On Supercurrents, Anomalies and (Effective) Lagrangians

Jean-Pierre Derendinger

Albert Einstein Center for Fundamental Physics,
Institute for Theoretical Physics, Bern University

PASCOS 2013, Taipei, Taiwan, 20–26 November 2013

UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Introduction

- Currents, their (non)conservation equations, anomalies are important tools in effective Lagrangian descriptions.
- In supersymmetric gauge theories, a subject with several decades of history and also lasting controversies.
- Decades of publications, not cited here.
(*Exceptions are my choice*).

- Collaboration with N. Ambrosetti, D. Arnold, J. Hartong.
- Arnold, Hartong, J.-P. D., hep-th:1208.1648 / Nucl. Phys. B867 (2013) 370. AADH and DH to appear.
- “Old" works, 1991-95, wich Ferrara, Kounnas, Zwirner, Chamseddine, Quirós, Burgess, Quevedo.

The superconformal case

Superconformal ($N = 1$) theory: $SU(2,2|1)$ conserved currents.
In particular:

- $T_{\mu\nu} = T_{\nu\mu}$, energy-momentum tensor, $\partial^\mu T_{\mu\nu} = 0$ $\quad (10 - 4 = 6_B)$
- $S_{\mu\alpha}$, supercurrent, $\partial^\mu S_{\mu\alpha} = 0$ $\quad (16 - 4 = 12_F)$
- $j_\mu^{(R)}$, chiral $U(1)_R$ symmetry, $\partial^\mu j_\mu^{(R)} = 0$ $\quad (4 - 1 = 3_B)$
- Total: $\underline{9_B + 12_F}$

appear in a supermultiplet with superfield description

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = 0 \qquad\qquad J_\mu = (\overline{\sigma}_\mu)^{\dot\alpha\alpha} J_{\alpha\dot\alpha} \text{ real}$$

(Ferrara, Zumino, 1975)

Supercurrent equation:

- J_μ: $\quad 4 \times (8_B + 8_F) = \underline{32_B + 32_F}$ fields.
- $\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = 0$: $\quad 2 \times (12_B + 12_F) = \underline{24_B + 24_F}$ conditions.
- Then: $\underline{8_B + 8_F}$ fields/operators in the supercurrent structure.

The superconformal case

The supercurrent structure $D^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0$ includes $\underline{1_B + 4_F}$ supplementary conditions for superconformal invariance, to leave $8_B + 8_F$ operators/fields:

- $(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} S_{\mu\alpha} = 0$ (4_F conditions), for conformal supersymmetry;
- $T^\mu{}_\mu = 0$ (1_B), *related to* dilatation/scale invariance, $\partial^\mu j_\mu^{(dil.)} = 0$
- Dilatation current not in $J_{\alpha\dot{\alpha}}$, its relation to $T_{\mu\nu}$ depends on possible improvements of $T_{\mu\nu}$.

Expect to find in the superconformal case:

$$j_\mu^{(dil.)} = x^\nu T_{\mu\nu} \qquad\qquad \partial^\mu j_\mu^{(dil.)} = T^\mu{}_\mu = 0$$

and $T^\mu{}_\mu = 0$ implies scale invariance and conformal invariance, with conserved currents

$$K_\mu^\alpha = (2x^\alpha x^\nu - \eta^{\alpha\nu} x^2)\, T_{\mu\nu} \qquad\qquad \partial^\mu K_\mu^\alpha = 2x^\alpha T^\mu{}_\mu$$

The super-Poincaré case

Again:

- $T_{\mu\nu} = T_{\nu\mu}$, energy-momentum tensor, $\partial^\mu T_{\mu\nu} = 0$ $(10 - 4 = 6_B)$
- $S_{\mu\alpha}$, supercurrent, $\partial^\mu S_{\mu\alpha} = 0$ $(16 - 4 = 12_F)$

In general, broken $U(1)_R$, scale, conformal, special supersymmetry invariances

- $\partial^\mu j_\mu^{(R)}, \quad \partial^\mu j_\mu^{(dil.)}, \quad \dots$ non zero in general

Supercurrent structure: $\partial^\mu T_{\mu\nu} = \partial^\mu S_{\mu\alpha} = 0$ *required*

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \Delta_\alpha \qquad \overline{DD}\Delta_\alpha = 0 \qquad \dots$$

Source, anomaly superfields $(8_B + 8_F)$: (sufficient, not the most general)

$$\Delta_\alpha = D_\alpha X + \chi_\alpha \qquad \overline{\Delta}_{\dot\alpha} = -\overline{D}_{\dot\alpha}\overline{X} + \overline{\chi}_{\dot\alpha}$$

$$\overline{D}_{\dot\alpha} X = 0 \qquad \chi_\alpha = -\tfrac{1}{4}\overline{DD} D_\alpha U \qquad U^\dagger = U$$

The super-Poincaré case

Counting fields: $\mathbf{32}_B + \mathbf{32}_F$ in $J_{\alpha\dot\alpha}$, $\mathbf{8}_B + \mathbf{8}_F$ in Δ_α,
$-(\mathbf{24}_B + \mathbf{24}_F)$ conditions in $\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = \Delta_\alpha$

$\Longrightarrow$ $\underline{\mathbf{16}_B + \mathbf{16}_F}$ fields/operators [Komargodski, Seiberg, earlier literature]

Counting component fields:

$\mathbf{6}_B$ in (conserved) $T_{\mu\nu}$, $\mathbf{12}_F$ in (conserved) $S_{\mu\alpha}$, $\mathbf{4}_B$ in $j_\mu^{(R)}$, $\mathbf{8}_B + \mathbf{8}_F$ in source superfields X and χ_α

reduced to $\mathbf{16}_B + \mathbf{16}_F$ by $\mathbf{2}_B + \mathbf{4}_F$ conditions:

$$T^\mu{}_\mu = \tfrac{1}{4} D + \tfrac{3}{2}\,\mathrm{Re}\, f_X \qquad \partial^\mu j_\mu^{(R)} = -\tfrac{3}{2}\,\mathrm{Im}\, f_X$$

$$(\sigma^\mu \overline{S}_\mu)_\alpha = 6\sqrt{2}\,\psi_{X\,\alpha} + 2i\,\lambda_\alpha$$

- Not the most general structure (see for instance Kusenko, 2010-11), sufficient for our needs.

Super-Poincaré supercurrent structure

- Supercurrent structure systematically derived for a given theory.
- Not unique, a (continuous) family of supercurrent structures with significant currents.
- Supersymmetric improvement formula, an identity:

$$2\overline{D}^{\dot{\alpha}}[D_\alpha, \overline{D}_{\dot{\alpha}}]\mathcal{G} = D_\alpha \overline{DD}\mathcal{G} + 3\overline{DD}D_\alpha\mathcal{G}$$

Then:

$$J_{\alpha\dot{\alpha}} \longrightarrow J_{\alpha\dot{\alpha}} + 2\,[D_\alpha, \overline{D}_{\dot{\alpha}}]\mathcal{G}$$

$$X \longrightarrow X + \overline{DD}\mathcal{G} \qquad\qquad \chi_\alpha \longrightarrow \chi_\alpha + 3\,\overline{DD}D_\alpha\mathcal{G}$$

- An identity, without dynamical content.

(If C_μ added to a current j_μ, the source field of $\partial^\mu j_\mu$ also receives $\partial^\mu C_\mu$.)

Super-Poincaré supercurrent structure

Some questions:

- Which energy-momentum tensor in which supercurrent structure ?
- Poincaré supermultiplets also representations of superconformal $N = 1$.
- Makes sense to assign $U(1)_R$ charge q and scale dimension w to chiral superfields.
- Which $U(1)_R$–current (q dependent) in which supercurrent structure ?
- Which dilatation current (w dependent), which relation with $T_{\mu\nu}$?
- Relevant questions for the description of scale and R anomalies, crucial tools in effective Lagrangians constructions.
- Any relation between w and q in the supercurrent structure ?
  Superconformal case requires: $q = w$ *(in my convention)*.

Systematic construction of supercurrent structures

For an arbitrary (real, gauge-invariant) function $\mathcal{H}(\hat{L}, \Phi, \overline{\Phi}e^{\mathcal{A}})$ of:

- Chiral Φ and gauge $\mathcal{A}$ superfields,
- A real $\hat{L} = L - 2\Omega(\mathcal{A})$, L linear, Ω CS superfield, $\overline{DD}\Omega = \mathrm{Tr}\,\mathcal{W}\mathcal{W}$

A gauge-invariant superfield identity:

$$
\begin{aligned}
&2\overline{D}^{\dot{\alpha}}\left[(\overline{\mathcal{D}}_{\dot{\alpha}}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot{\alpha}}\hat{L})(D_\alpha\hat{L})\right] \\
&\qquad = -\hat{L}\,\overline{DD}D_\alpha\mathcal{H}_L - (\overline{DD}\mathcal{H}_\Phi)\,\mathcal{D}_\alpha\Phi - \overline{DD}D_\alpha(\mathcal{H} - \hat{L}\mathcal{H}_L) \\
&\qquad\quad -2\,\widetilde{\mathrm{Tr}}\,\mathcal{W}\mathcal{W}\,D_\alpha\mathcal{H}_L - 4\,\mathcal{H}_Y\,\overline{\Phi}e^{A}\mathcal{W}_\alpha\Phi,
\end{aligned}
$$

Use then field equations of theory

$$
\mathcal{L} = \int d^2\theta d^2\overline{\theta}\,\mathcal{H}(\hat{L}, \Phi, \overline{\Phi}e^{\mathcal{A}}) + \int d^2\theta\,W(\Phi) + \int d^2\overline{\theta}\,\overline{W}(\overline{\Phi})
$$

to obtain the “natural" supercurrent structure …

I: The “natural" supercurrent structure

$$\begin{aligned}
\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} &= D_\alpha X + \chi_\alpha \\
J_{\alpha\dot\alpha} &= -2(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) + 2\mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) \\
&\quad -4\,\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}}) \\
X &= 4\,W \;=\; -\tfrac{4}{3}\widetilde{\Delta}_{(0)} \\
\chi_\alpha &= \overline{D}\overline{D}D_\alpha(\mathcal{H} - \hat{L}\mathcal{H}_L) \;=\; -\tfrac{1}{2}\overline{D}\overline{D}D_\alpha\Delta_{(0)}
\end{aligned}$$

$$\begin{aligned}
\Delta_{(w)} &= 2\hat{L}\mathcal{H}_L + w\Phi\mathcal{H}_\Phi + w\overline{\Phi}\mathcal{H}_{\overline{\Phi}} - 2\mathcal{H}, && (\Delta_{(w)}\ \text{real}), \\
\widetilde{\Delta}_{(w)} &= wW_\Phi\Phi - 3W, && (\overline{D}_{\dot\alpha}\widetilde{\Delta}_{(w)} = 0),
\end{aligned}$$

measure the breaking of scale invariance in theory $\mathcal{L}$ if scale dimension w is assigned to Φ ($\hat{L}$ has canonical dimension 2)

II: The "w–improved" supercurrent structure

- Assign scale dimensions w to Φ (in general reducible)
- Improve with identity $2\overline{D}^{\dot\alpha}[D_\alpha, \overline{D}_{\dot\alpha}]\mathcal{G} = D_\alpha\overline{DD}\mathcal{G} + 3\overline{DD}D_\alpha\mathcal{G}$

  using $\mathcal{G} = -\frac{w}{6}(\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}})$ and field equations:

$$\begin{aligned}
\overline{D}^{\dot\alpha}\widetilde{J}_{\alpha\dot\alpha} &= D_\alpha\widetilde{X} + \widetilde{\chi}_\alpha \\
\widetilde{J}_{\alpha\dot\alpha} &= -2(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) + 2\mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) \\
&\quad -4\,\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}}) - \frac{w}{3}[D_\alpha, \overline{D}_{\dot\alpha}](\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \\
\widetilde{X} &= -\frac{4}{3}\widetilde{\Delta}_{(w)} + \frac{w}{6}\overline{DD}(\mathcal{H}_\Phi\Phi - \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \\
\widetilde{\chi}_\alpha &= -\frac{1}{2}\overline{DD}D_\alpha\Delta_{(w)}
\end{aligned}$$

Supercurrent structures

- R–current with charge q for Φ: in improved structure with $w = q$.
- The Belinfante energy-momentum tensor is in the natural structure $w = 0$ only.
- $w \neq 0$: energy-momentum tensor is “improved": $T_{\mu\nu}^{(w)}$
- In general: $j_\mu^{(dil.)} = \mathcal{V}_\mu^{(w)} + x^\nu T_{\mu\nu}^{(w)}$ $\quad\mathcal{V}_\mu^{(w)}$: virial current
- Scale invariance (for $w \neq 0$ only): virial current only vanishes
  if $\mathcal{H}$ has $U(1)_q$ invariance, with $q = w$.
- With $U(1)_q$ invariance:

$$\mathcal{V}_\mu^{(w)} = -\frac{1}{2}\left[\partial_C \Delta^{(w)}|_{\theta=0}\right]\partial_\mu C \qquad C = \hat{L}|_{\theta=0}$$

- Without $U(1)_q$: scale invariance does not imply conformal invariance

On effective actions in $N = 1$ theories

NSVZ β function: $$\beta(g^2) = -\frac{g^4}{16\pi^2}\frac{A}{1-\frac{g^2}{8\pi^2}B}$$

[Novikov, Shifman, Vainshtein, Zakharov]

NSVZ:

- $A = b_0 + \gamma T(R)$, $b_0 = 3C(G) - T(R)$, γ: anomalous dimension.
- $B = C(G)$
- Not unique, depends on the convention defining the renormalization factor Z in $\gamma = -M\partial_M \ln Z$. NSVZ:

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, Z\overline{\Phi}e^{\mathcal{A}}\Phi + \frac{1}{4g^2}\int d^2\theta\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \mathrm{h.c.}$$

For instance: $Z = g^{-2}$ for $N = 2$ super-Yang Mills,
and then $B = C(G) - T(R) = 0$ (and $\gamma = 0$).

Can we derive the NSVZ beta function with a field-coupling $C = g^2(M)$ *from anomaly-matching ?*

On effective actions in $N = 1$ theories

Write the super Yang-Mills Lagrangian as a full superspace integral:

$$\mathcal{L}_{SYM} = \int d^2\theta d^2\overline{\theta}\, \hat{L} \qquad \hat{L} = L - 2\Omega \qquad \overline{DD}\, L = 0 \quad \overline{DD}\Omega = \mathrm{Tr}\, \mathcal{W}\mathcal{W}$$

Gauge invariance of $\hat{L}$: $\delta L = 2\,\delta\Omega$

In $\mathcal{L}_{SYM}$ the linear superfield L is non-dynamical. But in

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \mathcal{H}(\hat{L}, \overline{\Phi} e^{\mathcal{A}}, \Phi) + \int d^2\theta\, W(\Phi) + \int d^2\overline{\theta}\, \overline{W}(\overline{\Phi})$$

it is in general propagating and since

$$\mathcal{L} = -\frac{1}{4g^2}\; \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \qquad\qquad \frac{1}{g^2} = \frac{\partial}{\partial C}\mathcal{H}(C, \overline{z}, z)$$

the real scalar $C = L|_{\theta=0}$ is the (non-holomorphic !) gauge coupling field

and also the string loop-counting dilaton field (Ferrara, Cecotti, Villasante, 1987)

On effective actions in $N = 1$ theories

- Two gauge-invariant superfields $\hat{L}$ (real) and $\mathrm{Tr}\,\mathcal{W}\mathcal{W}$ (chiral)

  in effective Lagrangians for $N = 1$ gauge theories, matching both chiral $U(1)_R$ and dilatation (virial) anomalies.

- Start with an invariant Lagrangian

  $$\mathcal{L}_0 = \int d^2\theta d^2\overline{\theta}\,\mathcal{H}(\hat{L}, \overline{\Phi}e^{\mathcal{A}}\Phi) + \int d^2\theta\, W(\Phi) + \mathrm{h.c.}$$

- Add anomaly terms: (in a Wilson effective action)

  $$A\int d^2\theta d^2\overline{\theta}\,(\hat{L}\ln\hat{L} - \hat{L}) + \frac{B}{4}\ln\left(\frac{\mu}{M}\right)\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \mathrm{h.c.}$$

- Source superfields include: $\chi_\alpha = -A\overline{DD}D_\alpha\hat{L} + \ldots$

- Gauge coupling: $\frac{1}{g_W^2} = \mathcal{H}_C + A\ln C + B\ln\left(\frac{\mu}{M}\right)$ as in NSVZ.

- Coefficients A and B evaluated from $U(1)_R$ and dilatation (virial current) anomalies (formally 1-loop)

- Super-Yang Mills: $\mathcal{H} = \ln\hat{L}$.

Non dynamical (super)fields ?

An example: *the Fayet-Iliopoulos term*:

Theory

$$\int d^2\theta d^2\overline{\theta}\, [\mathcal{K}(\overline{\Phi}_i e^{q_i \mathcal{A}} \Phi_i) + \xi \mathcal{A}] + \int d^2\theta \left[\frac{1}{4}\mathcal{W}\mathcal{W} + W(\Phi_i)\right] + \text{h.c.}$$

has a gauge-invariant supercurrent structure $\overline{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = D_\alpha X + \chi_\alpha$ with source superfields

$$X = -\frac{4}{3}\widetilde{\Delta} \qquad \chi_\alpha = -\frac{1}{2}\overline{DD}D_\alpha(\Delta - 2\,\xi\mathcal{A}) = -\frac{1}{2}\overline{DD}D_\alpha\Delta - 4\,\xi\mathcal{W}_\alpha$$

with $\Delta = w_i \mathcal{K}_i \Phi_i - 2\mathcal{K}$ and $\widetilde{\Delta} = w_i W_i \Phi_i - 3W$.

- ξ in χ_α: scale invariance broken, $U(1)_R$ untouched, as expected.

Is there a Ferrara-Zumino gauge-invariant version ?

Non dynamical (super)fields ?

Fayet-Iliopoulos term: the gauge-invariant FZ structure:

Consider instead

$$\int d^2\theta d^2\overline{\theta}\,[\mathcal{K}(\overline{\Phi}_i e^{q_i\mathcal{A}}\Phi_i)+\xi(\mathcal{A}+S+\overline{S})]+\int d^2\theta\left[\frac{1}{4}\mathcal{W}\mathcal{W}+W(\Phi)\right]+\text{h.c.}$$

Gauge variations: $\delta\mathcal{A}=-\Lambda-\overline{\Lambda}$ $\qquad \delta S=\Lambda$ $\qquad \Phi_i'=e^{q_i\Lambda}\Phi_i$

- $\mathcal{A}+\mathcal{S}+\overline{\mathcal{S}}$ gauge invariant, $\mathcal{W}_\alpha$ and dynamical equations unchanged, S is *non-dynamical*.

Use the supersymmetric improvement formula:

$$J_{\alpha\dot\alpha}\rightarrow J_{\alpha\dot\alpha}+\tfrac{1}{3}[D_\alpha,\overline{D}_{\dot\alpha}](\Delta-2\xi(\mathcal{A}+S+\overline{S}))$$

$$X\rightarrow X+\tfrac{1}{6}\overline{DD}(\Delta-2\xi(\mathcal{A}+S+\overline{S}))\qquad \chi_\alpha\rightarrow 0$$

Since the improvement formula is an identity (field equations are not used), physics unchanged, a perfectly legitimate procedure.

Non dynamical (super)fields ?

Some remarks:

- A derivative Lagrangian contributes in general to the canonical energy-momentum tensor and currents by improvement terms (charges unchanged). . .
- Coupling to supergravity with the S superfield:
  - If the superpotential is R symmetric ($\widetilde{\Delta} = 0$), couple the supercurrent structure with $\chi_\alpha \neq 0$ to new minimal supergravity and transform to old minimal. The S field in old minimal is a gauge artifact produced by a gauge transformation of the chiral compensator S_0 (non dynamical).
  - If the superpotential is not R symmetric, couple to $16 + 16$ supergravity and $4_B + 4_F$ supplementary degrees of freedom propagate (like in the reduced NS sector of superstrings. . .).

  See also Komargodski, Seiberg.
- Conformal supergravity approach: two scales needed, κ (or M_P) and ξ. Two scalar fields needed, to gauge fix dilatations and provide ξ in the $M_P \rightarrow \infty$ global supersymmetry limit.