Thermalization process of dark matter axions

Ken'ichi Saikawa
Tokyo Institute of Technology (Titech)

Collaborate with T. Noumi (RIKEN), R. Sato (KEK) and M. Yamaguchi (Titech)

based on
[1] KS, M. Yamaguchi, hep-ph/1210.7080. [PRD87, 085010 (2013)]
[2] T. Noumi, KS, R. Sato, M. Yamaguchi, hep-ph/1310.0167. (submitted to PRD)

Abstract

- Discuss the possibility that QCD axions form a Bose-Einstein condensate (BEC)
- Investigate elementary processes in the system of coherently oscillating axions
  - Estimate the interaction rate KS, Yamaguchi, PRD87, 085010 (2013) (based on Newtonian approximation)
  - Reanalyze it on the ground of general relativity Noumi, KS, Sato, Yamaguchi, 1310.0167

Axion

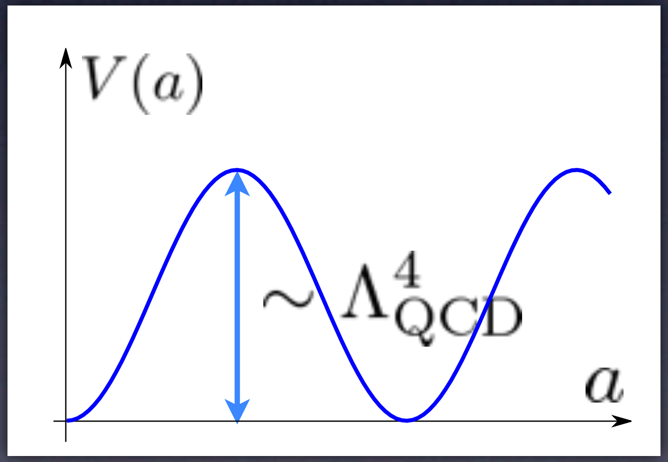
- motivated as a solution of strong CP problem
- Spontaneous breaking of continuous Peccei-Quinn symmetry at $T \simeq F_a \simeq 10^{9-12}\mathrm{GeV}$ "axion decay constant"
  - Nambu-Goldstone theorem
    → emergence of the (massless) particle ≡ axion

Weinberg(1978), Wilczek(1978)

- Axion has a small mass (QCD effect)
  → pseudo-Nambu-Golstone boson

$$m_a \sim \frac{\Lambda_{\mathrm{QCD}}^2}{F_a} \sim 6 \times 10^{-6}\mathrm{eV}\left(\frac{10^{12}\mathrm{GeV}}{F_a}\right)$$

$$\Lambda_{\mathrm{QCD}} \simeq \mathcal{O}(100)\mathrm{MeV}$$

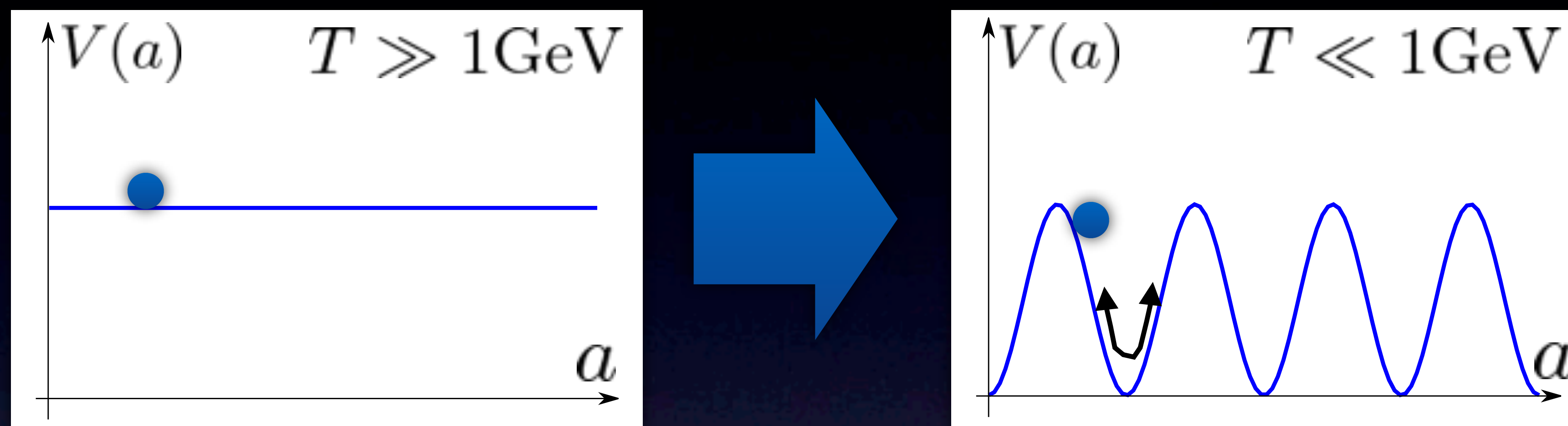


- Tiny coupling with matter + non-thermal production
  → good candidate of cold dark matter

Production mechanism

Preskill, Wise, Wilczek (1983); Abbott, Sikivie (1983); Dine, Fischler (1983)

- Misalignment mechanism



The axion mass "turns on" at $m_a(t_1) = H(t_1)$ ($T_1 \sim 1\mathrm{GeV}$)

- EOM for homogeneous axion field

$$\left(\frac{d^2}{dt^2} + \frac{3}{2t}\frac{d}{dt} + m_a^2\right)\langle a\rangle = 0$$

$$\Rightarrow \quad m_a A^2 \propto R^{-3}(t) \quad , \quad \langle a\rangle = A(t)\cos(m_a t)$$

$R(t)$: scale factor of the universe

$$\Rightarrow \quad \rho_a(t) = \frac{1}{2} m_a^2 \langle a\rangle^2 \propto R^{-3}(t)$$

behave like non-relativistic matter

Axion BEC dark matter ?

- Peculiarities of axion dark matter
  - Non-thermal production

$$H \lesssim m_a \quad (t = t_1) \qquad t_1 \sim 10^{-7}\mathrm{sec}$$

$$\delta v \sim \frac{\delta p}{m_a} \sim \frac{R(t_1)}{R(t_0)} \frac{1}{m_a t_1} \sim 3 \times 10^{-17} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{0.81}$$

small velocity dispersion
("cold" dark matter)

  - Large occupation number

$$\mathcal{N} \sim n_a \frac{(2\pi)^3}{\frac{4\pi}{3}(m_a \delta v)^3} \sim 10^{61} \left(\frac{F_a}{10^{12}\mathrm{GeV}} \right)^{2.75}$$

($n_a \sim m_a F_a^2 (R(t_1)/R(t_0))^3$: number density of axions)

- A possibility that axions exist in the form of Bose-Einstein condensate (BEC) Sikivie, Yang, PRL103, 111301 (2009)
- Observable signatures (distinction between axions and WIMPs) ?
  - Effects on phase space structure of galactic halo (?)
    Sikivie, Phys. Lett. B695, 22 (2011); Banik, Sikivie, astro-ph.GA/1307.3547
  - Effects on cosmological parameters (?)
    Erken, Sikivie, Tam, Yang, PRL108, 061304 (2012)

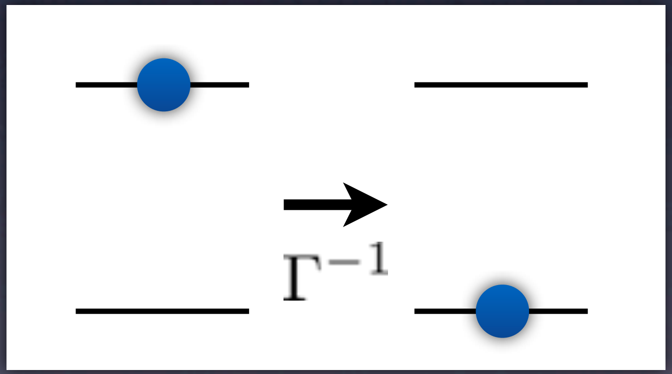
Can axions form a BEC ?

- They must be in thermal equilibrium
- Can axions thermalize in the expanding universe ?
- Naive expectation :
  They develop toward thermal equilibrium if
  the transition rate Γ exceeds the expansion rate H

$$\Gamma \sim \dot{\mathcal{N}}_{\mathbf{p}}/\mathcal{N}_{\mathbf{p}} > H$$

$\mathcal{N}_{\mathrm{p}}$: occupation number

(state labeled by three momentum p)



- Identify elementary processes to estimate Γ !

Previous study

Erken, Sikivie, Tam, Yang, PRD85, 063520 (2012)

- Time evolution of quantum operators in the Heisenberg picture

$$H = \sum_i \omega_i a_i^\dagger a_i + \sum_{i,j,k,l} \frac{1}{4} \Lambda_{kl}^{ij} a_k^\dagger a_l^\dagger a_i a_j$$

l : label of the state (momentum)

$\mathcal{N}_l = a_l^\dagger a_l$

$$\dot{\mathcal{N}}_l = i[H, \mathcal{N}_l]$$

$$= i \sum_{i,j,k} \frac{1}{2} (\Lambda_{ij}^{kl} a_i^\dagger a_j^\dagger a_k a_l e^{-i\Omega_{ij}^{kl} t} - \mathrm{H.c.})$$

$$+ \sum_{k,i,j} \frac{1}{2} |\Lambda_{ij}^{kl}|^2 [\mathcal{N}_i \mathcal{N}_j (\mathcal{N}_l + 1)(\mathcal{N}_k + 1)$$

$$- \mathcal{N}_l \mathcal{N}_k (\mathcal{N}_i + 1)(\mathcal{N}_j + 1)] \frac{2}{\Omega_{ij}^{kl}} \sin(\Omega_{ij}^{kl} t) + \ldots$$

$$\Omega_{ij}^{kl} \equiv \omega_k + \omega_l - \omega_i - \omega_j$$

Leading contribution in the condensed regime

$\Omega_{ij}^{kl} t \ll 1$

$\dot{\mathcal{N}}_l \sim \mathcal{O}(\Lambda_{ij}^{kl})$

reduce to Boltzmann eq. in the particle kinetic regime

$\Omega_{ij}^{kl} t \gg 1$

$\dot{\mathcal{N}}_l \sim \mathcal{O}(|\Lambda_{ij}^{kl}|^2)$

- Axions : condensed regime ($\Omega_{ij}^{kl} \sim m_a \delta v^2 < t^{-1}$)

  ➡ enhancement of interaction rate $\Gamma \sim \dot{\mathcal{N}}/\mathcal{N} \sim \mathcal{O}(\Lambda)$

- What about the quantum-mechanical averages $\langle \dot{\mathcal{N}}_l(t) \rangle$?

In-in formalism

Weinberg, PRD72, 043514 (2005)

- Analytic methods to calculate the time evolution of the expectation value of a quantum operator

$$\langle \mathrm{in}|\mathcal{O}(t)|\mathrm{in}\rangle = \langle\mathcal{O}\rangle + i\int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{O}]\rangle + (i)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \langle [H_I(t_1), [H_I(t_2), \mathcal{O}]]\rangle + \dots$$

$$\mathcal{O} = \mathcal{N}_n \equiv \frac{a_n^\dagger a_n}{V}$$: number operator

V : volume of the 3-dim space

$$H_I(t) = \frac{1}{V^4}\sum_{ijkl}\frac{1}{4}\Lambda_{kl}^{ij} e^{-i\Omega_{kl}^{ij}t} a_k^\dagger a_l^\dagger a_i a_j$$ (ignore axion # violating process)

- Gravitational quartic interaction (Newtonian approx.)

$$\Lambda_{kl}^{ij} = -4\pi G m_a^2 \left(\frac{1}{|\mathbf{p}_k - \mathbf{p}_i|^2} + \frac{1}{|\mathbf{p}_k - \mathbf{p}_j|^2}\right) V\delta_{i+j,k+l} \Leftarrow H_I = -\frac{G}{2}\int d^3x d^3x' \frac{\rho(\mathbf{x},t)\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$$

$\rho(\mathbf{x},t)$: energy density of axions

- Calculation procedure

Specify $H_I(t)$ and $|\mathrm{in}\rangle$ → Compute $\langle\mathcal{N}_{\mathbf{p}}(t)\rangle$ via perturbative expansion

Coherent vs number state

- $|\mathrm{in}\rangle$ = a state which represents the coherent oscillation of axions
- For axions "wavy fields"

use a coherent state $$|\{\alpha\}\rangle = \prod_i e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^n}{n!\sqrt{V^n}} (a_i^\dagger)^n |0\rangle$$

$$a_i|\alpha_i\rangle = V^{1/2}\alpha_i|\alpha_i\rangle$$ with $a_i|0\rangle = 0$

- Field amplitude

hereafter,
ϕ : axion field

$$\phi = \frac{1}{V}\sum_n \frac{1}{\sqrt{2E_{p_n}}}(e^{ip_n\cdot x}a_n + e^{-ip_n\cdot x}a_n^\dagger)$$

$$\langle\{\alpha\}|\phi|\{\alpha\}\rangle = \sum_n \frac{1}{\sqrt{2m_a V}}(e^{-im_a t + i\mathbf{p_n}\cdot\mathbf{x}}\alpha_n + e^{im_a t - i\mathbf{p_n}\cdot\mathbf{x}}\alpha_n^*)$$

$$= \sum_n \sqrt{\frac{2}{m_a V}}|\alpha_n|\cos(m_a t - \mathbf{p_n}\cdot\mathbf{x} - \beta)$$ classical field trajectory

- For other species "point particles" (photons, baryons, WIMPs,...)

use a number state $$|\{\mathcal{N}\}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(a_k^\dagger)^{\mathcal{N}_k}|0\rangle$$

Evolution of occupation number

KS, Yamaguchi, PRD87, 085010 (2013)

$$\langle \mathrm{in}|\mathcal{N}_p(t)|\mathrm{in}\rangle = \langle \mathcal{N}_p\rangle + i\int_{t_0}^{t} \langle [H_I(t_1), \mathcal{N}_p]\rangle + \mathcal{O}(H_I^2) + \dots$$

$$i\int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{N}_p]\rangle \xrightarrow{t-t_0\to\infty} -\frac{1}{2V^2}\sum_j\sum_k\sum_l \left[\Lambda_{kl}^{pj}\frac{e^{-i\Omega_{kl}^{pj}t}}{\Omega_{kl}^{pj}}\alpha_k^*\alpha_l^*\alpha_j\alpha_p + \mathrm{c.c.}\right]$$

for $\quad |\mathrm{in}\rangle = \prod_i e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a_i^\dagger)^n|0\rangle$

coherent state

$$i\int_{t_0}^{t} dt_1 \langle [H_I(t_1), \mathcal{N}_p]\rangle = 0$$

for $\quad |\mathrm{in}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(a_k^\dagger)^{\mathcal{N}_k}|0\rangle$

number state

- First order term is relevant if

  (1) condensed regime $\Omega_{kl}^{pj}t \ll 1$ $\qquad \Omega_{ij}^{kl} \equiv \omega_k + \omega_l - \omega_i - \omega_j$

  (c.f. $e^{-i\Omega_{kl}^{pj}t} \approx 0$ for particle kinetic regime $\Omega_{kl}^{pj}t \gg 1$)

  (2) coherent state representation $|\mathrm{in}\rangle = |\{\alpha\}\rangle$

Transition rate

- Transition rate of coherently oscillating components

$$\Gamma \equiv \frac{1}{\mathcal{N}_p(t)} \frac{d\mathcal{N}_p(t)}{dt} \simeq \Lambda n_a$$

n_a : number density of axions

$\Lambda^{kl}_{pj} = \Lambda V \delta_{k+l,p+j}$: coefficient in the interaction term

From gravitational self-interaction

$$H_I = -\frac{G}{2}\int d^3x d^3x' \frac{\rho(\mathbf{x},t)\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$$

$$\Rightarrow \quad \Gamma_g \simeq \frac{4\pi G m_a^2 n_a}{(\delta p)^2} \quad \propto 1/R(t)$$

$$\delta p \sim m_a \delta v \propto 1/R(t)$$

- Exceed the expansion rate at

$$\Gamma_g \gtrsim H \quad \Rightarrow \quad T \simeq 2 \times 10^3 \mathrm{eV} \left(\frac{F_a}{10^{12}\mathrm{GeV}}\right)^{0.56}$$

Formation of BEC at $T \sim \mathrm{keV}$?

Note: the result is based on the zero temperature QFT
(we did not show the establishment of BE distribution)

Interaction with other species

- Interaction with other species b

$$H_{I,b}(t) = \frac{1}{V^4}\sum_{ijkl}\frac{1}{4}{\Lambda_b}^{ij}{}_{kl}e^{-i\Omega^{ij}_{kl}t}a^\dagger_k b^\dagger_l a_i b_j$$

- Assume b particles are represented as a number state

$$|\mathrm{in}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(b^\dagger_k)^{\mathcal{N}_k}|\{\alpha\}\rangle$$

while $|\{\alpha\}\rangle = \prod_i^K e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a^\dagger_i)^n|0\rangle$

- First order term exactly vanishes

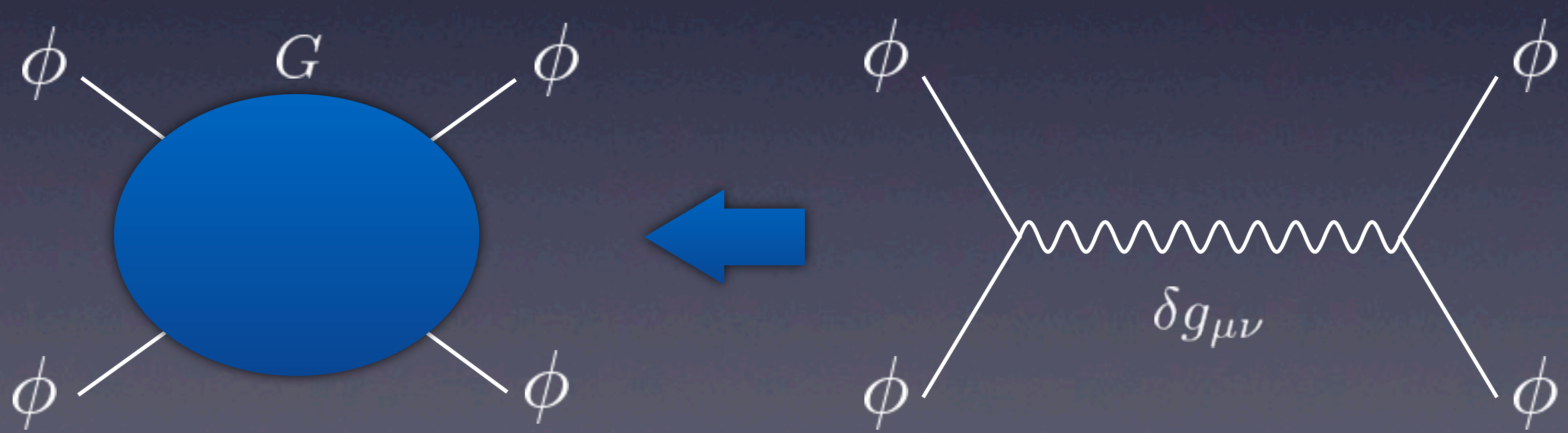
$$\langle[H_{I,b}(t), \mathcal{N}_p]\rangle = 0$$

- Interaction with other species is second order effect.
    - Axions do not have thermal contact with other particles → does not conflict with standard cosmology

General relativistic formulation

Noumi, KS, Sato, Yamaguchi, 1310.0167

- Calculations in previous slides : assumed Newtonian approximation
  → It breaks down for the modes $k/R \lesssim H$
- Reformulate in general relativistic framework
- Schematics:
  Effective quartic interaction from graviton exchange
  (contraction of cubic $\delta g_{\mu\nu}\phi^2$ interactions)



Action of the scalar-graviton system

- Assumption :
  Effect of axions on the background evolution is negligible
  (i.e. early radiation dominated era)

  → FRW background geometry is supported by other fluid (radiations)

  dynamical d.o.f. : ϕ , $g_{\mu\nu}$, $\delta\rho_{\mathrm{rad}}$

- Use Effective Field Theory (EFT) approach to derive the action

  Cheung, Fitzpatrick, Kaplan, Senatore, Creminelli, JHEP03(2008)014

  - Take the unitary gauge ($\delta\rho_{\mathrm{rad}} = 0$, $\rho_{\mathrm{rad}} = \bar{\rho}_{\mathrm{rad}}(t)$)
  - Time diffeomorphisms (diffs) is broken by $\bar{\rho}_{\mathrm{rad}}(t)$
  - Use residual spatial diffs to constrain the action for $g_{\mu\nu}$

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_a^2\phi^2\right] + \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\mathrm{Pl}}^2\mathcal{R} + M_{\mathrm{Pl}}^2\dot{H}g^{00} - M_{\mathrm{Pl}}^2(3H^2+\dot{H}) + \frac{M_2^4}{2}(g^{00}+1)^2\right]$$

M_2 : a theoretical parameter related to the sound speed c_s

$$c_s^2 = \frac{-M_{\mathrm{Pl}}^2\dot{H}}{-M_{\mathrm{Pl}}^2\dot{H} + 2M_2^4}$$

Effective quartic interactions

- ADM decomposition

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$h_{ij} = R^2(t)\underline{e^{2\zeta}(e^{\gamma})_{ij}}$$

fluctuations around FRW background

$\zeta, \ \gamma_{ij}$: dynamical fields

$N, \ N^i$: Lagrange multipliers

- Relevant interactions in the regime $m_a \gg H, \ k/R$
  (Eliminating auxiliary fields $N, \ N^i$)

$$H_{I,\zeta\phi^2} \simeq \int d^3x R^3 \left[\frac{1}{2H}\dot{\zeta}(\dot{\phi}^2 + m_a^2\phi^2) - \frac{3}{2}\zeta(\dot{\phi}^2 - m_a^2\phi^2) \right] + \mathcal{O}(H^2\zeta\phi^2)$$

$$H_{I,\phi^4} \simeq \int d^3x R^3 \left[\frac{1}{16M_{\rm Pl}^2 H^2 \tilde{\epsilon}}(\dot{\phi}^2 + m_a^2\phi^2)^2 \right] + \mathcal{O}(m_a^2\phi^4/M_{\rm Pl}^2)$$

where $\tilde{\epsilon} = 2c_s^{-2}$

$$H_{I,\gamma\phi^2} = \int d^3x R^3 \left[-\frac{1}{2}\gamma_{ij}\frac{\partial_i\phi\partial_j\phi}{R^2} \right] \sim \mathcal{O}(H^2\gamma\phi^2)$$

(subdominant)

Contributions for $\langle \mathcal{N}_{\mathbf{p}}(t) \rangle$

$$\langle \mathcal{N}_{\mathbf{p}}(t) \rangle \simeq \langle \mathcal{N}_{\mathbf{p}}(t_0) \rangle + i \int_{t_0}^{t} dt_1 \langle [H_{I,\phi^4}(t_1), \mathcal{N}_{\mathbf{p}}] \rangle$$

$$+ i^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \underline{\langle [H_{I,\zeta\phi^2}(t_1), [H_{I,\zeta\phi^2}(t_2), \mathcal{N}_{\mathbf{p}}]] \rangle}$$

contraction of cubic interactions

$$\simeq \langle \mathcal{N}_{\mathbf{p}}(t_0) \rangle + i \int_{t_0}^{t} dt_1 \langle [H_{\text{eff}}(t_1), \mathcal{N}_{\mathbf{p}}] \rangle$$

- Effective Hamiltonian for tree-level analysis

$$H_{\text{eff}}(t) = \int \left(\prod_{i=1}^{4} \frac{d^3 k_i}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_3} - \mathbf{k_4}) F(t; |\mathbf{k_1} - \mathbf{k_3}|) a^\dagger_{\mathbf{k_1}} a^\dagger_{\mathbf{k_2}} a_{\mathbf{k_3}} a_{\mathbf{k_4}}$$

$$F(t; k) = -\frac{2\pi G m_a^2}{R^3(t)} \frac{R^2(t)}{k^2} f\left(\frac{k}{k_H(t)}\right), \qquad f(x) = 1 - \cos x - x \sin x$$

$k_H(t) = R(t)H(t)/c_s$: sound horizon

- $f(k/k_H) \to 1 + (\text{highly oscillating terms})$ for $k/k_H \gg 1$

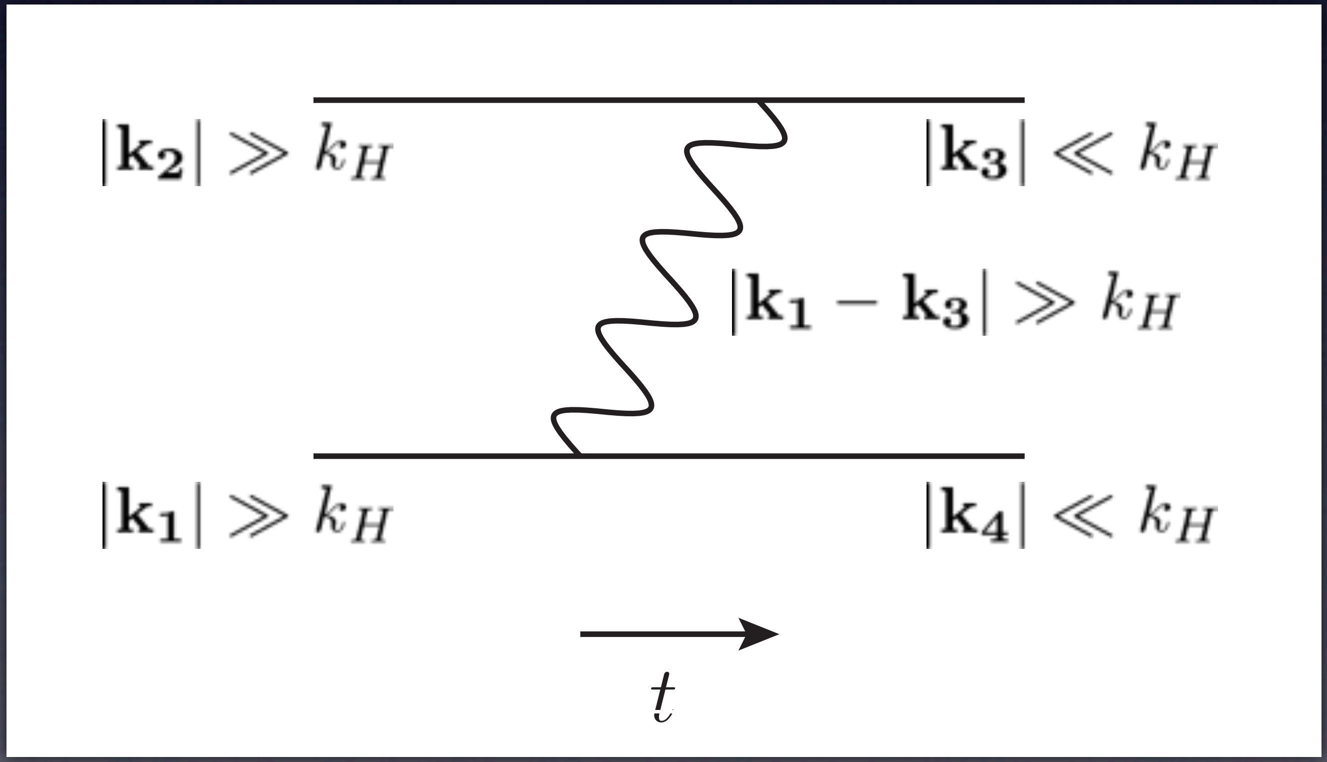
- Reproduces the result in Newtonian approx.

$$F(t; |\mathbf{k_1} - \mathbf{k_3}|) \to -\frac{2\pi G m_a^2 / R^3(t)}{|\mathbf{k_1} - \mathbf{k_3}|^2 / R^2(t)} \quad \Rightarrow \quad \Gamma_g \sim \frac{G m_a^2 n_a}{(\delta p)^2}$$

Transition into BEC ?

- Time scale of the process whose momentum transfer satisfies $|\mathbf{k}_1 - \mathbf{k}_3| \gg k_H(t) = R(t)H(t)/c_s$ can be estimated as $\sim \Gamma_g^{-1}$
- Allows transition between sub-horizon & super-horizon modes

example :



- Relevance to the gravitational thermalization ?

Summary

- Estimate interaction rate of axions in coherent state & condensed regime
- Gravitational self-interaction of cold axions becomes relevant at $T \sim \mathcal{O}(1)\mathrm{keV}$
- Interaction between axions and other particle species is highly suppressed
- Reanalyze in general relativistic framework
  - Derive effective quartic interactions of massive scalar fields
  - Transition between modes inside and outside the horizon can occur rapidly

Backup slides

"Zero modes"

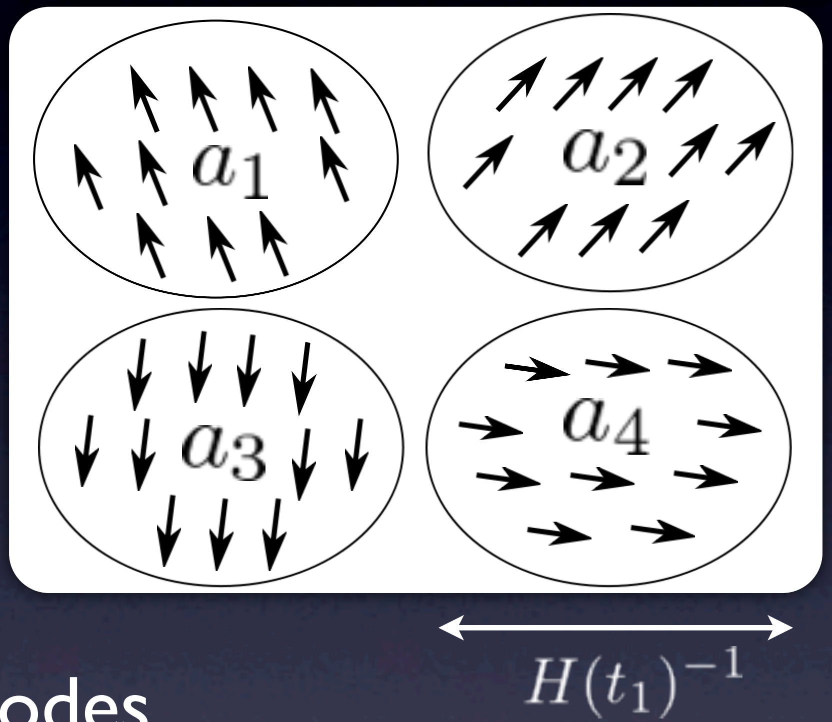
- Initial time t_1 (QCD phase transition) :

  amplitudes of oscillation might be uncorrelated

  beyond the horizon

  → axions have non-zero (but small) momenta

  $$p(t_1) \lesssim H(t_1) \sim m_a(t_1)$$



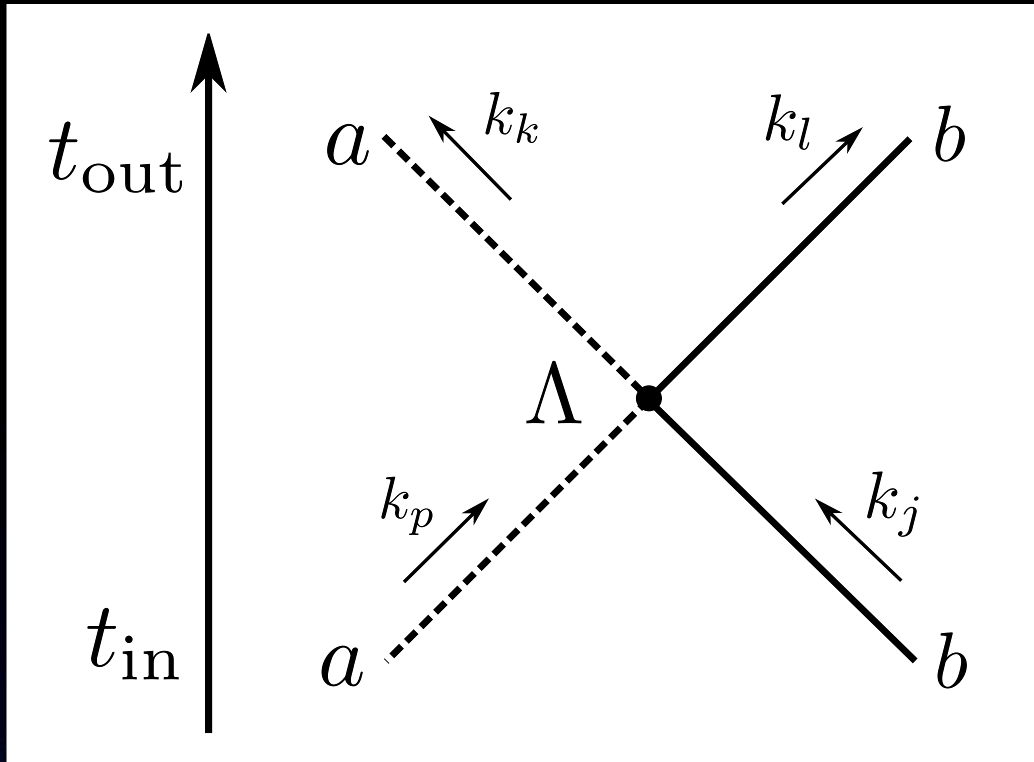
- Assume plural (say K) oscillating modes

$$|\{\alpha\}\rangle = \prod_i^K e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^n}{n!\sqrt{V^n}} (a_i^\dagger)^n |0\rangle$$

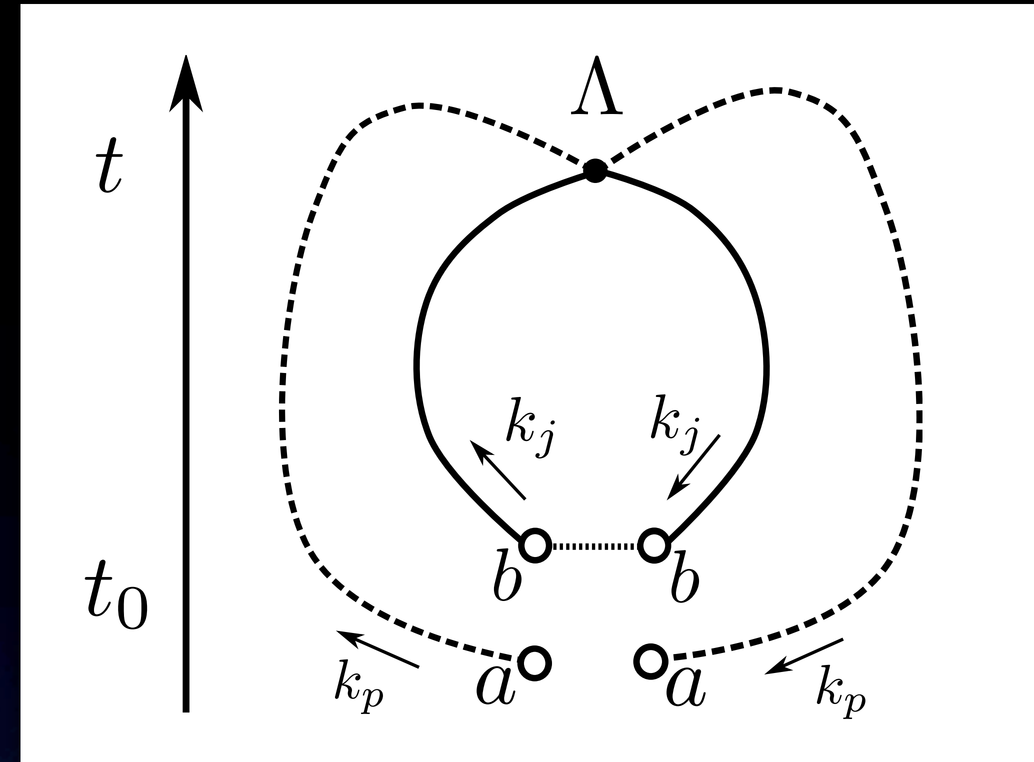
$|\mathbf{p}_i| \lesssim H(t_1) \sim m_a(t_1)$ for $i = 1, \ldots, K$

- $|\alpha_i|^2 \leftrightarrow$ momentum distribution

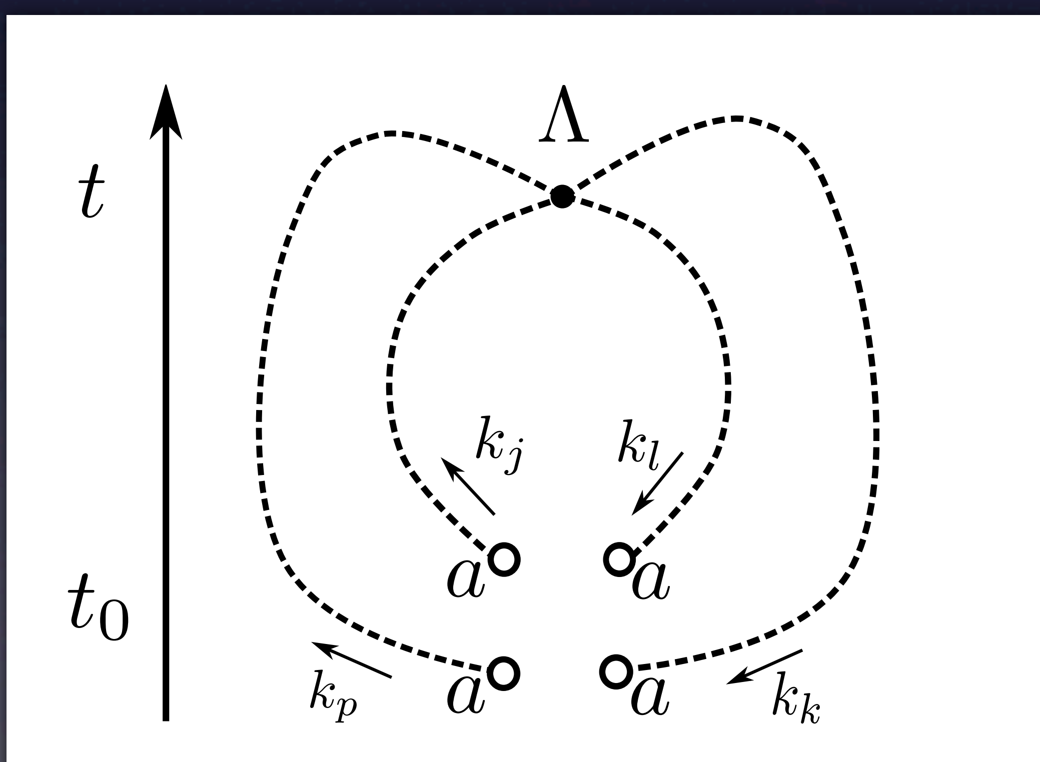
$$n_a = \frac{1}{V}\sum_n \langle\{\alpha\}|\mathcal{N}_n|\{\alpha\}\rangle = \frac{1}{V}\sum_i^K |\alpha_i|^2 \equiv \sum_i^K n_{c,i}$$



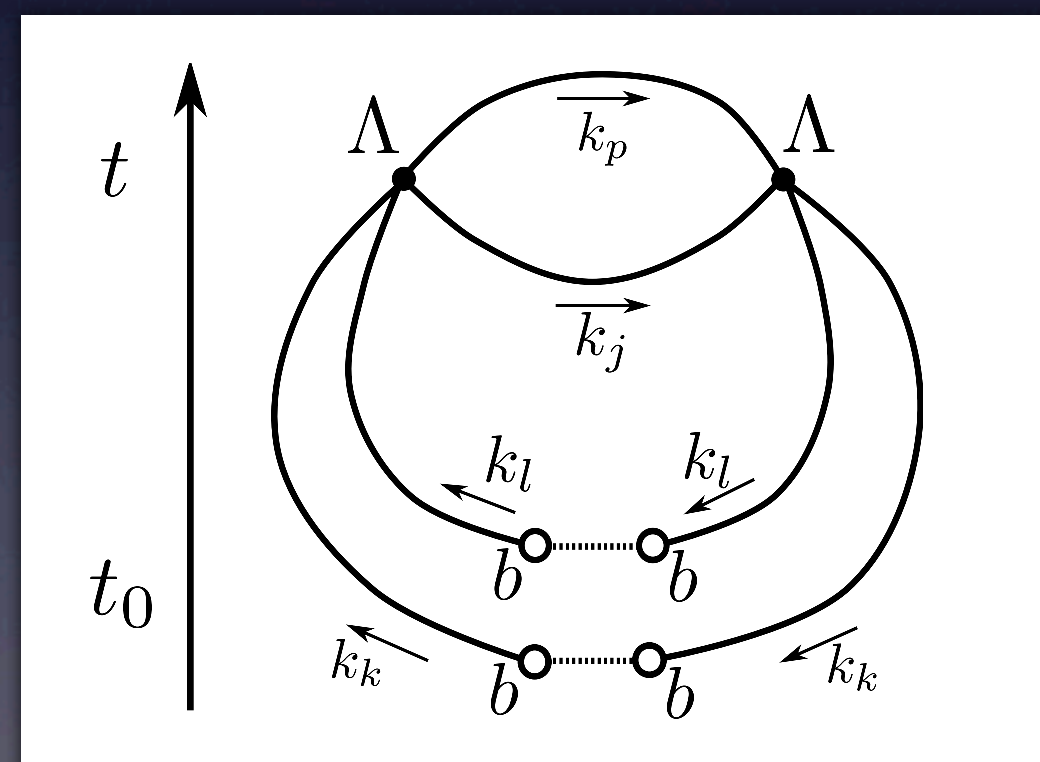
usual a+b→a+b process



1st order a+b→a+b (vanish)



1st order a+a→a+a (non-zero)



2nd order b+b→b+b (non-zero)

Interaction rate for modes outside the horizon

- For the opposite limit $|\mathbf{k_1} - \mathbf{k_3}| \ll k_H$ (possible only for $k_1, k_2, k_3, k_4 \ll k_H$)

$$F(t; |\mathbf{k_1} - \mathbf{k_3}|) \to \frac{\pi G m_a^2 c_s^2}{R^3(t) H^2(t)} \quad \Rightarrow \quad \Gamma \sim \frac{G m_a^2 n_a}{H^2}$$

- Similar result can be obtained when we consider the evolution of the background field $\phi_{\rm cl} = \langle \alpha_{\mathbf{p=0}} | \phi | \alpha_{\mathbf{p=0}} \rangle$

From the effective action for $\phi_{\rm cl}$

$$e^{i\Gamma_{\rm eff}[\phi_{\rm cl}]} = \int \mathcal{D}\phi' \mathcal{D}\zeta \mathcal{D}\gamma_{ij} e^{iS[\phi', \zeta, \gamma_{ij}; \phi_{\rm cl}]}$$

$$\Gamma_{\rm eff}[\phi_{\rm cl}] = \int d^4x R^3 \left[\frac{1}{2}\dot{\phi}_{\rm cl}^2 - \frac{1}{2} m_a^2 \phi_{\rm cl}^2 - \frac{1}{16 H^2 M_{\rm Pl}^2 \tilde{\epsilon}} \left(\dot{\phi}_{\rm cl}^2 + m_a^2 \phi_{\rm cl}^2 \right)^2 + \dots \right]$$

$$\Rightarrow \quad \ddot{\phi}_{\rm cl} + 3H\dot{\phi}_{\rm cl} + m_a^2 \phi_{\rm cl} \approx -\frac{c_s^2 m_a^2}{4 H^2 M_{\rm Pl}^2} \phi_{\rm cl} \left(\dot{\phi}_{\rm cl}^2 + m_a^2 \phi_{\rm cl}^2 \right)$$

$$\Rightarrow \quad \Gamma \sim \frac{\dot{N}}{N} \sim -\frac{c_s^2 m_a^3 \phi_{\rm cl}^2}{2 H^2 M_{\rm Pl}^2} \sim \frac{G m_a^2 n_a}{H^2}$$

where $N = \frac{R^3 \rho_{\rm free}}{m_a}, \qquad \rho_{\rm free} = \frac{1}{2}\dot{\phi}_{\rm cl}^2 + \frac{1}{2} m_a^2 \phi_{\rm cl}^2$

- Not the "transition" between superhorizon modes, but the "correction" to the evolution of the background field