An analytical treatment of neutrino oscillation probabilities

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Contents

Revised version of arxiv:0704.1531v2 (to appear in future)

1. Introduction

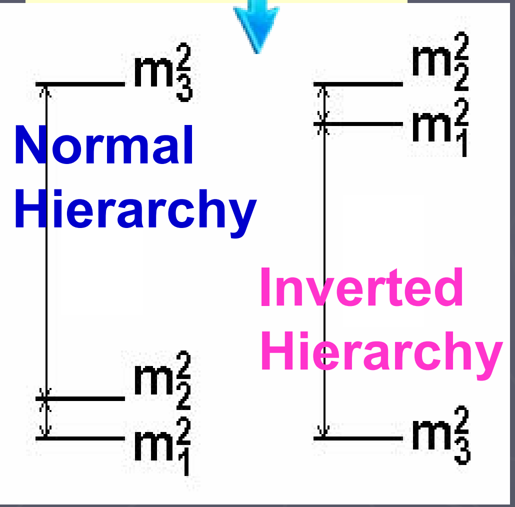
1.1 Scheme of 3 flavor ν oscillation

Mixing matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Functions of mixing angles θ_{12}, θ_{23}, θ_{13}, and CP phase δ

Both hierarchy patterns are allowed



All 3 mixing angles have been measured (2012):

ν_{solar}+KamLAND (reactor) → $\theta_{12} \cong \frac{\pi}{6}, \ \Delta m_{21}^2 \cong 8 \times 10^{-5} \mathrm{eV}^2$

ν_{atm}+K2K,MINOS(accelerators) → $\theta_{23} \cong \frac{\pi}{4}, \ |\Delta m_{32}^2| \cong 2.5 \times 10^{-3} \mathrm{eV}^2$

DCHOOZ+Daya Bay+Reno (reactors), T2K+MINOS, others → $\theta_{13} \cong \pi / 20$

Analytical expressions for ν oscillation probability are useful to discuss qualitative behaviors in various cases.

- **ν oscillation probability in matter is complicated beyond the 2-flavor case.**
- **The results which are obtained so far are mainly for the case of matter with constant density**

In this talk treatment of more generalized cases is discussed.

1.2 Exact oscillation probability in matter with constant density

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \left(U\mathcal{E}U^{-1} + \boxed{\mathcal{A}}\right)\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

matter effect

$$\begin{aligned}
\mathcal{E} &\equiv \mathrm{diag}\left(E_1, E_2, E_3\right)\\
\mathcal{A} &\equiv \mathrm{diag}\left(A, 0, 0\right)\\
\tilde{\mathcal{E}} &\equiv \mathrm{diag}\left(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\right)\\
A &\equiv \sqrt{2}G_F N_e\\
\Delta\tilde{E}_{jk} &\equiv \tilde{E}_j - \tilde{E}_k
\end{aligned}$$

$$U\mathcal{E}U^{-1} + \mathcal{A} = \tilde{U}\tilde{\mathcal{E}}\tilde{U}^{-1}$$

$$\mathbf{E_j} \equiv \sqrt{\vec{\mathbf{p}}^2 + \mathbf{m}_j^2}$$

$$\mathbf{U} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}$$

effective mixing matrix elements in matter

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} \; &- \; 4\sum_{j<k} \mathrm{Re}\left(\tilde{X}_j^{\alpha\beta}\tilde{X}_k^{\alpha\beta *}\right)\sin^2\left(\frac{\Delta\tilde{E}_{jk}L}{2}\right)\\
&+ \; 2\sum_{j<k} \mathrm{Im}\left(\tilde{X}_j^{\alpha\beta}\tilde{X}_k^{\alpha\beta *}\right)\sin\left(\Delta\tilde{E}_{jk}L\right)
\end{aligned}$$

Probability of ν oscillation can be expressed in terms of the energy eigenvalues $\tilde{E}_j$ and bilinear forms $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}_{\alpha j}\tilde{U}_{\beta j}^*$ of effective mixing matrix elements in matter

1.3 Formulation by Kimura-Takamura-Yokomakura (KTY PLB537:86,2002)



$$\left(\tilde{U}\tilde{U}^{-1}\right)_{\alpha\beta} = (1)_{\alpha\beta} \implies \sum_j \tilde{X}_j^{\alpha\beta} = \delta_{\alpha\beta}$$

$$\left(\tilde{U}\tilde{\mathcal{E}}\tilde{U}^{-1}\right)_{\alpha\beta} = \left(U\mathcal{E}U^{-1} + \mathcal{A}\right)_{\alpha\beta} \implies \sum_j \tilde{E}_j\tilde{X}_j^{\alpha\beta} = \sum_j E_j X_j^{\alpha\beta} + A\delta_{\alpha e}\delta_{e\beta}$$

$$\left(\tilde{U}\tilde{\mathcal{E}}^2\tilde{U}^{-1}\right)_{\alpha\beta} = \left[\left(U\mathcal{E}U^{-1} + \mathcal{A}\right)^2\right]_{\alpha\beta} \implies \sum_j \tilde{E}_j^2\tilde{X}_j^{\alpha\beta} = \sum_j E_j^2 X_j^{\alpha\beta} + \cdots$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ \tilde{E}_1 & \tilde{E}_2 & \tilde{E}_3 \\ \tilde{E}_1^2 & \tilde{E}_2^2 & \tilde{E}_3^2 \end{pmatrix} \begin{pmatrix} \tilde{X}_1^{\alpha\beta} \\ \tilde{X}_2^{\alpha\beta} \\ \tilde{X}_3^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\beta} \\ \sum_j E_j X_j^{\alpha\beta} + A\delta_{\alpha e}\delta_{e\beta} \\ \sum_j E_j^2 X_j^{\alpha\beta} + \cdots \end{pmatrix}$$

Simultaneous equation for $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}_{\alpha j}\tilde{U}_{\beta j}^*$ **can be solved**

$$\rightarrow \begin{pmatrix} \tilde{X}_1^{\alpha\beta} \\ \tilde{X}_2^{\alpha\beta} \\ \tilde{X}_3^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(\tilde{E}_2\tilde{E}_3,\ -(\tilde{E}_2+\tilde{E}_3),\ 1) \\ \dfrac{-1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(\tilde{E}_3\tilde{E}_1,\ -(\tilde{E}_3+\tilde{E}_1),\ 1) \\ \dfrac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(\tilde{E}_1\tilde{E}_2,\ -(\tilde{E}_1+\tilde{E}_2),\ 1) \end{pmatrix} \begin{pmatrix} \delta_{\alpha\beta} \\ \left[U\mathcal{E}U^{-1} + \mathcal{A}\right]_{\alpha\beta} \\ \left[\left(U\mathcal{E}U^{-1} + \mathcal{A}\right)^2\right]_{\alpha\beta} \end{pmatrix}$$

Thus the problem of obtaining the exact analytical oscillation probability is reduced to obtaining only the eigenvalues $\tilde{E}_j$ **!**

2.Extension of KTY's formulation

● **It can be generalized to the case with adiabatically varying mass matrix in L=∞ limit:**

$$i\frac{d}{dt}\psi(t) = \tilde{U}(t)\,\tilde{\mathcal{E}}(t)\,\tilde{U}^{-1}(t)$$

$$\psi(t_2) = \tilde{U}(t_2)\,\exp\left(-i\int_{t_1}^{t_2}\tilde{\mathcal{E}}(t)dt\right)\,\tilde{U}^{-1}(t_1)\psi(t_1)$$

$$\begin{aligned} A(\nu_\alpha\to\nu_\beta) &= \left[\tilde{U}(t_2)\,\exp\left(-i\int_{t_1}^{t_2}\tilde{\mathcal{E}}(t)dt\right)\,\tilde{U}^{-1}(t_1)\right]_{\beta\alpha} \\ &= \sum_j \tilde{U}(t_2)_{\beta j}\exp\left(-i\int_{t_1}^{t_2}\tilde{E}_j(t)dt\right)\tilde{U}(t_1)^*_{\alpha j} \end{aligned}$$

Average over rapid oscillations

$$\begin{aligned} P(\nu_\alpha\to\nu_\beta) &= |A(\nu_\alpha\to\nu_\beta)|^2 \\ &= \sum_{j,k}\tilde{U}(t_2)_{\beta j}\tilde{U}(t_2)^*_{\beta k}\tilde{U}(t_1)^*_{\alpha j}\tilde{U}(t_1)_{\alpha k}\exp\left(-i\int_{t_1}^{t_2}\Delta\tilde{E}_{jk}(t)dt\right) \\ &\to \sum_j \left|\tilde{U}(t_1)_{\alpha j}\right|^2\left|\tilde{U}(t_2)_{\beta j}\right|^2 \qquad \left(\exp\left(-i\int_{t_1}^{t_2}\Delta\tilde{E}_{jk}(t)dt\right)\to\delta_{jk}\right) \\ &= \left(|U_{\beta 1}|^2, |U_{\beta 2}|^2, |U_{\beta 3}|^2\right)\begin{pmatrix}|\tilde{U}_{\alpha 1}|^2\\ |\tilde{U}_{\alpha 2}|^2\\ |\tilde{U}_{\alpha 3}|^2\end{pmatrix} \end{aligned}$$

(in vacuum at t=t_2)

● To generalize to the nonadiabatical cases, the mixing matrix element $\tilde{U}_{\alpha j}$ itself is required.

In KTY's formulation the bilinear form $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}_{\alpha j}\tilde{U}^*_{\beta j}$ can be obtained, but $\tilde{U}_{\alpha j}$ itself cannot be.

⇒ It turns out that $\tilde{U}_{\alpha j}$ can be obtained from the bilinear form $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}_{\alpha j}\tilde{U}^*_{\beta j}$
: the main result of this talk

In the following, the notation $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}_{\alpha j}\tilde{U}^*_{\beta j}$ is used; we assume $\tilde{X}_j^{\alpha\beta}$ is known from KTY

From trivial identities, we have：

$$\left\{\begin{aligned} \tilde{U}_{ej} &= \sqrt{\tilde{X}_j^{ee}}\, e^{i\arg\tilde{U}_{ej}}, \\ \tilde{U}_{\mu j} &= \frac{\tilde{X}_j^{\mu e}}{\sqrt{\tilde{X}_j^{ee}}}\, e^{i\arg\tilde{U}_{ej}} \\ \tilde{U}_{\tau j} &= \frac{\tilde{X}_j^{\tau e}}{\sqrt{\tilde{X}_j^{ee}}}\, e^{i\arg\tilde{U}_{ej}} \end{aligned}\right.$$

→Up to phases, we obtain the following $\tilde{U}_{\alpha j}$:

$$\tilde{U}_0 \equiv \left(\begin{pmatrix} \sqrt{\tilde{X}_1^{ee}} \\ \tilde{X}_1^{\mu e}/\sqrt{\tilde{X}_1^{ee}} \\ \tilde{X}_1^{\tau e}/\sqrt{\tilde{X}_1^{ee}} \end{pmatrix}, \begin{pmatrix} \sqrt{\tilde{X}_2^{ee}} \\ \tilde{X}_2^{\mu e}/\sqrt{\tilde{X}_2^{ee}} \\ \tilde{X}_2^{\tau e}/\sqrt{\tilde{X}_2^{ee}} \end{pmatrix}, \begin{pmatrix} \sqrt{\tilde{X}_3^{ee}} \\ \tilde{X}_3^{\mu e}/\sqrt{\tilde{X}_3^{ee}} \\ \tilde{X}_3^{\tau e}/\sqrt{\tilde{X}_3^{ee}} \end{pmatrix} \right)$$

To get the standard parameterization, we multiply a diagonal phase matrix both from left and right as follows:

$$\tilde{U} \equiv e^{i\alpha}\, e^{i\beta\lambda_3}\, e^{i\gamma\lambda'_8}\, \tilde{U}_0\, e^{i\gamma'\lambda'_8}\, e^{i\beta'\lambda_3}$$

$$\begin{aligned}
\arg \tilde{U}_{e1} &= 0 \\
\arg \tilde{U}_{e2} &= 0 \\
\arg \tilde{U}_{\mu 3} &= 0 \\
\arg \tilde{U}_{\tau 3} &= 0 \\
\arg\det \tilde{U}_{\alpha j} &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha &= -\frac{1}{3}\arg\det\tilde{U}_0, \\
\beta &= \frac{1}{2}\arg\det\tilde{U}_0 - \frac{1}{2}\arg\tilde{X}_3^{\tau e}, \\
\beta' &= 0, \\
\gamma &= \frac{1}{6}\arg\det\tilde{U}_0 - \frac{1}{3}\arg\tilde{X}_3^{\mu e} + \frac{1}{6}\arg\tilde{X}_3^{\tau e}, \\
\gamma' &= -\frac{1}{3}\arg\det\tilde{U}_0 + \frac{1}{3}\arg\tilde{X}_3^{\mu e} + \frac{1}{3}\arg\tilde{X}_3^{\tau e}.
\end{aligned}$$

$$\begin{aligned}
\cos 2\tilde{\theta}_{12} &= \frac{\tilde{X}_1^{ee} - \tilde{X}_2^{ee}}{\tilde{X}_1^{ee} + \tilde{X}_2^{ee}}, \\
\cos 2\tilde{\theta}_{13} &= 1 - 2\tilde{X}_3^{ee}, \\
\cos 2\tilde{\theta}_{23} &= \frac{|\tilde{X}_3^{\tau e}|^2 - |\tilde{X}_3^{\mu e}|^2}{|\tilde{X}_3^{\tau e}|^2 + |\tilde{X}_3^{\mu e}|^2}, \\
\tilde{\delta} &= \alpha + \beta + \gamma - 2\gamma' = \arg\det\tilde{U}_0 - \arg\tilde{X}_3^{\mu e} - \arg\tilde{X}_3^{\tau e}
\end{aligned}$$

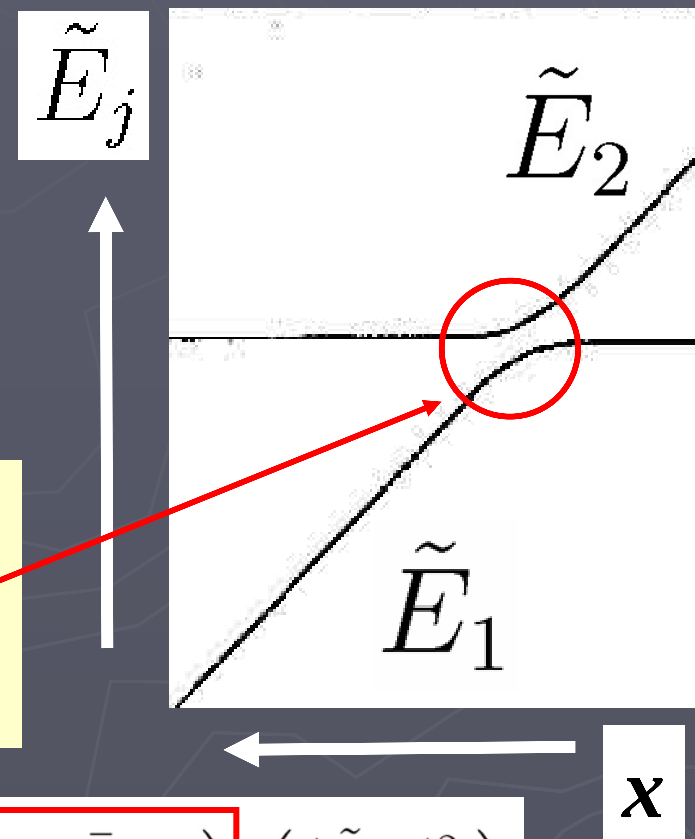
At a locally given density, we can obtain the analytical expression for the effective mixing angles.

3. An example of non-adiabatic cases

Two flavor case

Adiabatic case

$$P(\nu_\alpha \to \nu_\beta) \;=\; \left(|U_{\beta 1}|^2, |U_{\beta 2}|^2\right) \begin{pmatrix} |\tilde{U}_{\alpha 1}|^2 \\ |\tilde{U}_{\alpha 2}|^2 \end{pmatrix}$$



Non-adiabatic case: we can describe it by inserting the probability of jumping

$$P(\nu_\alpha \to \nu_\beta) \;=\; \left(|U_{\beta 1}|^2, |U_{\beta 2}|^2\right) \begin{pmatrix} 1-\bar{P}_c & \bar{P}_c \\ \bar{P}_c & 1-\bar{P}_c \end{pmatrix} \begin{pmatrix} |\tilde{U}_{\alpha 1}|^2 \\ |\tilde{U}_{\alpha 2}|^2 \end{pmatrix}$$

Probability of jumping

Kuo-Pantaleone, RMP61 ('89) 937

$$P_c = \frac{\exp\left[-(\pi/2)F\gamma\right] - \exp\left[-(\pi/2)F\gamma/\sin^2\theta\right]}{1-\exp\left[-(\pi/2)F\gamma/\sin^2\theta\right]}$$

$$\gamma = \frac{\Delta E \sin^2 2\theta}{\cos 2\theta |d \log N_e/dx|_{res}}$$

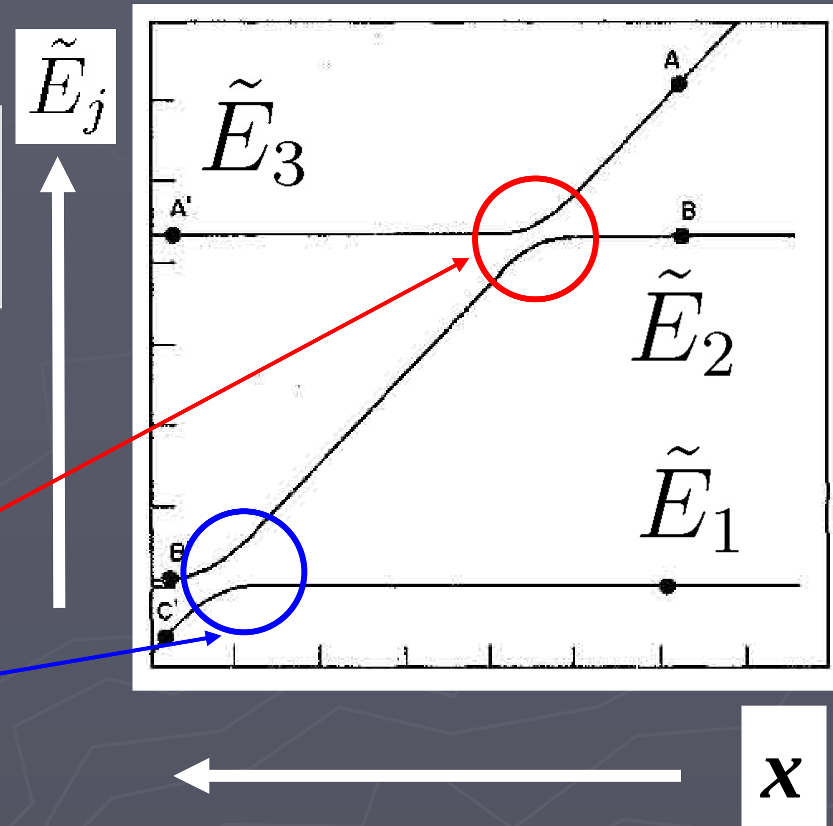
The probability of jumping depends on details of the density profile

A (density profile)	F
$A \propto \exp(-r)$	$1-\tan^2\theta$
$A \propto r$	1
$A \propto 1/r$	$(1-\tan^2\theta)^2/(1+\tan^2\theta)$
$A \propto r^n$	$2\sum_{m=0}^{\infty} \begin{bmatrix} (1/n)-1 \\ 2m \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ m+1 \end{bmatrix} (\tan 2\theta)^{2m}$

Three flavor case

Adiabatic case

$$P(\nu_\alpha \to \nu_\beta) = \left(|U_{\beta 1}|^2, |U_{\beta 2}|^2, |U_{\beta 3}|^2\right) \begin{pmatrix} |\tilde{U}_{\alpha 1}|^2 \\ |\tilde{U}_{\alpha 2}|^2 \\ |\tilde{U}_{\alpha 3}|^2 \end{pmatrix}$$



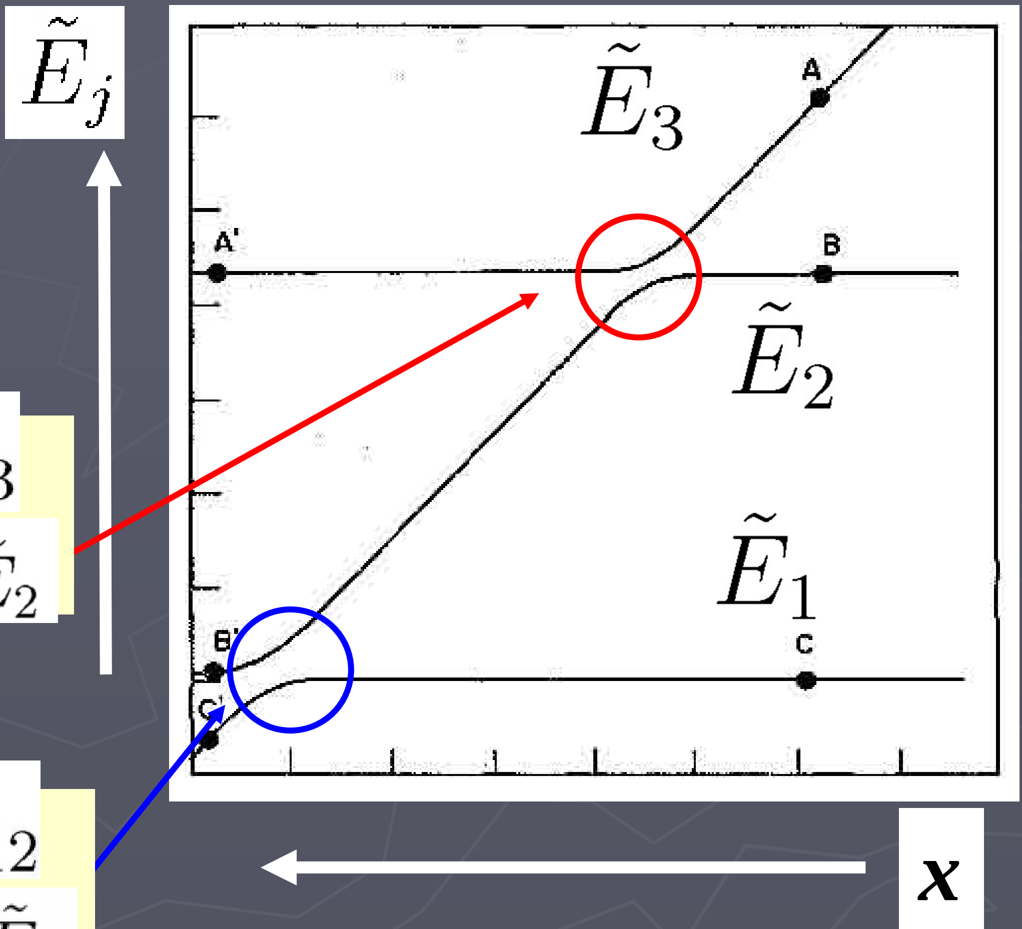
Non-adiabatic case: we can describe it by inserting the probability of jumping at two regions

$$P(\nu_\alpha \to \nu_\beta) = \left(|U_{\beta 1}|^2, |U_{\beta 2}|^2, |U_{\beta 3}|^2\right) \begin{pmatrix} 1-\bar{P}_c^L & \bar{P}_c^L & 0 \\ \bar{P}_c^L & 1-\bar{P}_c^L & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\bar{P}_c^H & \bar{P}_c^H \\ 0 & \bar{P}_c^H & 1-\bar{P}_c^H \end{pmatrix} \begin{pmatrix} |\tilde{U}_{\alpha 1}|^2 \\ |\tilde{U}_{\alpha 2}|^2 \\ |\tilde{U}_{\alpha 3}|^2 \end{pmatrix}$$

To describe non-adiabatic transitions, the expressions for the effective mixing angles are useful.

The expression for $\tilde{\theta}_{23}$ can be used for $\tilde{E}_3 \simeq \tilde{E}_2$

The expression for $\tilde{\theta}_{12}$ can be used for $\tilde{E}_2 \simeq \tilde{E}_1$

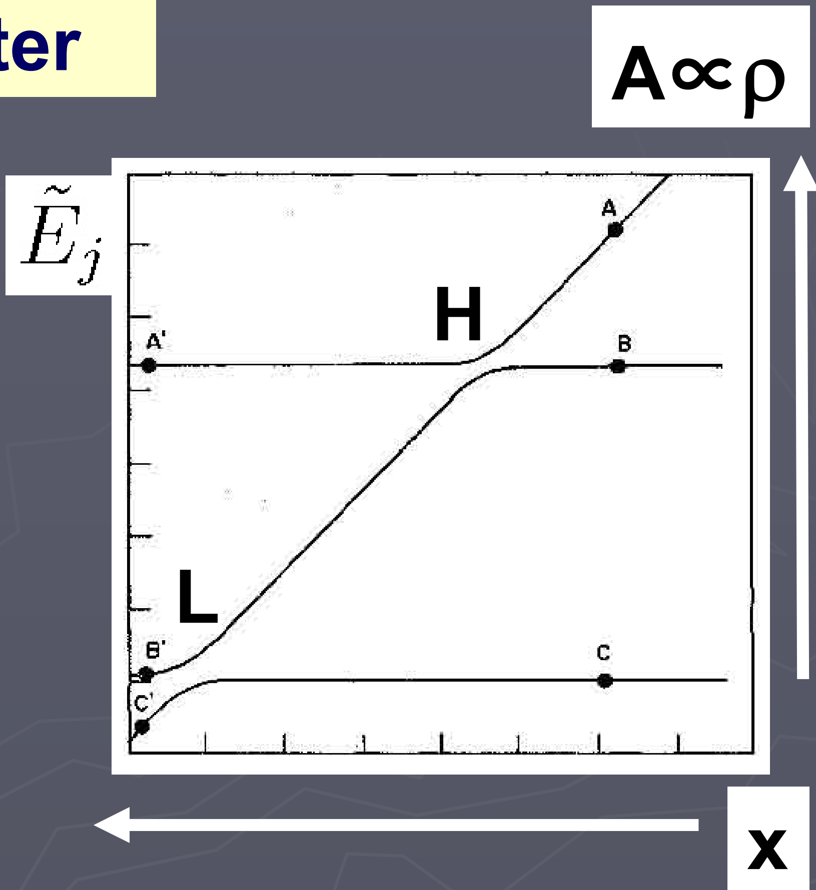


Example:Supernova ν with Non-Standard Interactions in matter

For simplicity self interactions of ν are not considered

$$\mathcal{A}_{NP} = A \begin{pmatrix} 1+\epsilon_{ee} & 0 & \epsilon_{e\tau} \\ 0 & 0 & 0 \\ \epsilon_{e\tau}^{*} & 0 & \epsilon_{\tau\tau} \end{pmatrix}$$

From other ν experiments μ components can be ignored.



λ_9=diag(1,0,-1) γ=arg($\varepsilon_{e\tau}$)

$$\mathcal{A}_{NP} = A\begin{pmatrix} 1+\epsilon_{ee} & 0 & \epsilon_{e\tau} \\ 0 & 0 & 0 \\ \epsilon_{e\tau}^* & 0 & \epsilon_{\tau\tau} \end{pmatrix} = A\, e^{i\gamma\lambda_9} e^{-i\beta\lambda_5} \mathrm{diag}\,(\lambda_{e'}, 0, 0)\, e^{i\beta\lambda_5} e^{-i\gamma\lambda_9}$$

$$\tan\beta = \frac{|\epsilon_{e\tau}|}{1+\epsilon_{ee}} \qquad \lambda_{e'} = \frac{A(1+\epsilon_{ee})}{\cos^2\beta} \qquad \epsilon_{\tau\tau} = \frac{|\epsilon_{e\tau}|^2}{1+\epsilon_{ee}}$$

Constraints from high energy ν_{atm}

$$U\mathcal{E}U^{-1} + \mathcal{A}_{NP} = e^{i\gamma\lambda_9} e^{-i\beta\lambda_5} \mathrm{diag}(1,1,e^{i\arg U'_{\tau 3}}) \left[U''\mathcal{E}U''^{-1} + \mathrm{diag}\,(\lambda_{e'},0,0)\right] \times \mathrm{diag}(1,1,e^{-i\arg U'_{\tau 3}})\, e^{i\beta\lambda_5} e^{-i\gamma\lambda_9}$$

Due to NSI, the effective mixing angles are modified:

$$\begin{aligned} \sin\theta''_{13} &= |U''_{e3}| = |c_\beta e^{-i\gamma} U_{e3} + s_\beta e^{i\gamma} U_{\tau 3}| \\ \tan\theta''_{12} &= |U''_{e2}|/|U''_{e1}| = |c_\beta e^{-i\gamma} U_{e2} + s_\beta e^{i\gamma} U_{\tau 2}|/|c_\beta e^{-i\gamma} U_{e1} + s_\beta e^{i\gamma} U_{\tau 1}| \\ \tan\theta''_{23} &= |U''_{\mu 3}|/|U''_{\tau 3}| = U_{\mu 3}/|c_\beta e^{-i\gamma} U_{\tau 3} - s_\beta e^{i\gamma} U_{e3}| \end{aligned}$$

From these, we get the probability which takes into account non-adiabatic transitions

$$P(\nu_e \to \nu_e) = \left(|U_{e1}|^2, |U_{e2}|^2, |U_{e3}|^2\right) \begin{pmatrix} 1-\bar{P}_c^L & \bar{P}_c^L & 0 \\ \bar{P}_c^L & 1-\bar{P}_c^L & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\bar{P}_c^H & \bar{P}_c^H \\ 0 & \bar{P}_c^H & 1-\bar{P}_c^H \end{pmatrix} \begin{pmatrix} |\tilde{U}_{e1}|^2 \\ |\tilde{U}_{e2}|^2 \\ |\tilde{U}_{e3}|^2 \end{pmatrix}$$

$$\bar{P}_c^L = \Theta\left(c_{13}^2 \lambda_{e'} - \Delta E_{21} \cos 2\theta''_{12}\right) P_c^L$$

$$P_c^L = \frac{\exp(-\frac{\pi}{2}\gamma_L) - \exp(-\frac{\pi}{2}\gamma_L / s''^2_{12})}{1 - \exp(-\frac{\pi}{2}\gamma_L / s''^2_{12})}$$

$$\gamma_L = \frac{\Delta E_{21} \sin^2 2\theta''_{13} \cos 2\theta''_{12}}{c''^2_{12} |d \log N_e / dx|_{res}}$$

$$\bar{P}_c^H = \Theta\left(\lambda_{e'} - \Delta E_{31} \cos 2\theta''_{13}\right) P_c^H$$

$$P_c^H = \frac{\exp(-\frac{\pi}{2}\gamma_H) - \exp(-\frac{\pi}{2}\gamma_H / s''^2_{13})}{1 - \exp(-\frac{\pi}{2}\gamma_H / s''^2_{13})}$$

$$\gamma_H = \frac{\Delta E_{31} \sin^2 2\theta''_{13} \cos 2\theta''_{13}}{c''^2_{13} |d \log N_e / dx|_{res}}$$

4. Summary

- From trivial identities, we can (in principle) obtain the analytical expression for the mixing angles and the CP phase.
- Using such expressions, probability for ν flavor transitions including non-adiabatic processes can be analytically obtained.
- As a demonstration, one example was discussed: Supernova ν with Non-Standard Interactions in matter