A regular modified version of Schwarzschild metric

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1. Introduction

The Schwarzschild spacetime - the exact spherically-symmetric solution of Einstein's equations.

$r = 0$ - singularity $(R^a{}_{bcd} \to \infty)$,

Dymnikova (GRG 24, 235 (1992); PLB 472, 33 (2000)) - a nonsingular BH solution, which is deS at small r but KS at large r. It is sourced by an anisotropic $T^a{}_b$ with

$$T^0{}_0 = T^1{}_1 \quad \text{and} \quad T^2{}_2 = T^3{}_3,$$

i.e. $\rho = -p_r$ and $p_\theta = p_\varphi$.

Bonanno and Reuter (PRD 62, 043008 (2000), arXiv: hep-th/0002196) studied the quantum gravitational effects on the dynamics of geometry.

The BH evaporation stops when $M_{cr} \simeq m_P \longrightarrow$ a cold soliton-like remnant is formed $\longrightarrow$

the classical singularity at $r=0$ is removed $\longrightarrow$ the final state is an extremal charged BH space. Nicolini (J. Phys. A 38, L631 (2005); IJMPA 24, 1229 (2009)) used the non-commutativity of spacetime to study the radiating KS black holes.

The evaporation ends at $T=0$ extremal BH with no singularity at $r=0$ (a regular dS core) with $p_r = -\rho$, a kind of "quantum" outward push induced by non-commuting coordinate quantum fluctuations.

Xiang et al (arXiv:1305.3851 [gr-qc]) proposed a model where BH will not evaporate entirely $\longrightarrow$ a Planck order remnant remains. The modified KS geometry has a finite Kretschmann scalar.

2. Modified Schwarzschild metric.

Consider Xiang et al metric

$$ds^2 = -(1+2\phi)dt^2 + \frac{dr^2}{1+2\phi} + r^2 d\Omega^2,$$

with $\phi(r) = -\frac{m}{r} e^{-\varepsilon(r)}$, $\varepsilon(r) > 0$

$\varepsilon(r)$ – the damping factor.

assumption: take $\phi(r) = -\frac{m}{r} e^{-\frac{\kappa}{r}}$

Choose $\kappa > 0$ such that $g_{tt} = 0$ has only one root. We have

$$e^{\kappa x} = 2mx \qquad (x = 1/r)$$

Hence, $x_H = 1/\kappa = e/2m$. Therefore

$$ds^2 = -\left(1 - \frac{2m}{r} e^{-\frac{2m}{re}}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} e^{-\frac{2m}{re}}} + r^2 d\Omega^2$$

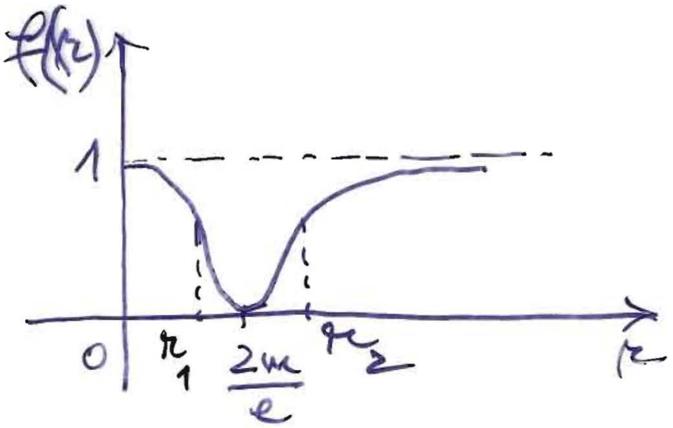
with a horizon at $r_H = 1/x_H = 2m/e$.

Properties:

$f(r) \equiv -g_{tt} > 0$ (no signature flip)

$f(r) \xrightarrow[r \to \infty]{r \to 0} 1$

$f(r)$ inflexion points: $r = \frac{m}{e}(2 \pm \sqrt{2})$



$$r_1 = r_H - \frac{m\sqrt{2}}{e}$$

$$r_2 = r_H + \frac{m\sqrt{2}}{e}$$

$$f'\left(\frac{2m}{e}\right) = 0$$

Take $u^b = (1/\sqrt{f}, 0, 0, 0)$ – a static observer

Hence

$$a^b \equiv u^a \nabla_a u^b = \left(0, \frac{m\left(1-\frac{r_H}{r}\right)}{r^2} e^{-\frac{r_H}{r}}, 0, 0\right)$$

with $a^r \xrightarrow[r \to r_H]{r \to 0} 0.$

$r < r_H$ — repulsion.

$r > r_H$ — attraction.

The proper acceleration is

$$a \equiv \sqrt{a^b a_b} = \frac{m\left|1-\frac{r_H}{r}\right| e^{-\frac{r_H}{r}}}{r^2\sqrt{1-\frac{2m}{r}e^{-\frac{r_H}{r}}}}$$

The surface gravity is given by

$$\varkappa = \sqrt{a_b a^b}\cdot\sqrt{-g_{tt}}\Big|_H = 0,$$

$\longrightarrow$ $T_H = 0$. (no Hawking radiation).

$\longrightarrow$ extremal BH ("frozen horizon).

However, a is finite on H:

$$\sqrt{a^b a_b}\Big|_H = \frac{e\sqrt{2}}{4m}$$

Note that: $R^{abcd}R_{abcd}$, $R^a_{\ a} = \frac{8m^3}{e^2 r^5} e^{-\frac{r_H}{r}}$ are everywhere nonsingular.

3. Anisotropic stress tensor

What $T^a_{\ b}$ do we need on the RHS of Einstein's equations?

One finds that

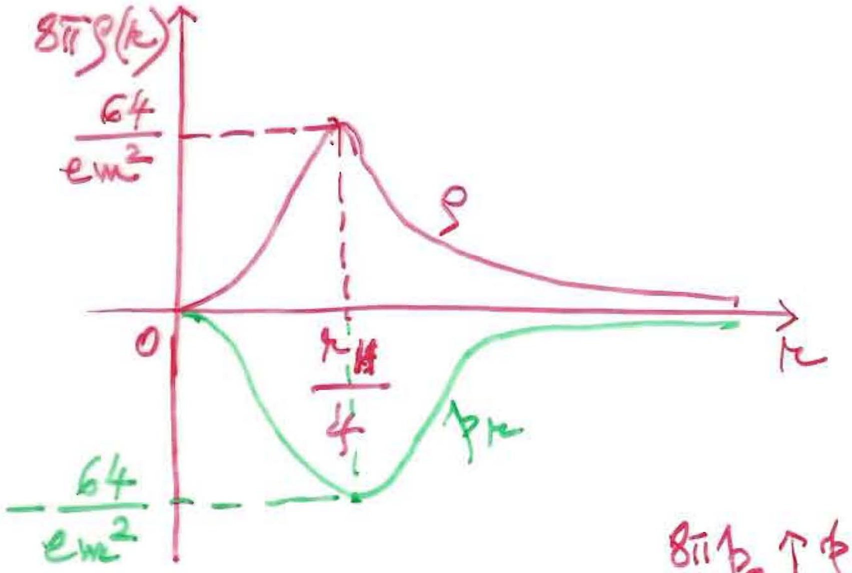
$$\rho = -p_r = \frac{m^2}{2\pi e r^4}\, e^{-\frac{2m}{er}}$$

$$p_\theta = p_\varphi = \frac{m^2}{2\pi e r^4}\left(1 - \frac{m}{er}\right) e^{-\frac{2m}{er}}$$

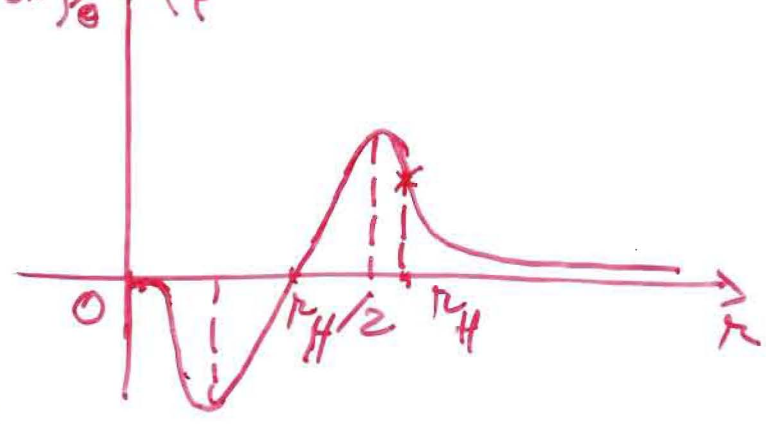
$p_\theta \neq p_r \longrightarrow$ anisotropic fluid.

$p_r = -\rho \longrightarrow$ as for deS space (DE)

$p_r, p_\theta, p_\varphi, \rho$ — are finite when $r \to 0$ or $r \to \infty$.



at $r \gg r_H$,
$\rho \approx p_\theta$.
$\rho > 0, \forall r > 0$



$p_\theta|_H = \frac{1}{2}\rho_H = \frac{e^2}{64\pi m^2}$

$p_\theta(r_H/2) = 0$.

$|p_\theta| \geq \rho$, if $r \leq r_H/4$.

For $r \gg r_H$, $\rho \simeq (1/2\pi)\cdot\left(\frac{m}{r^2}\right)^2$, and

$a \simeq \frac{m}{r^2}$, namely $\rho \sim a^2$, as in Newtonian gravitation and Electrostatics. We have, indeed, for a point charge q at rest

$$T^a_{(e)b} = \frac{q^2}{8\pi r^4}(-1,-1,1,1)$$

The Poisson equation for $\phi(r)$

$$\nabla^2\phi(r) \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{4m^2}{e r^4}\left(1-\frac{m}{er}\right)e^{-\frac{2m}{er}}$$

Or $\nabla^2\phi(r) = 4\pi \cdot 2p_\theta \neq 4\pi\rho$. In fact, for a perfect fluid

$\rho \to \rho + 3p$ (Padmanabhan, PRD81, 124040 (2010)).

In our situation

$\rho + 3p \to \rho + p_r + p_\theta + p_\varphi = 2p_\theta$, and therefore we have $2p_\theta$ as source of the field.

4. Komar energy

It is given by

$$W = 2\int\left(T_{ab} - \frac{1}{2}g_{ab}T\right)N u^a u^b \sqrt{\gamma}\, d^3x,$$

where N – the lapse function, γ – the determinant of the spatial 3-metric

We have, for our metric

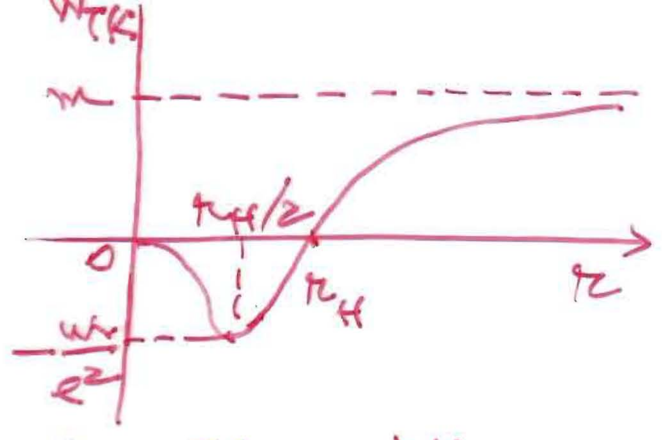
$$W_{TK} = \int_0^r \frac{4m^2}{e r^4}\left(1-\frac{2m}{er}\right) e^{-\frac{2m}{er}}\, r^2 dr \equiv W(r)$$

The substitution $x = 1/r$ yields

$$W_{TK} = m\left(1-\frac{r_H}{r}\right) e^{-\frac{r_H}{r}},$$

with $W \xrightarrow{r \to 0} 0$ and $W \xrightarrow{r\to\infty} m$.

Hence, the total Komar energy of the fluid equals the mass of the central body (see also Bonanno and Reuter). Moreover, W becomes negative for $r < r_H$ and $W_{min} = -m/e^2$ at $r = r_H/2$.



$W_{ADM} = m$

$$W_{ADM}(r) = \int_0^r 4\pi r^2 \rho(r)\, dr = m e^{-\frac{2m}{er}}$$

For the Misner-Sharp (MS) mass, we have

$$1 - \frac{2 W_{MS}}{R} = g^{ab} R_{,a} R_{,b}$$

(R - the areal radius), whence we find that $W_{MS}(r) = W_{ADM}(r)$.

Notice that $W_{TK} = a^r \cdot r^2$, a dependence obtained recently by Skakala (arXiv: 1308.2550 [gr-qc]) for the quasi-local mass inside the boundary

$\partial\Sigma_t \ (r=R)$:

$$M = \frac{1}{4\pi}\int_{\partial\Sigma_t} N\, n_b\, a^b \sqrt{\sigma}\, d^2x ,$$

where $n^b = (0, \sqrt{f}, 0, 0)$ and $\sqrt{\sigma} = r^2 \sin\theta$.

Even though W_{TK} and T cancel on $r = r_H$, the entropy $S = |W_{TK}|/2T$ is finite

$$S_H = \frac{|W|}{2T}\Big|_H = \frac{|W|\pi r^2}{\kappa}\Big|_H = \pi r_H^2$$

and we see that $S_H = A_H/4$.

5. Horizon stress tensor

We have a jump of the extrinsic curvature K_{ab} when the horizon is crossed.

$\longrightarrow$ a nonzero surface stress tensor.

By means of

$$K_{ab} = -\frac{f'}{2\sqrt{f}} u_a u_b + \frac{\sqrt{f}}{r} \sigma_{ab} ,$$

(Kolekar and Padmanabhan, PRD 85, 024004; H.C. PLA 376, 2817 (2012), ISMP Conf. Ser. 3, 455 (2011)), one obtains

$$K_{tt} = -\frac{f'}{2}\sqrt{f} \ ; \ K_{\theta\theta} = \frac{K_{\varphi\varphi}}{\sin^2\theta} = r\sqrt{f} ,$$

$$K \equiv h^{ab} K_{ab} = \frac{f'}{2\sqrt{f}} + \frac{2\sqrt{f}}{r} ,$$

where $u_a = (\sqrt{f}, 0, 0, 0)$; $h_{ab} = g_{ab} - n_a n_b$.

is the induced metric on $r = \text{const.}$ surface. $n_a = (0, 1/\sqrt{f}, 0, 0)$ is its normal vector and $g_{ab} = h_{ab} + u_a u_b$.

We suppose the horizon is a membrane with a stress tensor S_{ab} on it

$$S_{ab} = \rho_H u_a u_b + p_H g_{ab}.$$

S_{ab} is obtained from Lanczos eq.

$$8\pi S_{ab} = [K h_{ab} - K_{ab}],$$

where $[K_{ab}] = K^+_{ab} - K^-_{ab}$.

For the mean curvature we have

$$K = \frac{m\left(1 - \frac{r_H}{r}\right) e^{-\frac{r_H}{r}}}{r^2 \sqrt{1 - \frac{2m}{r} e^{-\frac{r_H}{r}}}} + \frac{2}{r}\sqrt{1 - \frac{2m}{r} e^{-\frac{r_H}{r}}},$$

taken at $r = r_H$.

The 2nd term has no jump when the horizon is crossed. For the 1st term we have different side limits as $r \to r_H$.

One obtains $K_+ = -K_- = e\sqrt{2}/4m$, which is the same as $(a^b n_b)_+ = -(a^b n_b)_-$ on H.

Therefore, $S_{tt} = 0$, $S_{\theta\theta} = m/2\pi e\sqrt{2}$.

Hence

$$\rho_H = 0, \quad p_H = \frac{e}{8\pi\sqrt{2}m}.$$

($\rho_H = 0$ has also been obtained by Kolekar et al. (PRD 85, 064031 (2012))

However, their surface pressure $p_H \to \infty$ because only the denominator of $f'/2\sqrt{f}$ is null on H. In our model $(f'/2\sqrt{f})_H$ is finite. For a Solar mass BH of $m = 2\cdot 10^{33}$ g, one obtains

$$p_H = \frac{c^6}{G^2}\cdot\frac{e}{8\pi\sqrt{2}m} \simeq 10^{42}\ \text{erg}/\text{cm}^2.$$

6. Near horizon approximation

We develop $f(r)$ in a power series around $r = r_H$, up to the 2nd order

$$f(r) \simeq f(r_H) + (r - r_H) f'(r_H) + \frac{1}{2}(r - r_H)^2 f''(r_H)$$

We have

$$f''(r) = -\frac{4m}{r^3}\left(1 - \frac{4m}{er} + \frac{2m^2}{e^2 r^2}\right) e^{-\frac{2m}{er}},$$

$f(r_H) = 0$, $f'(r_H) = 0$. Hence

$$f(r) = \frac{(r - r_H)^2}{2 r_H^2}\ ;\left(f''(r_H) = \frac{1}{r_H^2}\right)$$

Keeping in mind that $r_H^2 = 1/2a_H^2$, the near horizon metric appears as

$$ds^2 = -a_H^2 (r - r_H)^2 dt^2 + \frac{dr^2}{a_H^2 (r - r_H)^2} + r_H^2\, d\Omega^2$$

This is equivalent to the near-horizon approximation of the

Bonanno and Reuter critical quantum BH. It resembles the Robinson - Bertotti space, for the product $AdS_2 \times S^2$. (S^2 curvature is r_H but the AdS_2 curvature is $r_H \cdot \sqrt{2}$).
Therefore, the near-horizon metric is not conformally flat.

7. Conclusions

- the model is classical.
- the modified KS metric is non-singular everywhere.
- there is no signature flip at $r = r_H$.
- the source is an anisotropic fluid with $p_r = -\rho$
- p_θ and p_φ fluctuate for $r < r_H$ but for $r \gg r_H$ we have $p_\theta = \rho$.
- $r < r_H$ - repulsion, $r > r_H$ - attraction
- $\varkappa = 0$, $\longrightarrow T_{BH} = 0$
- $W_{TK}|_H = 0$, however $S = A_H / 4$.
- there is a jump of the extrinsic curvature on the horizon

$a^r|_H = 0$, but $\sqrt{a_b a^b}|_H = e\sqrt{2}/4m$.

- for $r \gg r_H$, the stress tensor resembles the Maxwell tensor

$$\frac{q^2}{8\pi r^4}(-1, -1, 1, 1).$$