to the WZW part of the chiral Goldstone boson lagrangian:

$$\Delta a \int d^4x \sqrt{-g} [\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \square\tau + 2(\partial\tau)^4] + \dots$$

The last term contributes to dilaton-dilaton scattering

### 3) The general proof of the a-theorem

The dilaton argument can be generalized to include any massive flow including perturbations by relevant operators. The coupling of the dilaton to the trace of the energy momentum tensor is linear and the general statement is that the trace obeys an exact sum rule :

$$\int_0^\infty ds \frac{\text{Im}S(s, M)}{s^3} = a_{UV} - a_{IR}$$

where

$$S(s, M) = \langle T[T_i^i(k_1)T_j^j(k_2)T_l^l(-k_1)T_m^m(-k_2)] \rangle$$

and we are in “forward kinematics” with

$$s = (k_1 + k_2)^2 \quad \text{and} \quad k_1^2 = k_2^2 = 0$$

$M =$  RG invariant mass scale

The sum rule is convergent (using the conformal invariant nature of the UV and IR) in both limits and proves the a-theorem generally due to the positivity of the imaginary part .

## 4) The relation between scale (SFT) and conformal invariant fixed points.

Poincare invariant flows can in principle involve fixed points which have as symmetry group just

$ISO(d-1, 1) \rtimes \mathbb{R}^+$  i.e. Poincare + dilations.

The end points of all known unitary flows have fixed points with the full conformal symmetry

$SO(d, 2)$ . The existence of unitary SFT would lead to new phases / theories in four dimensions with nonvanishing beta-functions at the end points.

In two dimensions Polchinski(1988) has shown that all unitary SFT are automatically CFT .

In four dimensions we have strong arguments for the same happening but there are still some loopholes .

Starting with a generic SFT one knows (Wess 1960) that the trace of the energy momentum obeys a local equation :

$$T_{\mu}^{\mu} = \partial^{\nu} V_{\nu}$$

where  $V^{\mu}$  is local , the “virial current” . If the virial current is a gradient i.e.

$$T_{\mu}^{\mu} = \square L$$

then the e.m. tensor can be “improved” such that the new one has vanishing trace i.e. the SFT is “secretely” conformal .

Therefore the SFT is a CFT if the trace of the e.m. tensor carries an explicit  $p^2$  factor.

One uses the structure of a universal counterterm when the SFT is put in a metric background (related to yet another Weyl anomaly):

$$\log \Lambda \int d^4x \sqrt{-g} R^2$$

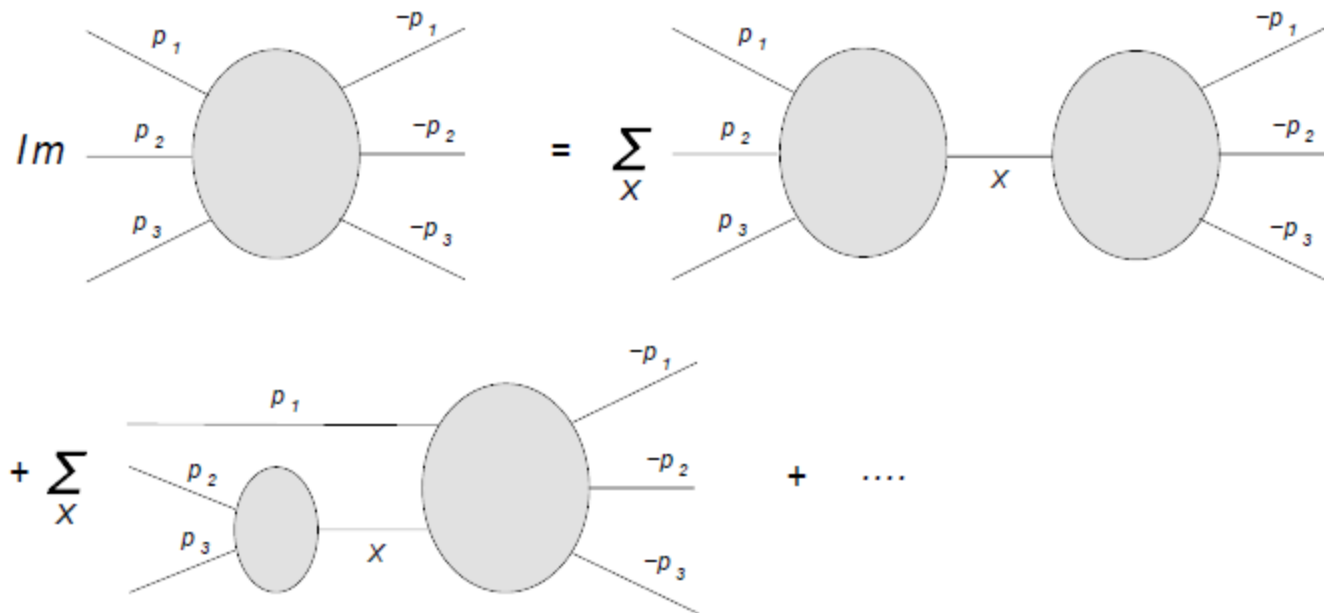
Coupling to a metric  $g_{\mu\nu} = (1 + \Psi)^2 \eta_{\mu\nu}$

which singles out the traces one gets :

$$\log \Lambda \int d^4 x \left\{ \frac{\square \Psi}{1 + \Psi} \right\}^2$$

which shows that the correlators of traces coupled to the sources  $\Psi$  vanish on shell. This, using the scale invariance of the theory implies the vanishing of the imaginary parts in the forward kinematics when the operators are on shell. From unitarity one gets a chain of discontinuity equations. For example the six point amplitudes discontinuity requires the vanishing of the left diagram:





Starting with the four point function one uses these relations inductively to show that :

$$\langle 0 | T(p_1)T(p_2)\dots T(p_n) | \text{any state} \rangle = 0 \quad n \geq 2 \quad \text{all } p_i^2 = 0$$

(the relation for  $n = 2$  was proven by Luty, Polchinski, Rattazzi(2012))

The complete proof would require the operatorial vanishing of the trace on shell, i.e. the relation holding also for  $n = 1$ . This is not happening in special free SFT which are “projected” through a higher gauge invariance. There are indications that interactive theories avoid this loophole and therefore unitary SFT are CFT in four dimensions.

# 5) Conclusions and open problems

a) The a-theorem is valid in four dimensions

b) Scale invariant unitary theories in four dimensions are also conformal invariant (with the exception of free theories having higher gauge invariances)

c) Monotonicity of the flow in four dimensions?

d) The rigorous elimination of the exceptions in b)

e) Limitations on a (Hoffmann-Maldacena-Zhiboedov) giving additional constraints.

f) Constraints in odd dimensions not relying on local anomalies...