Heterotic Model Building: 16 Special Manifolds

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Outline

- Motivation (Toric Calabi-Yau , KS list & Heterotic GUT models building)
- Physical Constraints (Anomaly cancellation , Poly-stability, & GUT spectrum)
- Search Algorithm and Result Analysis
- Conclusion and Outlook

Motivation

1 **Algorithmic string compactification**
A combination of the latest developments in computer algebra and algebraic geometry have been utilized to study the compactification of the heterotic string on smooth Calabi-Yau three-folds with holomorphic vector bundles satsifying the Hermitian Yang-Mills equations. *arXiv:hep-th/0702210 arXiv:1307.4787*

2 **Kreuzer-Skarke (KS) list of Toric Varieties**
These total 473,800,776 ambient toric four-folds, each coming from a reflexive polytope in 4-dimensions. Thus there are at least this many Calabi-Yau three-folds. *arXiv, hep-th/0002240*

Motivation

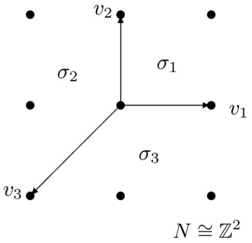
3. **The procedure of heterotic compactification**
   Given a generically simply connected Calabi-Yau three-fold $\widetilde{X}$, we need to find a freely-acting discrete symmetry group Γ, so that $\widetilde{X}/\Gamma$ is a smooth quotient. We then need to construct stable Γ-equivariant bundles $\widetilde{V}$ on the cover $\widetilde{X}$ so that on the quotient $X = \widetilde{X}/\Gamma$, $\widetilde{V}$ descends to a bona fide bundle V. It is the cohomology of V, coupled with Wilson lines valued in the group Γ, that gives us the particle content which we need to compute. In other words, we need to find Calabi-Yau manifolds X with non-trivial fundamental group $\pi_1(X) \simeq \Gamma$. Often, the manifolds $\widetilde{X}$ and X are referred to as "upstairs" and the "downstairs" manifolds, to emphasize their quotienting relation.
4. Of the some 500 million manifolds in the KS list, only 16 have non-trivial fundamental group. *arXiv, math/0505432*

The CY Construction over Toric Varieties

n dim. toric variety is generalization of WP space: $X = \dfrac{\mathbb{C}^r - Z}{(\mathbb{C}^*)^{r-n} \times G}$

Ex. $\mathbb{CP}^2 = \dfrac{\mathbb{C}^3 - \{z_1 = z_2 = z_3 = 0\}}{(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)}$ $\quad \lambda \in \mathbb{C}^*$

$$1\, v_1 + 1\, v_2 + 1\, v_3 = 0$$



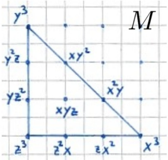
variety X	⟷	fan Σ
z_i, $D_i = \{z_i = 0\}$	⟷	v_i
$D_i \cap D_j \neq 0$	⟷	$v_i,\ v_j \in \sigma$
$z_i \sim \lambda^{q_i} z_i$	⟷	$\sum_i q_i v_i = 0$

Nils-Ole Walliser
String Phenomenology 2011, August 24th, 2011

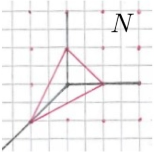
The CY Construction over Toric Varieties

Polytope dual to *P** $P = \{x \in M_{\mathbb{R}} | \langle x, y \rangle \geq -1 \quad \forall y \in P^*\}$

Dual pair of *reflexive* lattice polytopes: $P \subset M \,, \quad P^* \subset N$



mirror sym.



N-lattice : toric variety from fan over faces of *P** : $\Sigma \longrightarrow X_\Sigma$

M-lattice : Laurent polynomials $f = \sum_{m \in P \cap M} c_m \chi^m$ ⟶ CY hypersurf.

$$h_{11}(\chi) = h_{21}(\chi^*)$$
$$= l(P^*) - 1 - \dim P - \sum_{\mathrm{cod}(\theta^*)=1} l^*(\theta^*) + \sum_{\mathrm{cod}(\theta^*)=2} l^*(\theta^*) l^*(\theta)$$

[Batyrev '93]

The CY Construction over Toric Varieties

the downstairs manifold X_3,

$$\left(\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \hline 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{array}\right)$$

$n = 8$ vertices that lead to 8 homogeneous coordinates $x_{\rho=1,\cdots,8}$.

$$\begin{array}{c} x_2^2x_3^2x_4^2x_8^2\,,\ x_2^2x_3^2x_5^2x_8^2\,,\ x_1x_2^2x_3^2x_4x_5x_8\,,\ x_1^2x_2^2x_3^2x_4^2\,,\ x_2^2x_3x_4x_5x_6x_8^2\,,\ x_1x_2^2x_3x_4^2x_6x_8\,, \\ x_2^2x_4^2x_6^2x_8^2\,,\ x_2x_3^2x_4x_5x_7x_8^2\,,\ x_1x_2x_3^2x_4^2x_7x_8\,,\ x_2x_3x_4^2x_6x_7x_8^2\,,\ x_3^2x_4^2x_7^2x_8^2\,,\ x_1^2x_2^2x_3^2x_5^2\,, \\ x_1x_2^2x_3x_5^2x_6x_8\,,\ x_1^2x_2^2x_3x_4x_5x_6\,,\ x_2^2x_5^2x_6^2x_8^2\,,\ x_1x_2^2x_4x_5x_6^2x_8\,,\ x_1^2x_2^2x_4^2x_6^2\,,\ x_1x_2x_3^2x_5^2x_7x_8\,, \\ x_1^2x_2x_3^2x_4x_5x_7\,,\ x_2x_3x_5^2x_6x_7x_8^2\,,\ x_1x_2x_3x_4x_5x_6x_7x_8\,,\ x_1^2x_2x_3x_4^2x_6x_7\,,\ x_2x_4x_5x_6^2x_7x_8^2\,, \\ x_1x_2x_4^2x_6^2x_7x_8\,,\ x_3^2x_5^2x_7^2x_8^2\,,\ x_1x_3^2x_4x_5x_7^2x_8\,,\ x_1^2x_3^2x_4^2x_7^2\,,\ x_3x_4x_5x_6x_7^2x_8^2\,,\ x_1x_3x_4^2x_6x_7^2x_8\,, \\ x_4^2x_6^2x_7^2x_8^2\,,\ x_1^2x_2^2x_5^2x_6^2\,,\ x_1^2x_2x_3x_5^2x_6x_7\,,\ x_1x_2x_5^2x_6^2x_7x_8\,,\ x_1^2x_2x_4x_5x_6^2x_7\,,\ x_1^2x_3^2x_5^2x_7^2\,, \\ x_1x_3x_5^2x_6x_7^2x_8\,,\ x_1^2x_3x_4x_5x_6x_7^2\,,\ x_5^2x_6^2x_7^2x_8^2\,,\ x_1x_4x_5x_6^2x_7^2x_8\,,\ x_1^2x_4^2x_6^2x_7^2\,,\ x_1^2x_5^2x_6^2x_7^2\,. \end{array}$$

Kahler Cone

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Intersection Ring

$$D_1 = J_4,\ D_2 = J_3,\ D_3 = J_2,\ D_4 = J_1,\ D_5 = J_1,\ D_6 = J_2,\ D_7 = J_3,\ D_8 = J_4\ ,$$

$$d_{123}(X_3) = d_{124}(X_3) = d_{134}(X_3) = d_{234}(X_3) = 1$$

Intersection Numbers

$$h^{1,1}(X_3) = 4, \quad h^{1,2}(X_3) = 36\ ,$$

Topological Invariants

The CY Construction over Toric Varieties

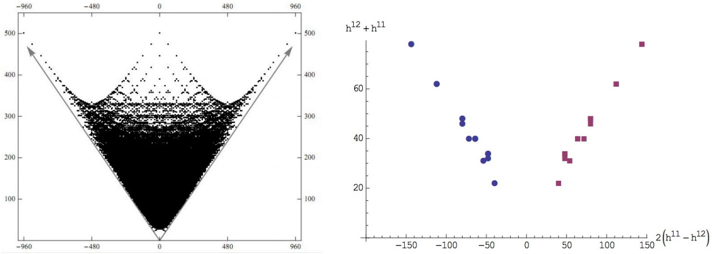


Figure 1: *The Hodge number plot:* $\{2(h^{1,1} - h^{2,1}), h^{1,1} + h^{2,1}\}$. *The left figure is for all the Calabi-Yau three-folds known to date and the right is for the sixteen non-simply-connected Calabi-Yau three-folds* X_i *as well as their mirrors; the blue round dots are for the original sixteen and the purple squares are for the mirrors.*

Physical Constraints

Bundle Structure:
we would like to consider Whitney sums of line bundles of the form

$$V = \bigoplus_{a=1}^{n} L_a \ , \quad L_a = \mathcal{O}_X(\mathbf{k}_a) \ , \tag{1}$$

which leads, generically, to the structure group $G = S(U(1)^n)$. For $n = 4, 5$ this structure group embeds into E_8 via the subgroup chains $S(U(1)^4) \subset SU(4) \subset E_8$ and $S(U(1)^5) \subset SU(5) \subset E_8$, respectively. This results in the commutants $H = SO(10) \times U(1)^3$ for $n = 4$ and $H = SU(5) \times U(1)^4$ for $n = 5$.

Physical Constraints

Anomaly Cancellation:

In general, anomaly cancelation can be expressed as the topological condition

$$\mathrm{ch}_2(V) + \mathrm{ch}_2(\hat{V}) - \mathrm{ch}_2(TX) = [C] \ , \tag{2}$$

A simple way to guarantee that this condition can be satisfied is to require that

$$\mathrm{c}_2(TX) - \mathrm{c}_2(V) \in \mathrm{Mori}(X) \ , \tag{3}$$

To compute the the second Chern class $c_2(V) = c_{2r}(V)\nu^r$ of line bundle sums (1) we can use the result

$$c_{2r}(V) = -\frac{1}{2} d_{rst} \sum_{a=1}^{n} k_a^s k_a^t \ , \tag{4}$$

Physical Constraints

Poly-stability:
In order to make the models consistent with supersymmetry, we need to verify that the sum of holomorphic line bundles is poly-stable.
Poly-stability of a bundle (coherent sheaf) $\mathcal{F}$ is defined by means of the *slope*

$$\mu(\mathcal{F}) \equiv \frac{1}{\mathrm{rk}(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J \wedge J \ , \tag{5}$$

The bundle $\mathcal{F}$ is called *poly-stable* if it decomposes as a direct sum of stable pieces,

$$\mathcal{F} = \bigoplus_{a=1}^{m} \mathcal{F}_a \ , \tag{6}$$

Since $c_1(V) = 0$, we have $\mu(V) = 0$ and, hence, the slopes of all constituent line bundles L_a must vanish.

Physical Constraints

GUT SU(5):
In this case we start with a line bundle sum (1) of rank five ($n = 5$) and associated structure group $G = S(U(1)^5)$. This leads to a four-dimensional gauge group $H = SU(5) \times S(U(1)^5)$. The four-dimensional spectrum consists of the following $SU(5) \times S(U(1)^5)$ multiplets:

$$\mathbf{10}_a \ , \ \ \overline{\mathbf{10}}_a \ , \ \ \overline{\mathbf{5}}_{a,b} \ , \ \ \mathbf{5}_{a,b} \ , \ \ \mathbf{1}_{a,b} \ . \qquad (7)$$

The most basic phenomenological constraint to impose on this spectrum is chiral asymmetry of three in the $\mathbf{10}$–$\overline{\mathbf{10}}$ sector. This translates into the condition

$$\mathrm{ind}(V) = -3 \ ,$$

$\overline{\mathbf{10}}$ multiplets and their standard-model descendants are phenomenologically unwanted we should impose that $\mathrm{ind}(L_a) \leq 0$ for all a.

Physical Constraints

GUT SU(5):

A similar argument can be made for the $\overline{\mathbf{5}}$–$\mathbf{5}$ multiplets. We should require that $\text{ind}(L_a \otimes L_b) \leq 0$ for all $a < b$ which implies that

$$-3 \leqslant \text{ind}(L_a \otimes L_b) \leqslant 0 \; , \tag{8}$$

Table 1 summarizes both the consistency constraints explained earlier and the phenomenological constraints discussed in this subsection.

Physics	Background geometry
Gauge group	$c_1(V) = 0$
Anomaly	$c_2(TX) - c_2(V) \in$ Mori(X)
Supersymmetry	$\mu(L_a) = 0$, for $1 \leq a \leq 5$
Three generations	$\text{ind}(V) = -3$
No exotics	$-3 \leq \text{ind}(L_a) \leq 0$, for $1 \leq a \leq 5$; $-3 \leq \text{ind}(L_a \otimes L_b) \leq 0$, for $1 \leq a < b \leq 5$

Physical Constraints

GUT SU(4):

Table 2 summarizes the consistency constraints explained earlier and the phenomenological constraints discussed above. These constraints will be used to classify rank four line bundle sums on our 16 manifolds.

Physics	Background geometry
Gauge group	$c_1(V) = 0$
Anomaly	$\mathrm{ch}_2(TX) - \mathrm{ch}_2(V) \in$ Mori(X)
Supersymmetry	$\mu(L_a) = 0,$ for $1 \le a \le 4$
Three generations	$\mathrm{ind}(V) = -3$
No exotics	$-3 \le \mathrm{ind}(L_a) \le 0,$ for $1 \le a \le 4$

Table : *Consistency and phenomenological constraints on rank four line bundles*

Search Algorithm

- We firstly generate all the single line bundles, $L = \mathcal{O}_X(\mathbf{k})$ with entries k^r in a certain range and with their index between -3 and 0. Then we compose these line bundles into rank four or five sums imposing the constraints detailed in Table 1 and 2, respectively, as we go along and at the earliest possible stage.
  The other issue is related to multiple triangulations, or multiple phases, which can arise when de-singularising the Calabi-Yau manifolds.
- Indeed, X_6 and X_{14} can be desingularised in two and three different ways, respectively. In general, the intersection ring can depend on which phase is considered. However, in cases where different phases carry the same intersection data they essentially describe a single manifold and we should, therefore, join the corresponding Kähler cones.

Results

- SU(5) Amongst the favourable base manifolds $X_{i=1,\cdots,14}$, only X_1 has Picard number 1, X_2 and X_4 have Picard number 2, X_5, X_6, X_7, X_8, X_{14} have Picard number 3, and X_3, X_9, X_{10}, X_{11}, X_{12}, X_{13} have Picard number 4. It turns out that viable models arise on all the six manifolds with Picard number 4 and on two out of the five manifolds with Picard number 3, namely X_6 and X_{14}, in total 122 models.
- SU(4) It turns out that amongst the five Picard number 3 manifolds, X_7 does not admit any viable models, and the other four, X_5, X_6, X_8, X_{14} admit 5, 13, 9, 28 bundles, respectively. For all those cases, the scan has saturated according to our criterion and the complete set of viable models has been found. For the other six manifolds X_3, X_9, X_{10}, X_{11}, X_{12}, X_{13}, all with Picard number four, only X_9 is complete and admits 2207 bundles.

Examlple: X_9

$$\left(\begin{array}{cccccccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 & \tilde{x}_7 & \tilde{x}_8 \\ \hline 3 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \hline -4 & 4 & 0 & 0 & 0 & 0 & 2 & -2 \\ -1 & 2 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & -2 & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right)$$

$$L_1 = \mathcal{O}_X(-4,0,1,1), \;\; L_2 = \mathcal{O}_X(1,3,-1,-1), \;\; L_3 = L_4 = L_5 = \mathcal{O}_X(1,-1,0,0) \, .$$

$$c_2(TX) \;=\; (12,12,12,4) \qquad c_2(V) \;=\; (3,5,9,-7)$$

$$c_2(TX) - c_2(V) \;=\; (9,7,3,11)$$

$$\mathrm{ind}(L_1) = -3$$

$$\mathrm{ind}(L_2 \otimes L_3) = \mathrm{ind}(L_2 \otimes L_4) = \mathrm{ind}(L_2 \otimes L_5) = -1$$

$$\mathbf{10}_1, \; \mathbf{10}_1, \; \mathbf{10}_1, \; \overline{\mathbf{5}}_{2,3}, \; \overline{\mathbf{5}}_{2,4}, \; \overline{\mathbf{5}}_{2,5} \; .$$

Conclusion & Outlook

- We have studied heterotic model building on the sixteen families of torically generated Calabi-Yau three-folds with non-trivial first fundamental group.
- For $SU(5)$ we have succeeded in finding all such line bundle models on the 14 base spaces, thereby proving finiteness of the class computationally. The result is a total of 122 $SU(5)$ GUT models.
- For $SO(10)$ we have obtained a complete classification for all spaces up to Picard number three, resulting in a total of 55 $SO(10)$ GUT models. For the other six manifolds, all with Picard number four, only one (X_9) was amenable to a complete classification.

Conclusion & Outlook

- **Favorable Issue** The main technical obstacle to determine the full spectrum of these models – before and after Wilson line breaking – is the computation of line bundle cohomology on torically defined Calabi-Yau manifolds. We hope to address this problem in the future.
- **Symmetries** We consider the present work as the first step in a programme of classifying all line bundle standard models on the Calabi-Yau manifolds in the Kreuzer-Skarke list. A number of technical challenges have to be overcome in order to complete this programme, including a classification of freely-acting symmetries for these Calabi-Yau manifolds and the aforementioned computation of line bundle cohomology.