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EXTENDED CONFORMAL SYMMETRY AND RECURSION FORMULAE FOR NEKRASOV PARTITION FUNCTION

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arXiv:1306.1523, 1207.5658, 1110.5255

INTRODUCTION

Brief History of 4D Instanton physics and 2D integrable models

Seiberg-Witten (1994)
Solving Prepotential of N=2 gauge theory by Riemann surface

Donagi-Witten, Gorsky et. al. Itoyama-Morozov, Nakatsu-Takasaki, D'Hoker-Phong (1995~)
Connection with integrable model/hierarchy

Nekrasov, Nekrasov-Okounkov, Nakajima-Yoshioka (2003-4)
Derivation of instanton partition function
Contribution from fixed point labeled by Young diagrams
Structure of 2D CFT in the cohomology of moduli space

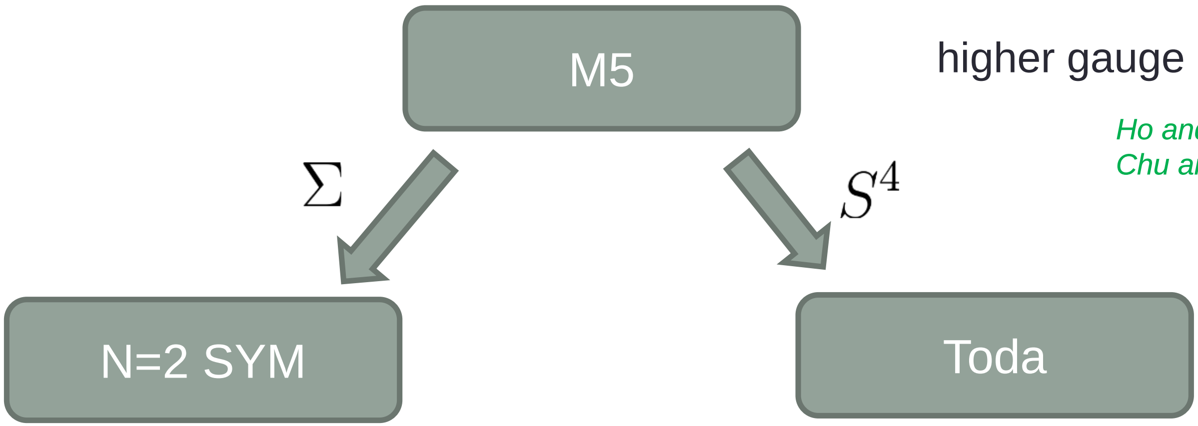
Nekrasov-Shatashvili (2009)
Discovery of a limit (NS limit) where integrablity shows up

Alday-Gaiotto-Tachikawa (2009)
Direct connection between instanton partition function and conformal block function of 2D CFT

AGT conjecture

Alday-Gaiotto-Tachikawa (2009)

Compactification of M5 brane



higher gauge symmetry?

Ho and Matsuo 2011
Chu and Ko, 2012

Nekrasov partition function = Correlation function

$$Z^{\mathrm{Nek}} = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{\vec{Y}} \qquad \langle V_\Delta(z_1) \cdots V_\Delta(z_L) \rangle$$

M-theory picture: Geometrical explanation of duality
Geometrical Langrands by Witten

Origin of Duality: A Naive picture

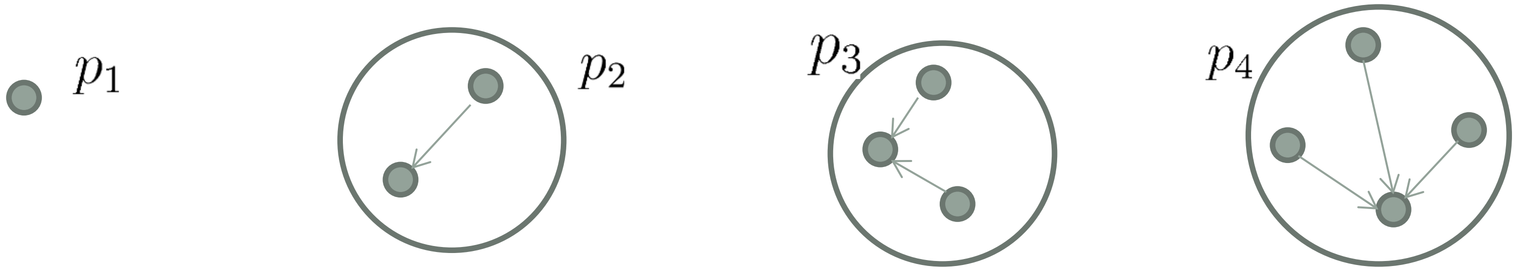
H. Nakajima
Short review: Morozov and Smirnov, arXiv:1307.2576

Configuration space of *k* instantons

$$\mathrm{Hilb}_k = (\mathbf{C}^2)^k / S_k$$

S_k: permutation group : Orbifold singularities

Overlap of some instantons gives rise to extra cycle by blow-up



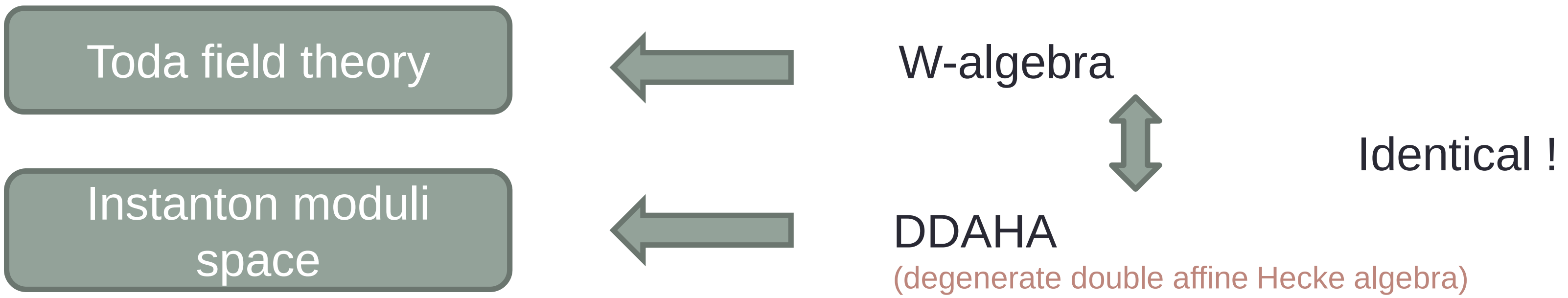
2(n-1)-cycle (p_n) appears when we blow up the singularity with n instantons.
Cohomology of Fock space is similar to free boson Fock space by assigning

$$a_{-n} \leftrightarrow p_n, \quad a_n \leftrightarrow n\partial_{p_n}$$

Free boson Fock space is parametrized by Young diagrams Y: origin of summation

More precise correspondence

Actually the symmetry behind the scene is more complicated,



It is recently recognized by mathematicians, the origin of the AGT correspondence is due to the coincidence of these two nonlinear symmetries.

Schiffmann and Vassero('12), Maulik and Okounkov ('13)

In this talk, we give an explicit realization of such correspondence in the form of recursion formulae for Nekrasov partition function

An incomplete list of references on the proof of AGT relation

Proof for pure super Yang-Mills

Schiffmann and Vassero('12), Maulik and Okounkov ('13)

Proof for N=2*

Fateev-Litvinov ('09)

Proof for quiver gauge theories

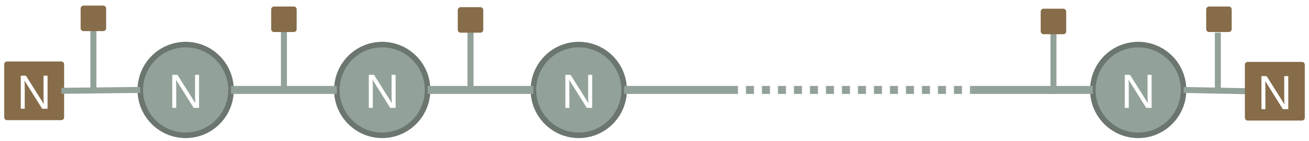
Alba-Fateev-Litvinov-Tarnopolskiy ('10)
Belavin-Belavin ('11)
Fateev-Litvinov ('11)
Morosov-Mironov-Shakirov ('11)
Zhang-M ('11), Kanno-M-Zhang ('12, '13)
Morosov-Smirnov ('13)

RECURSION FORMULA FOR NEKRASOV PARTITION FUNCTION

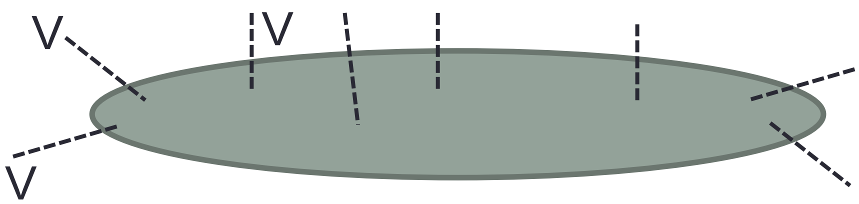
AGT conjecture for quiver

Alday-Gaiotto-Tachikawa, Wyllard ,Mironov-Morozov

Nekrasov Partition function for conformal inv. SU(N)x…xSU(N) quiver



is identical to conformal block function of 2D CFT described by SU(N) Toda field theory with L+3 vertex operator insertions



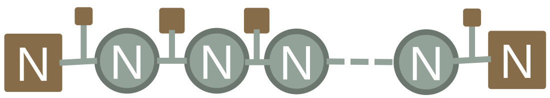
with identification of parameters

$$q_i \text{ : coupling const} \leftrightarrow \text{location of vertex}$$

$$\mu_i \text{ : mass for hyper} \leftrightarrow \text{momentum of vertex}$$

$$\vec{a}_i \text{ : VEV for vect. mult.} \leftrightarrow \text{weight of intermediate state}$$

Partition function takes the form of matrix multiplication



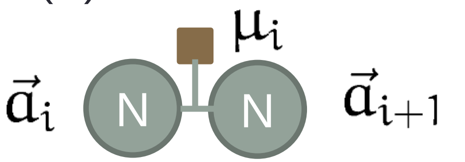
Partition function looks like matrix multiplication

$$Z^{\text{inst}} = \sum_{\vec{Y}_1,\cdots,\vec{Y}_\ell} \left(\prod_i q_i^{|\vec{Y}_i|}\right) \bar{V}_{\vec{Y}_1} Z_{\vec{Y}_1\vec{Y}_2} \cdots Z_{\vec{Y}_{\ell-1}\vec{Y}_\ell} V_{\vec{Y}_\ell}$$

$$\vec{Y} = (Y_1, \cdots, Y_N)$$

Set of Young diagrams which labels
the fixed points of localizations for SU(N)

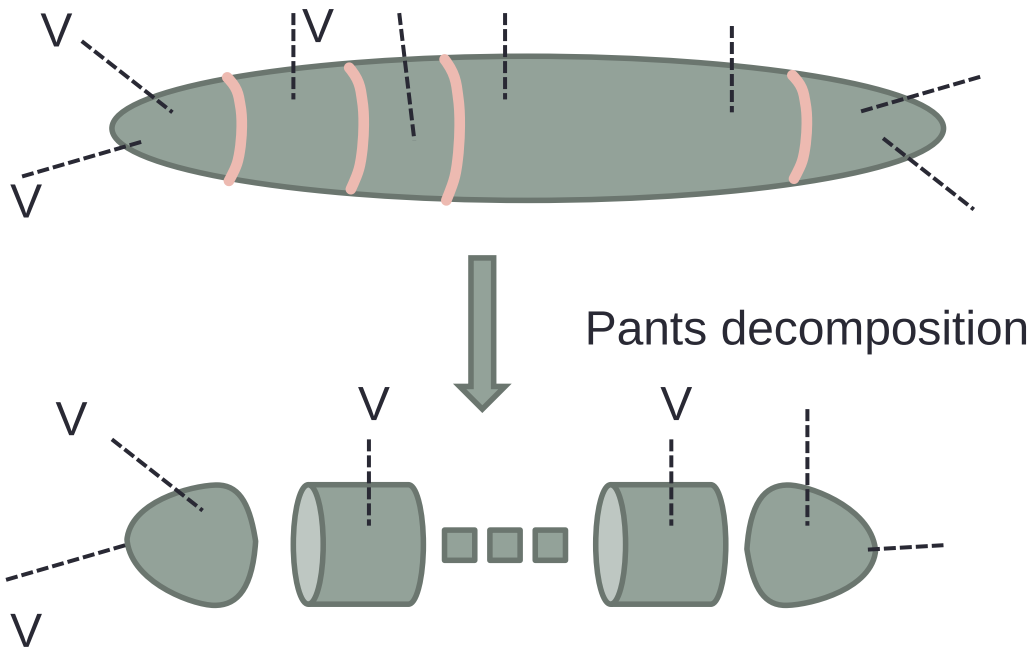
$$Z_{\vec{Y},\vec{W}} = Z(\vec{a},\vec{Y};\vec{a},\vec{W};\mu) = \frac{\prod_{p,q} g_{Y_p,W_q}(a_p - b_q - \mu)}{\prod_{p,q} g_{Y_pY_q}(a_p - a_q) g_{W_p,W_q}(b_p - b_q)}$$

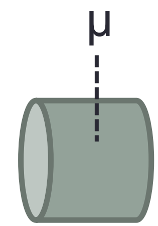


$$g_{Y,W}(x) = \prod_{(i,j)\in Y} (x + \beta(Y'_j - i + 1) + W_i - j) \prod_{(i,j)\in W} (-x + \beta(W'_j - i) + Y_i - j + 1)$$

Product of factors associated with each box of the Young diagram

Correlation function of Toda field theory





At the hole, we insert decomposition of "1"

$$1 = \sum_{\vec{Y}} |\vec{a}, \vec{Y}\rangle\langle\vec{a}, \vec{Y}|$$

$|\vec{a}, \vec{Y}\rangle$: orthonormal basis

Alba et. al. 2010

The same summation shows up if we identify

$$Z_{\vec{Y},\vec{W}} = \langle\vec{a}, \vec{Y}|V_{\kappa}|\vec{b}, \vec{W}\rangle$$

For an appropriate choice of $|\vec{a}, \vec{Y}\rangle$. This is the key step to prove AGT.

Recursion relation for $Z_{\vec{Y},\vec{W}}$

In the following, we claim a recursion formulae in the following form,

$$\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0$$

$\delta_{\pm 1,n}$ is defined by adding/subtracting a box in Y and W.
$U_{\pm 1,n}$ is a polynomial of $\vec{a}, \vec{Y}$.

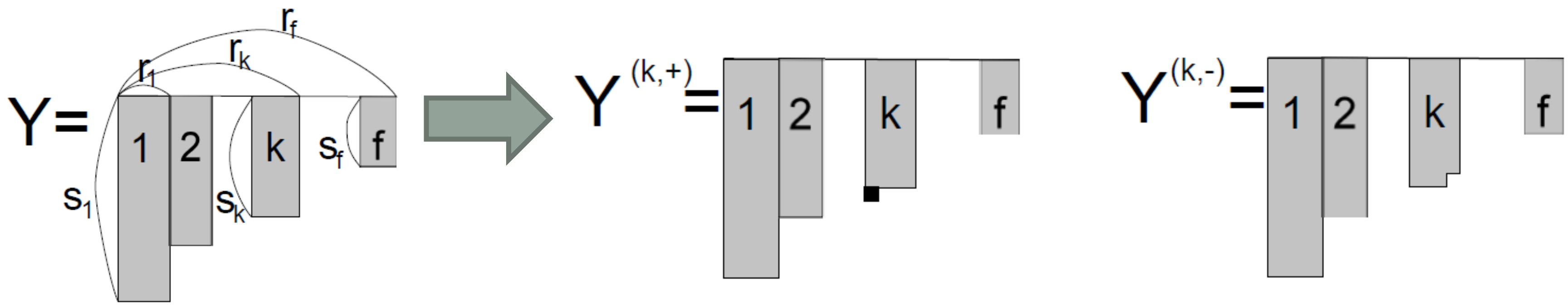
The structure constant which appears in $\delta_{\pm 1,n}$ is mostly identical to the action of DDAHA on the basis $|\vec{a}, \vec{Y}\rangle$.

It illustrates the nature of the basis in the representation theory.

Equivalence between DDAHA and W-algebra implies that above relation can be identified with the conformal Ward identities.

The proof is based on variation formulae

We evaluate the ratio of Nekrasov formula when we add/remove a box in Y and W.



For example:

$$
\begin{aligned}
\frac{g_{Y_p^{(k,+)}W_q}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} &= \frac{\prod_{\ell=1}^{\tilde f_q+1}(a_p-b_q-\mu+A_k(Y_p)-A_\ell(W_q)-\xi)}{\prod_{\ell=1}^{\tilde f_q}(a_p-b_q-\mu+A_k(Y_p)-B_\ell(W_q))}\,,\\
\frac{g_{Y_p^{(k,-)}W_q}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} &= \frac{\prod_{\ell=1}^{\tilde f_q}(a_p-b_q-\mu+B_k(Y_p)-B_\ell(W_q))}{\prod_{\ell=1}^{\tilde f_q+1}(a_p-b_q-\mu+B_k(Y_p)-A_\ell(W_q)-\xi)}\,,\\
\frac{g_{Y_pW_q^{(\ell,+)}}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} &= \frac{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+A_\ell(W_q)-A_k(Y_p))}{\prod_{k=1}^{f_p}(b_q-a_p+\mu+A_\ell(W_q)-B_k(Y_p)+\xi)}\,,\\
\frac{g_{Y_pW_q^{(\ell,-)}}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} &= \frac{\prod_{k=1}^{f_p}(b_q-a_p+\mu+B_\ell(W_q)-B_k(Y_p)+\xi)}{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+B_\ell(W_q)-A_k(Y_p))}\,.
\end{aligned}
$$

which looks very complicated..

with

$$
\begin{aligned}
A_k(Y) &= \beta r_{k-1}-s_k-\xi, \quad (k=1,\cdots,f+1)\,,\\
B_k(Y) &= \beta r_k - s_k, \quad (k=1,\cdots,f)\,,
\end{aligned}
$$

The precise form of variation:

$$\delta_{-1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N} \left(\sum_{k=1}^{f_p+1} (a_p+\nu+A_k(Y_p))^n \Lambda_p^{(k,+)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,+),p};\vec{b},\vec{W};\mu) \right.$$
$$\left. - \sum_{k=1}^{\tilde{f}_p} (b_p+\mu+\nu+B_k(W_p))^n \Lambda_p^{(k,-)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,-),p};\mu) \right) ,$$

$$\delta_{1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N} \left(- \sum_{k=1}^{f_p} (a_p+\nu+B_k(Y_p))^n \Lambda_p^{(k,-)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,-),p};\vec{b},\vec{W};\mu) \right.$$
$$\left. + \sum_{k=1}^{\tilde{f}_p} (b_p+\nu+\mu+A_k(W_p)+\xi)^n \Lambda_p^{(k,+)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,+),p};\mu) \right) ,$$

with

$$\Lambda_p^{(k,+)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q} \frac{a_p-a_q+A_k(Y_p)-B_\ell(Y_q)+\xi}{a_p-a_q+A_k(Y_p)-B_\ell(Y_q)} {\prod}'^{f_q+1}_{\ell=1} \frac{a_p-a_q+A_k(Y_p)-A_\ell(Y_q)-\xi}{a_p-a_q+A_k(Y_p)-A_\ell(Y_q)} \right) \right)^{1/2}$$

$$\Lambda_p^{(k,-)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q+1} \frac{a_p-a_q+B_k(Y_p)-A_\ell(Y_q)-\xi}{a_p-a_q+B_k(p)-A_\ell(q)} {\prod}'^{f_q}_{\ell=1} \frac{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)+\xi}{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)} \right) \right)^{1/2}$$

With some algebra, we arrive at the recursion formulae.

W-ALGEBRA AND DDAHA

Symmetry of Toda field theory

W-algebra

Fateev-Zamolodchikov '87

$$T(Z)T(Z') = \frac{c}{2(Z-Z')^4} + \frac{2}{(Z-Z')^2}T(Z') + \frac{1}{Z-Z'}T'(Z')$$

$$+ \left[\tfrac{3}{10}T''(Z') + \Lambda(Z')\right] + O((Z-Z')),$$

$$T(Z)W(Z') = \frac{3}{(Z-Z')^2}W(Z') + \frac{1}{Z-Z'}W'(Z') + O(1),$$

$$W(Z)W(Z') = \frac{c}{3(Z-Z')^6} + \frac{2}{(Z-Z')^4}T(Z') + \frac{1}{(Z-Z')^3}T'(Z')$$

$$+ \frac{1}{(Z-Z')^2}\left[\tfrac{3}{10}T''(Z') + 2b^2\Lambda(Z')\right]$$

$$+ \frac{1}{Z-Z'}\left[\tfrac{1}{15}T'''(Z') + b^2\Lambda'(Z')\right] + \text{reg.},$$

Famous but complicated nonlinear algebra
Characterization of orthonormal basis is very difficult

Symmetry of Instanton moduli space

DDAHA = Degenerate Double Affine Hecke Algebra

Schiffmann –Vasserot '12
Maulik-Okounkov '13

Generators: $D_{r,s}, \quad r \in \mathbf{Z},\, s \in \mathbf{Z}_{\geq 0}$

$$
\begin{aligned}
[D_{0,l}, D_{1,k}] &= D_{1,l+k-1}, \quad l \geq 1\,, \\
[D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, \quad l \geq 1\,, \\
[D_{-1,k}, D_{1,l}] &= E_{k+l} \quad l,k \geq 1\,, \\
[D_{0,l}, D_{0,k}] &= 0\,, \;\; k,l \geq 0\,,
\end{aligned}
$$

E_k : nonlinear combination of $D_{0,l}$

DAHA was introduced by Cherednik to analyze the properties of Macdonald polynomial. DDAHA is closely related to Calogero-Sutherland.

$$\mathrm{H}_{\mathrm{CS}} \sim \mathrm{D}_{0,2}$$

Representation by orthonormal basis

We introduce an orthonormal basis $|\vec{b},\vec{W}\rangle$ for the representation of DDAHA

$$
\begin{aligned}
D_{-1,l}|\vec{b},\vec{W}> &= (-1)^l \sum_{q=1}^{N}\sum_{t=1}^{\tilde{f}_q}(b_q + B_t(W_q))^l \Lambda_q^{(t,-)}(\vec{W})|\vec{b},\vec{W}^{(t,-),q}>\,,\\
D_{1,l}|\vec{b},\vec{W}> &= (-1)^l \sum_{q=1}^{N}\sum_{t=1}^{\tilde{f}_q+1}(b_q + A_t(W_q))^l \Lambda_q^{(t,+)}(\vec{W})|\vec{b},\vec{W}^{(t,+),q}>\,,\\
D_{0,l+1}|\vec{b},\vec{W}> &= (-1)^l \sum_{q=1}^{N}\sum_{\mu\in W_q}(b_q + c(\mu))^l|\vec{b},\vec{W}>\,,
\end{aligned}
$$

$$
c(\mu) = \beta i - j \text{ for } \mu = (i,j)
$$

which gives a representation of DDAHA.

As claimed, it is identical to the variations of Z
in the recursion formulae up to an anomaly term.

Comparison of the coefficients in recursion formula

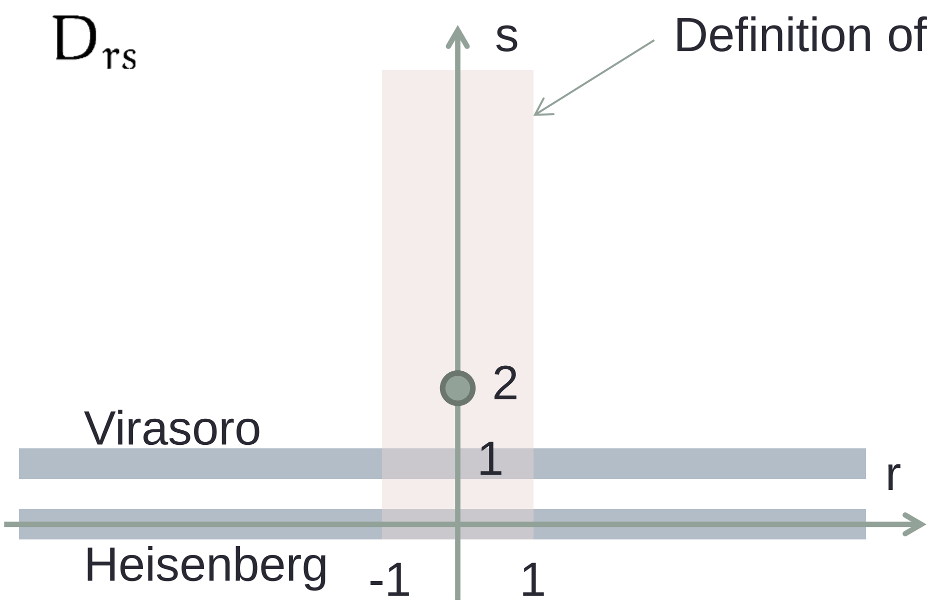
$$\delta_{-1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N}\left(\sum_{k=1}^{f_p+1}(a_p+\nu+A_k(Y_p))^n \Lambda_p^{(k,+)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,+),p};\vec{b},\vec{W};\mu)\right.$$
$$\left. - \sum_{k=1}^{\tilde{f}_p}(b_p+\mu+\nu+B_k(W_p))^n \Lambda_p^{(k,-)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,-),p};\mu)\right) ,$$

$$\delta_{1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N}\left(-\sum_{k=1}^{f_p}(a_p+\nu+B_k(Y_p))^n \Lambda_p^{(k,-)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,-),p};\vec{b},\vec{W};\mu)\right.$$
$$\left. + \sum_{k=1}^{\tilde{f}_p}(b_p+\nu+\mu+A_k(W_p)+\xi)^n \Lambda_p^{(k,+)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,+),p};\mu)\right) ,$$

The coefficients $\delta_{-1,n}$ coincide with $D_{-1,n}$ action on the bra and ket.
On the other hand the coefficient appearing in $\delta_{1,n}$ is slightly shifted by ξ.
This small mismatch will be explained by an exotic choice of vertex operator.

Comparison between DDAHA and W-algebra

D_{rs}



$D_{0,2}$ plays a special role "Hamiltonian" for Calogero-Sutherland

From $D_{0,2}$ and $D_{1,0}$, $D_{-1,0}$ one may reconstruct the action of other generators

Definition of Heisenberg (U(1)) and Virasoro

$$J_l = (-\sqrt{\beta})^{-l} D_{-l,0}, \quad J_{-l} = (-\sqrt{\beta})^{-l} D_{l,0}, \quad J_0 = E_1/\beta,$$

$$L_l = (-\sqrt{\beta})^{-l} D_{-l,1}/l + (1-l)c_0 \xi J_l/2,$$

$$L_{-l} = (-\sqrt{\beta})^{-l} D_{l,1}/l + (1-l)c_0 \xi J_{-l}/2,$$

$$L_0 = [L_1, L_{-1}]/2 = D_{0,1} + \frac{1}{2\beta}\left(c_2 + c_1(1-c_0)\xi + \frac{\xi^2}{6} c_0(c_0-1)(c_0-2)\right)$$

Equivalence of W algebra and DDAHA

Although W_n-algebra and DDAHA looks very different,
they are equivalent for the representation in $|\vec{Y}\rangle$

Schiffmann and Vaserot '12

It is nontrivial to show it directly since they are defined in orthogonal strips.

The proof was made by the computation of $D_{0,2}$
and properties of coproduct.

We give extra confirmation by the study of singular vector.
DDAHA and W-algebra has the same null-state in the Hilbert space.

$$D_{-1,n}|\vec{a}, \vec{Y}\rangle = 0 \qquad \text{for all } n$$

INTERPRETATION AS CONFORMAL WARD IDENTITY

Recursion formula and DDAHA

In the following, we would like to establish the identification,

$$Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)=\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle,$$

by showing the recursion formulae we obtained can be identified with the conformal Ward identity. In the language of Fock space, it is written as "obvious relations";

$$\begin{aligned}0= \quad & (\langle\vec{a}+\nu\vec{e},\vec{Y}|D_{n,m})V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle \\ & -\langle\vec{a}+\nu\vec{e},\vec{Y}|\,[D_{n,m},V(1)]\,|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle-\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)(D_{n,m}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle)\,.\end{aligned}$$

While the action of DDAHA on the bra and ket basis are known, the commutation relation with the vertex operator remains mysterious. We restrict our analysis to Virasoro and U(1) current.

Exotic choice of vertex operator

The vertex operator is written as a product of U(1) (Heisenberg) part and W_N part:

$$V = \tilde{V}^H V^W$$

The standard definition in terms of free boson is written as,

$$\tilde{V}^H_\kappa = e^{-\frac{\kappa}{N}\vec{e}\cdot\vec{\varphi}}, \quad V^W_\kappa = e^{-\kappa(\vec{e}_N - \frac{\vec{e}}{N})\vec{\varphi}},$$

$$\vec{e} := (1, \cdots, 1) \qquad \vec{e}_N = (0, \cdots, 0, 1)$$

We need modify the U(1) part of the vertex operator

Carlsson-Okounkov

$$V^H_\kappa(z) = e^{\frac{1}{\sqrt{N}}(NQ-\kappa)\phi_-} e^{\frac{-1}{\sqrt{N}}\kappa\phi_+},$$

$$\phi_+ = \alpha_0 \log z - \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^{-n}, \quad \phi_- = q + \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} z^n .$$

It correctly reproduce the correlation function of U(1) part:

$$\langle V^H_{\kappa_1}(z_1)\cdots V^H_{\kappa_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{\frac{-\kappa_i(NQ-\kappa_j)}{N}} .$$

But the conformal property of the vertex becomes much more complicated:

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{(NQ-\kappa)^2}{2N}(n+1)z^n V_\kappa(z) + \sqrt{N}Q\sum_{m=0}^{n} z^{n-m}V_\kappa(z)\alpha_m + (n+1)z^n \Delta_W V_\kappa(z), \;\; n \geq 0 ,$$

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{\kappa^2}{2N}(n+1)z^n V_\kappa(z) - \sqrt{N}Q\sum_{m=1}^{|n|} z^{n+m}\alpha_{-m}V_\kappa(z) + (n+1)z^n \Delta_W V_\kappa(z), \;\; n < 0 ,$$

The third term represent the anomaly due to the modification of the vertex.
It has the effect of correcting the shift of a coefficient in recursion relation.

We confirmed Ward identities for $L_{\pm 2}$, $L_{\pm 1}$, $J_{\pm 1}$ which are sufficient to prove the general case $L_{\pm n}$, $J_{\pm n}$ for general *n*.

SUMMARY

Summary

1. We derive an infinite set of recursion relations

$$\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} = U_{\pm 1,n} Z_{\vec{Y},\vec{W}}$$

where LHS is the variations by adding/subtracting a box in Y or W with some structure constant

2. The variation δ can be closely related to a quantum symmetry DDAHA (Degenerate Double Affine Hecke Algebra) which is equivalent to W_N algebra

3. They can be identified with the extended conformal Ward identities if we have the identification

$$Z_{\vec{Y},\vec{W}} = \langle \vec{a}, \vec{Y} | V_\kappa(1) | \vec{b}, \vec{W} \rangle$$

which gives a recursive proof of AGT for SU(2) gauge group

Thank you for your attention!

感謝您的關注!