Modular properties of 3D higher spin theory

(Based on 1308.2959)

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Wei Li (Max Planck Institute)

Modular Group and Modular Transformation

(Passive Point of View)

Complex plane $\xrightarrow{\text{quotient}}$ Torus

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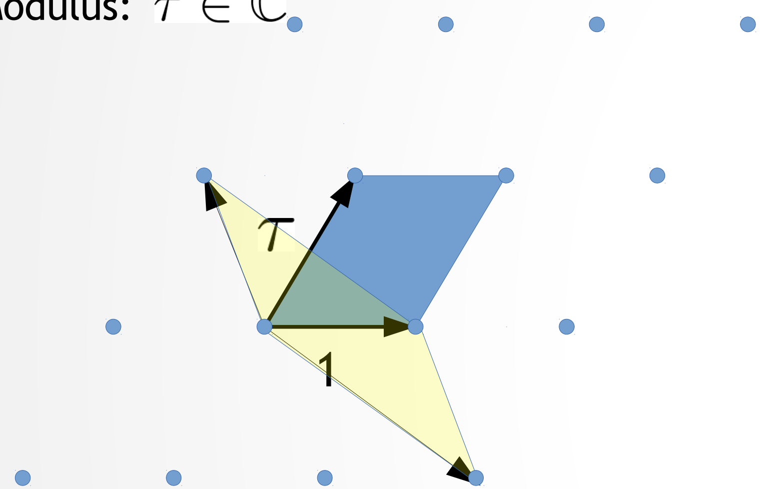

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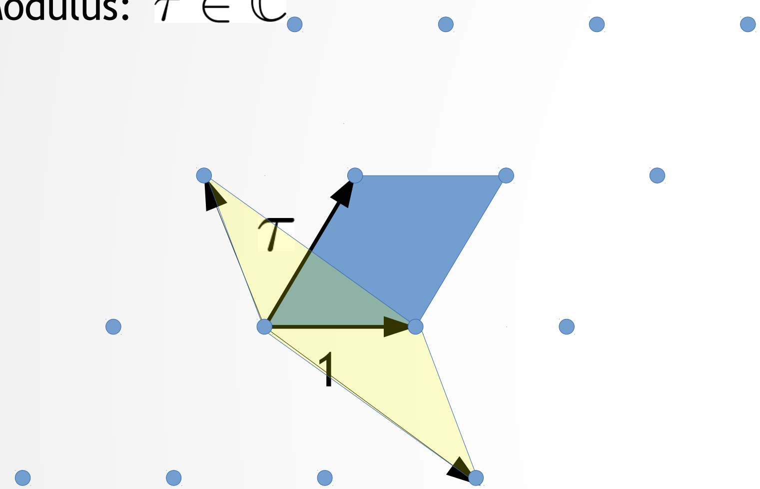

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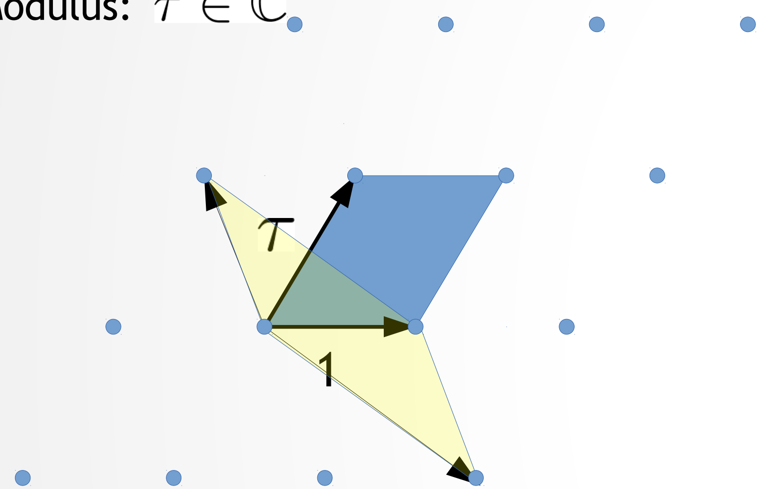

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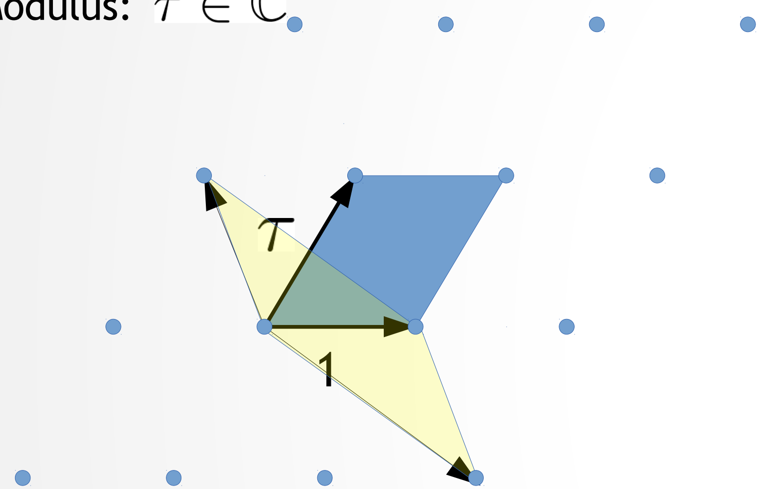

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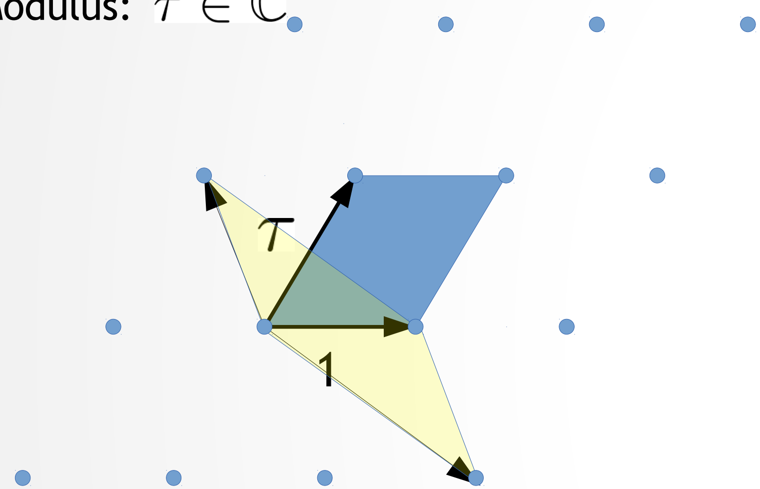

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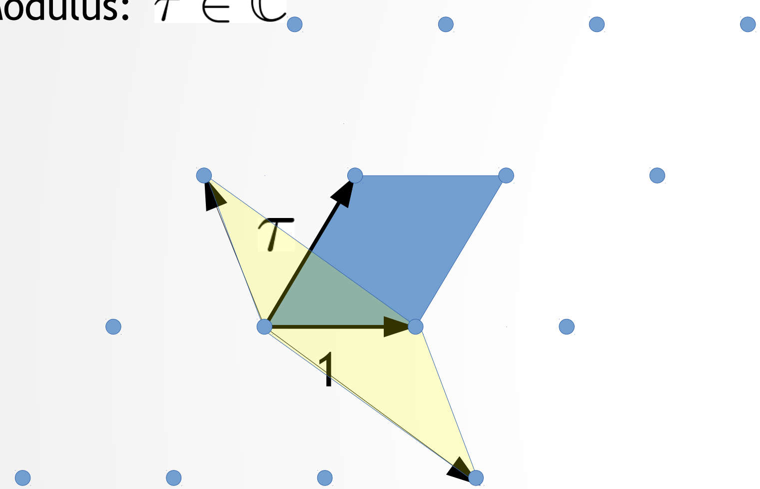

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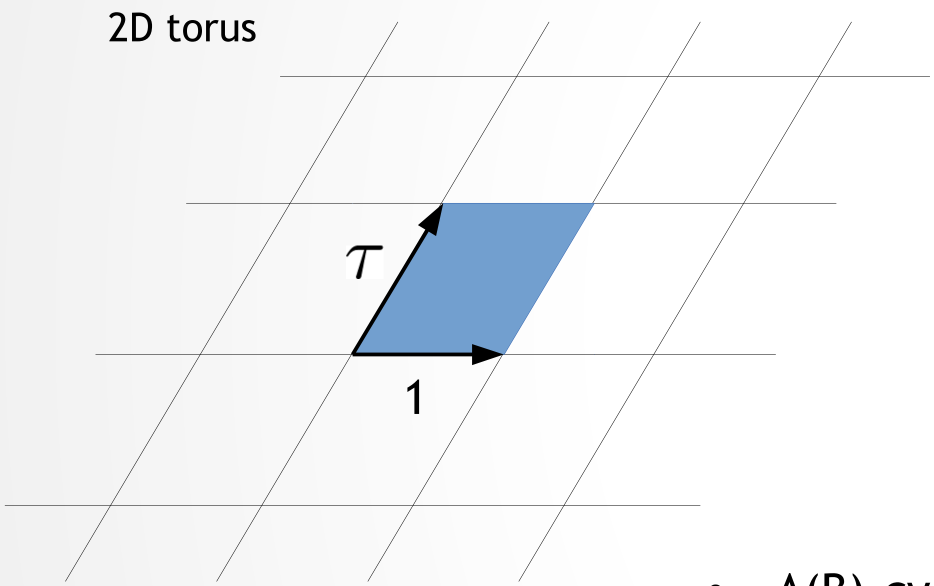
S-dual: $\tau \rightarrow \tau' = -\dfrac{1}{\tau}$

Solid Torus (A/B cycles)



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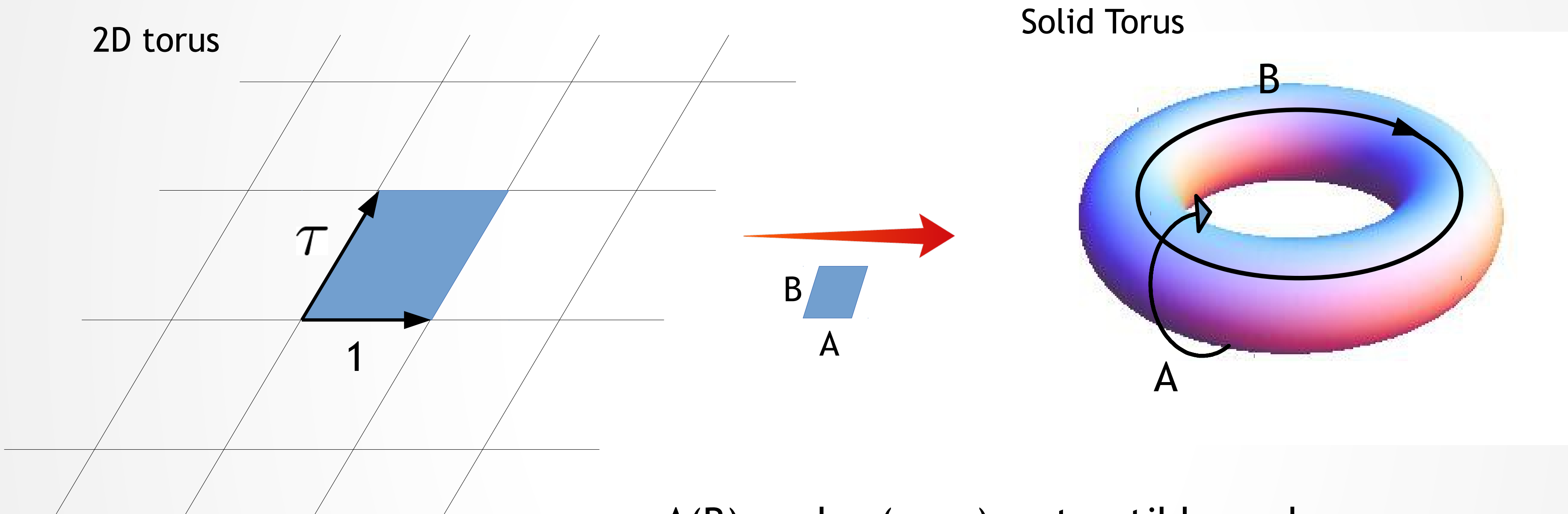

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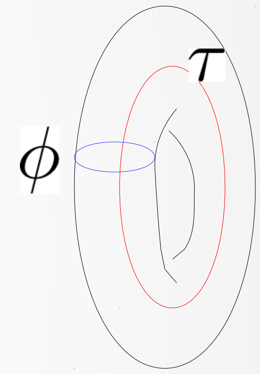
$$ds^2 = d\rho^2 + (e^{2\rho} + \frac{1}{16}e^{-2\rho})dz d\bar{z} - \frac{1}{4}(dz^2 + d\bar{z}^2) \qquad z \equiv \phi + it_E$$



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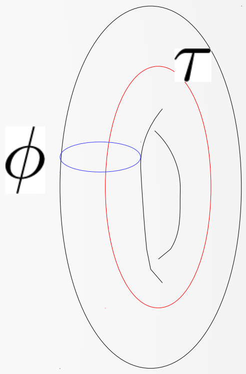

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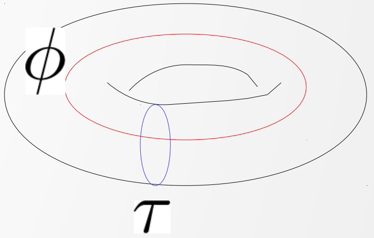

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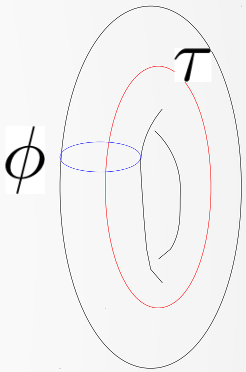

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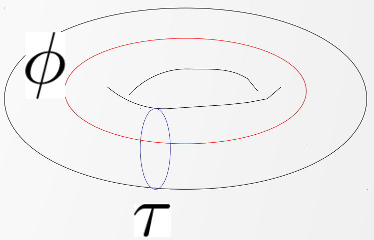

In general: $I^{(E)}_{\gamma}[\tau] = I^{(E)}_{AdS}[\frac{a\tau + b}{c\tau + d}]$

Vasiliev's Higher-Spin Theory [Vasiliev `91]

- An extension of ordinary gravity theory including an infinite tower of massless higher spin fields with spin $s \geq 3$ coupled non-linearly.
- The theory lives in AdS (or dS) space. The no-go theorems are evaded.
- In AdS3, the theory can be realized as a Chern-Simon gauge theory with an infinite-dimensional gauge algebra hs[λ].
- At λ=N, the algebra reduce to sl(N). The result theory is a nature generalization of the usual sl(2) Chern-Simon theory. This 3D Chern-Simon theory with sl(N) algebra will be the main topic of this talk.

Basics of 3D Higher Spin Theory

- In D=2+1(or 3), there is a gauge formulation of Einstein gravity in terms of the Chern-Simon Theory:

  The action of the Chern-Simon Theory:

$$S = S_{CS}[A] - S_{CS}[\bar{A}] \qquad S_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]$$

$$A, \bar{A} \in sl(2,R) \qquad k = \frac{l}{4G}$$

sl(2) algebra: $[L_m, L_n] = (m-n)L_{m+n}\,, \qquad m,n = -1,0,1.$

Equation of motion: $dA + A \wedge A = 0$

- A convenient gauge choice: $A = b^{-1} a\, b + b^{-1} db \qquad b = e^{\rho L_0}$

  $a = a_z dz + a_{\bar{z}} d\bar{z}$ is a gauge field lives on the boundary

  E.O.M. for a constant connection: $[a_z, a_{\bar{z}}] = 0$

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- Precise field content will depend on how one embed the gravity sector sl(2) into sl(N)

  Principal embedding:

  sl(N) generators → Embeded sl(2): $L_0\,,\ L_{\pm 1}$

  → Higher spin: $W_m^{(s)}$ $\quad s = 3, \cdots, N \quad m = -s+1, \cdots, s-1$

$$[L_m, W_n^{(s)}] = [(s-1)m - n]W_{m+n}^{(s)}$$

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- The only gauge invariant quantity: holonomy $\mathrm{Hol}_{\mathcal{C}}(A) \equiv \mathcal{P}e^{\oint_{\mathcal{C}} A}$

General Framework in sl(N) [de Boer, Jottar `13, Castro et al. `11]

- Highest/Lowest weight gauge convention:

Q and M are linear in charges and chemical potential respectively

$$a_z = L_1 + \mathbf{Q} \qquad \mathbf{Q} = \sum_{s=2}^{N} \frac{Q_s}{t^{(s)}} W^{(s)}_{-s+1}$$

$$a_{\bar{z}} = \mathbf{M} + (\text{terms} \sim W^{(s)}_{m \leq s-2}) \qquad \mathbf{M} = \frac{i}{2\tau_2} \sum_{s=3}^{N} \mu_s W^{(s)}_{s-1}$$

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- Smooth solutions are characterized by the holonomy condition along A-cycle:

$$\mathrm{Hol}_{\mathrm{A}}(A) \equiv \mathcal{P} e^{\oint_{\mathrm{A}} A} \in \text{ center of the SL(N)}$$

For a constant gauge field: $\mathrm{Hol}_{\mathrm{A}}(A) = b^{-1} e^{2\pi \omega_{\mathrm{A}}} b$

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- Condition constraint the vector of the eigenvalues of holonomy matrix: $\Lambda(\omega_{\mathrm{A}}) = i\ \vec n$

$$\vec n = (n_1, \ldots, n_N)\,, \quad n_i \in \begin{cases} \mathbb{Z} & N \text{ odd} \\ \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2} & N \text{ even} \end{cases}, \quad n_i \neq n_j \text{ for } i \neq j\,, \quad n_i + n_{N+1-i} = 0,$$

Modular Images of the Conical Surpluses

For a conical surplus, $\omega_{\mathrm{A}} = \omega_{\phi} = a_z + a_{\bar{z}}$

$$\text{CS:} \qquad i\ \vec{n} = \Lambda\left(\omega_{\phi}\left[\tau;\ \mu_s;\ Q_s\right]\right) = \Lambda\left(a_z\left[Q_s\right]\right) + \Lambda\left(a_{\bar{z}}\left[\tau;\ \mu_s;\ Q_s\right]\right)$$

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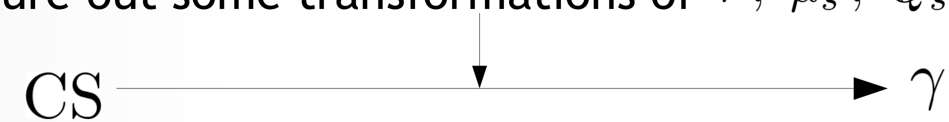
Goal: to figure out some transformations of $\tau\,,\ \mu_s\,,\ Q_s\,.$



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Modular transformation:

$$\tau \longmapsto \hat{\gamma}\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad \mu_s \longmapsto \frac{\mu_s}{(c\tau+d)^s}\,, \qquad Q_s \longmapsto (c\tau+d)^s Q_s\,,$$

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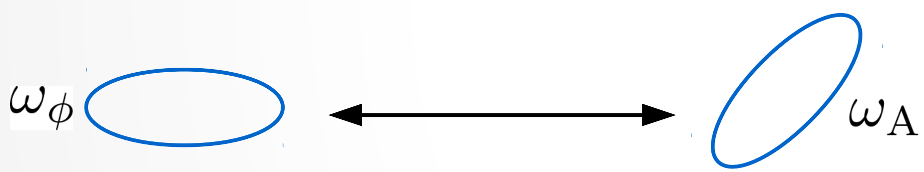

$$Q_t^{\mathrm{CS}} = q_t\left[\vec{n};\ \tau;\ \mu_s\right] \quad \Longleftrightarrow \quad Q_t^{\gamma} = \frac{1}{(c\tau+d)^t} q_t\left[\vec{n};\ \hat{\gamma}\tau;\ \frac{\mu_s}{(c\tau+d)^s}\right] \qquad t = 2, \ldots, N$$

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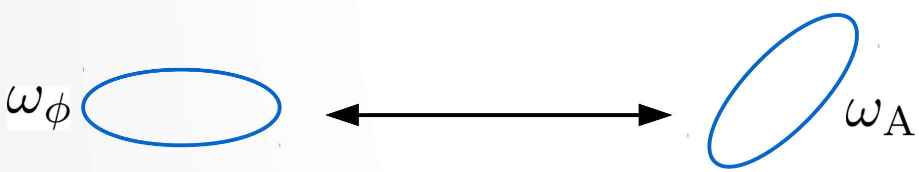

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Different solutions, different solid torus

Thermodynamics ("canonical" formalism) [de Boer, Jottar `13]

$$-\beta F\left[\tau;\ \mu_s\right] = \ln Z\left[\tau;\ \mu_s\right] \quad \approx \quad -\left. I^{(\mathrm{E})}\right|_{\text{on-shell}}$$

saddle point approximation

- Modulus τ act as the chemical potential of spin-2 charge
- μ_s: chemical potential for higher spin charge with s>2

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add boundary action to impose appropriate boundary condition

- Varying bulk action produce a boundary term: $\delta I^{(\mathrm{E})}_{\text{bulk}}[A]|_{\text{on-shell}} = -\frac{ik}{4\pi}\int_{\partial M} \mathrm{Tr}[a \wedge \delta a]$
- When varying the action, one need to vary τ (shape of the torus) explicitly. To do that, we can change the coordinate to the rigid torus and shift τ dependence to the gauge field, a, and then vary it.
- δa involves the variation of charges and chemical potentials including τ.

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Varying the whole action yield the desired form (including the part coming from $\bar{A}$):

$$\delta I^{(\rm E)}|_{\rm os} = \delta I^{(\rm E)}_{\rm bulk}|_{\rm os} + \delta I^{(\rm E)}_{\rm bndy}|_{\rm os} = -(2\pi i k)\left(T\ \delta\tau - \bar{T}\ \delta\bar{\tau} + \sum_{s=3}^{N}(Q_s\ \delta\mu_s - \bar{Q}_s\ \delta\bar{\mu}_s)\right)$$

$$T = \frac{1}{2}{\rm Tr}\left[(a_z)^2\right] + {\rm Tr}\left[a_z a_{\bar z}\right] - \frac{1}{2}{\rm Tr}\left[(\bar{a}_z)^2\right]$$

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- T is the energy momentum tensor conjugated to the modulus τ.
- T is ***not*** holomorphic and will depend on the higher spin charges if the chemical potential is not zero.
- In short, the highest/lowest weight gauge choice of the charge/chemical potential separation plus this particular boundary action yield a consistent thermodynamic system.

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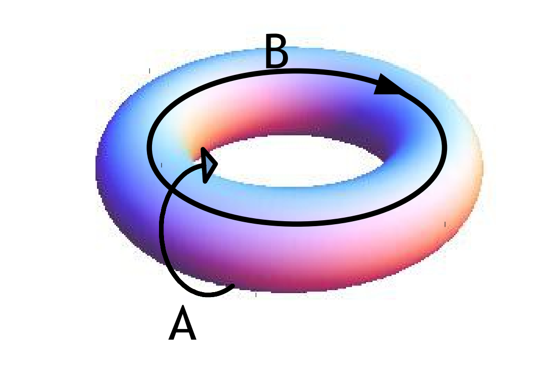

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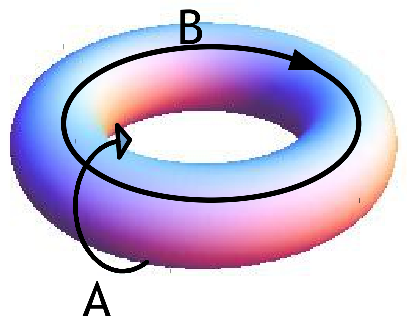

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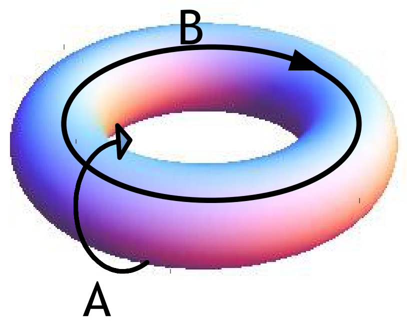

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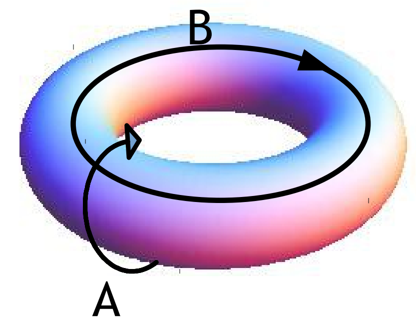

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- The free energy is: $-\beta F = -(I^{(\mathrm{E})}_{\mathrm{bulk}}|_{\mathrm{os}} + I^{(\mathrm{E})}_{\mathrm{bndy}}|_{\mathrm{os}})$

Modular Invariant Full Partition Function

- Simple result (obtained non-trivially): $F^{\gamma}\left[\vec{n};\ \tau;\ \mu_s\right] = F^{\mathrm{CS}}\left[\vec{n};\ \hat{\gamma}\tau;\ \frac{\mu_s}{(c\tau+d)^s}\right]\ .$

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