Mitchell Workshop on Collider and Dark Matter Physics

Better Mass Measurement in Cascade Decay

Using the boundary of many-body phase space

Jiang-Hao Yu

University of Texas at Austin

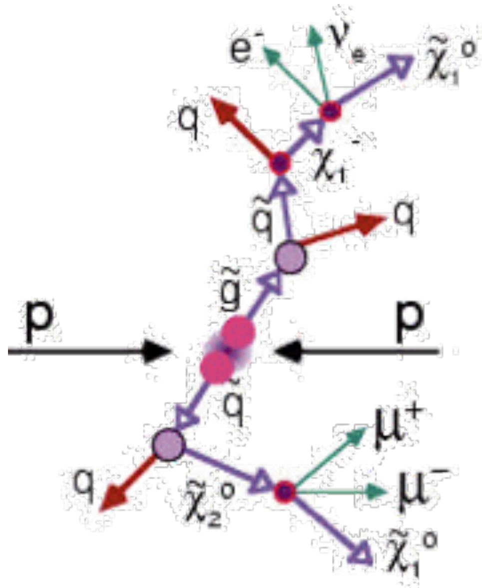
Prateek Agrawal, Can Kilic, Craig White, and JHY, PRD89(2014)015021
arXiv: 1308.6560

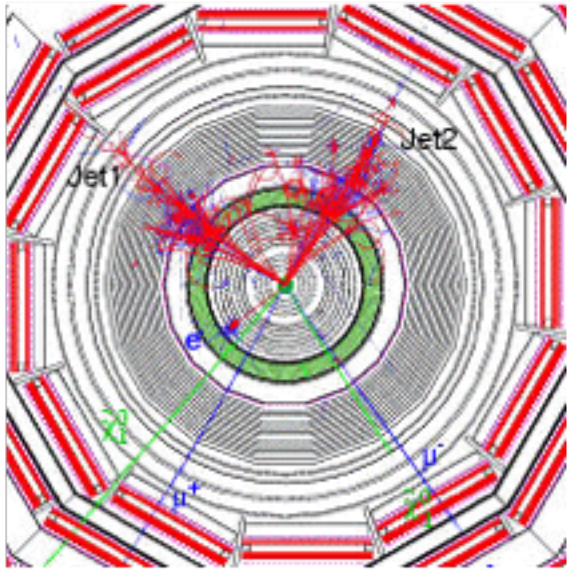
May 13, 2014 @Texas A&M University

Mass Measurement

- Crucial after discovery of a new particle
- Could discriminate different models and determine parameters of a model
- Difficult in SUSY, due to long cascade decay with transverse missing energy

In the cascade decay chain, sparticle mass in every step is unknown. Hard to reconstruct the system.





- How to measure masses of the immediate particles and final state LSP?

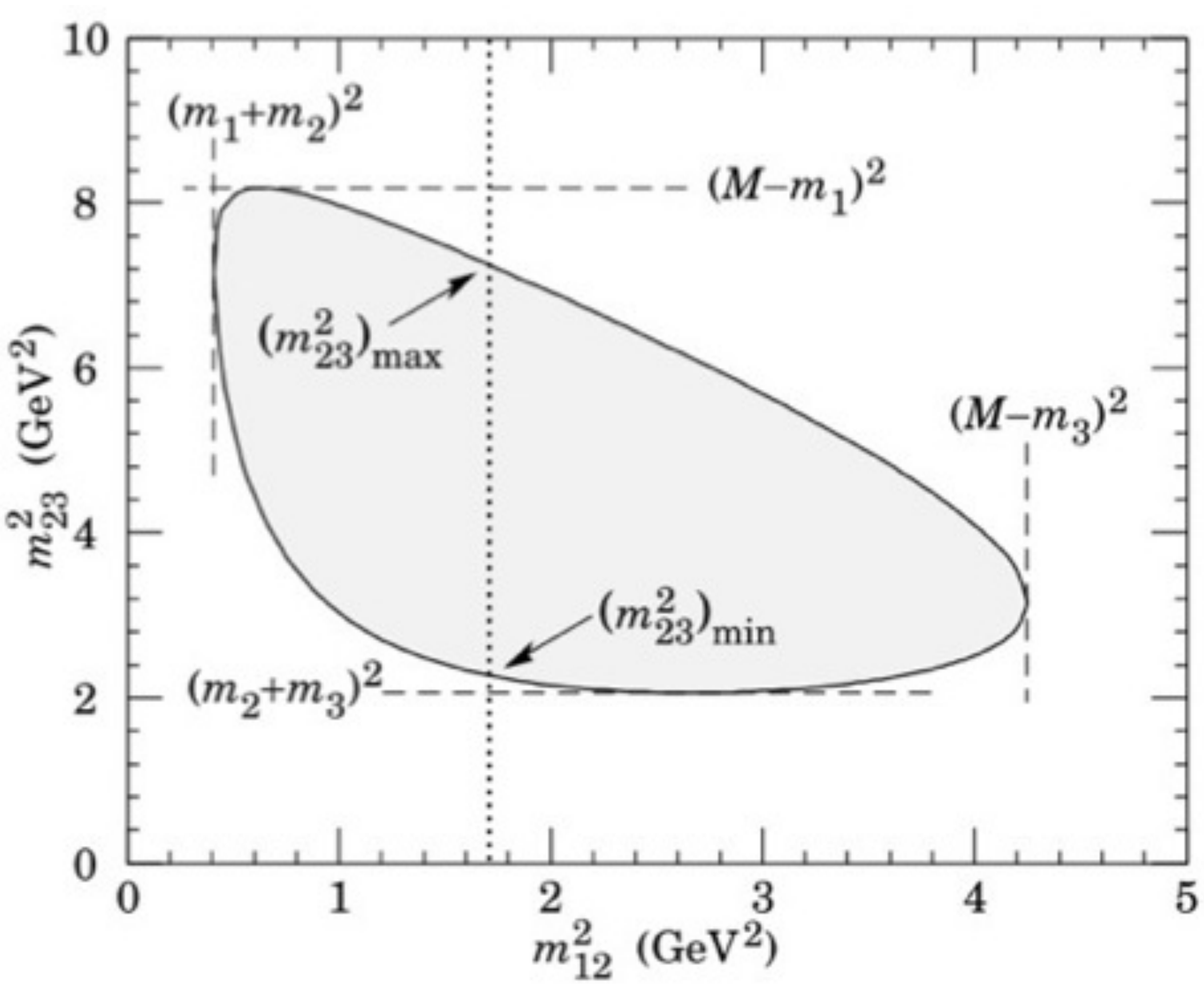
Simplest Cases

- Focus on one cascade decay chain
- Two body decay is trivial

$$X \to 1 + 2$$

Due to missing particle 2 look at transverse mass

- Three body phase space gives the Dalitz plot



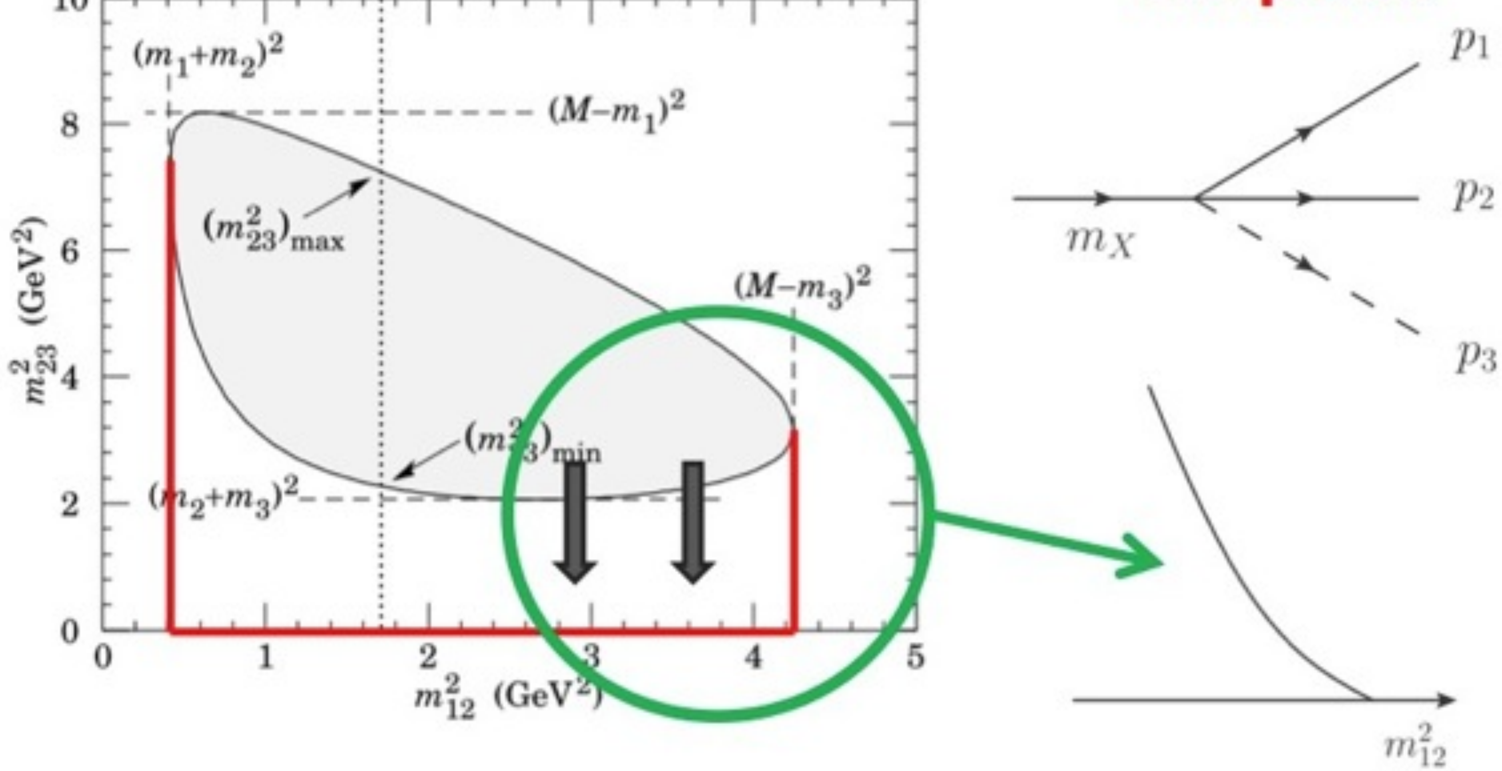
$$X \to 1 + 2 + 3$$

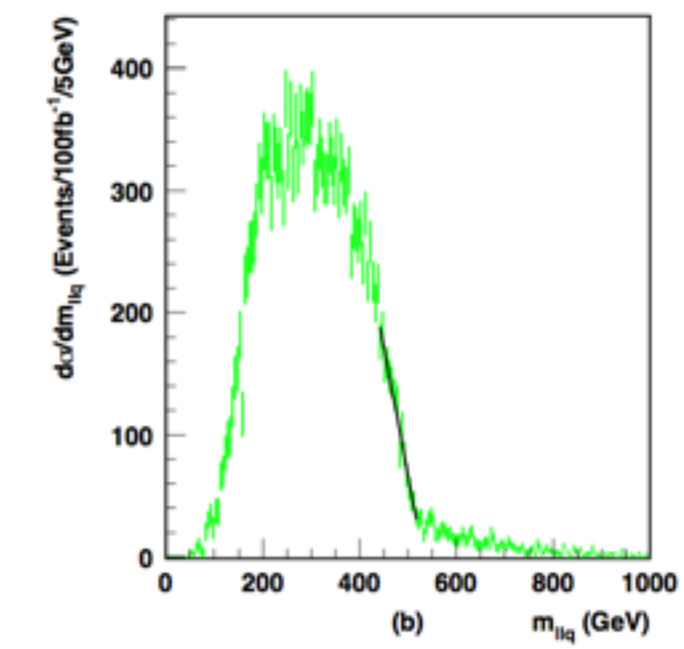
$$d\Pi_3 = \text{const.} \times \mathrm{dm}_{12}^2 \mathrm{dm}_{23}^2$$

Due to missing particle 3, m23 is not measurable. So we only know m12 (1D projection)

1D Projection: Endpoint and Edge

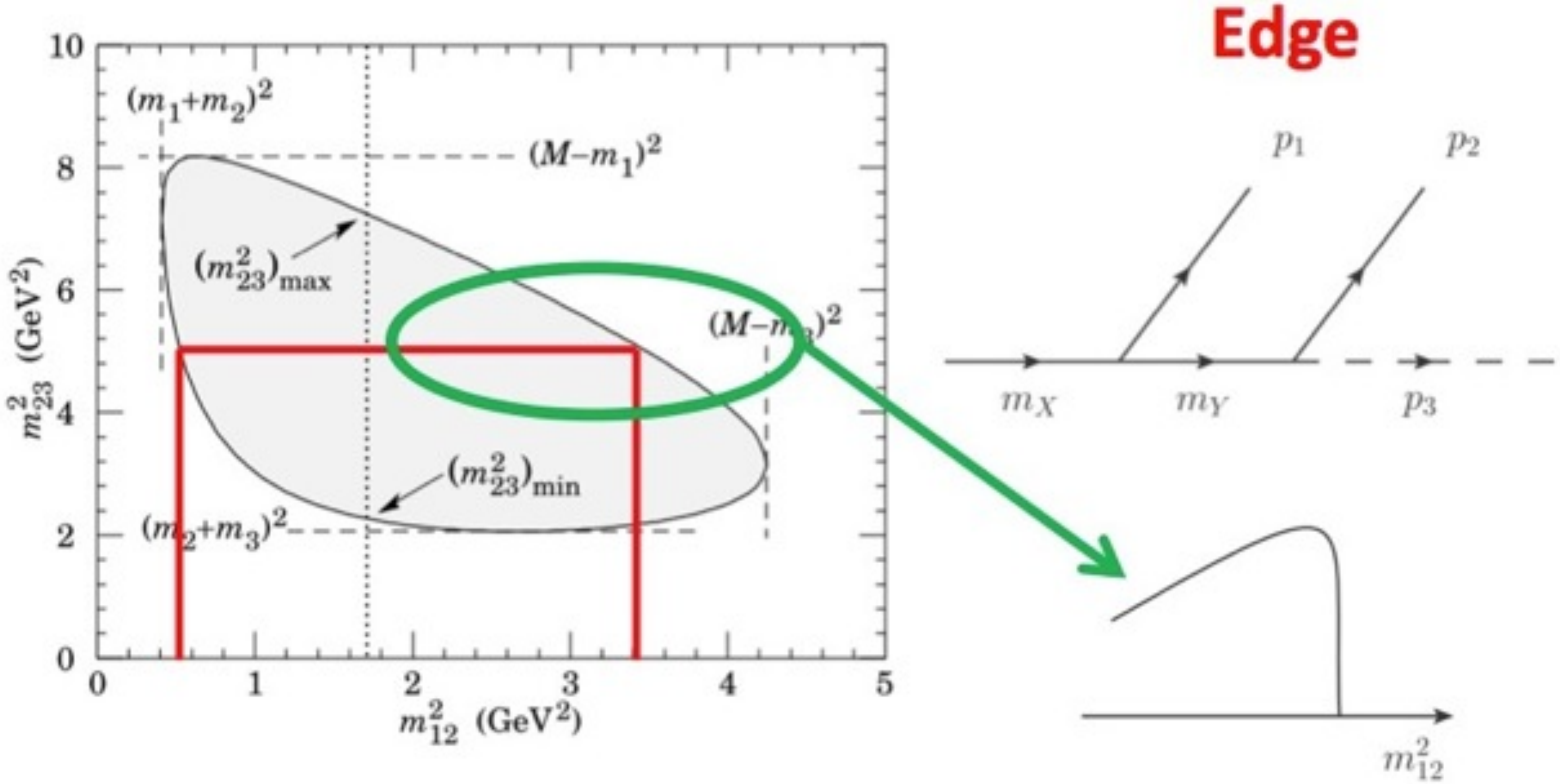
Endpoint

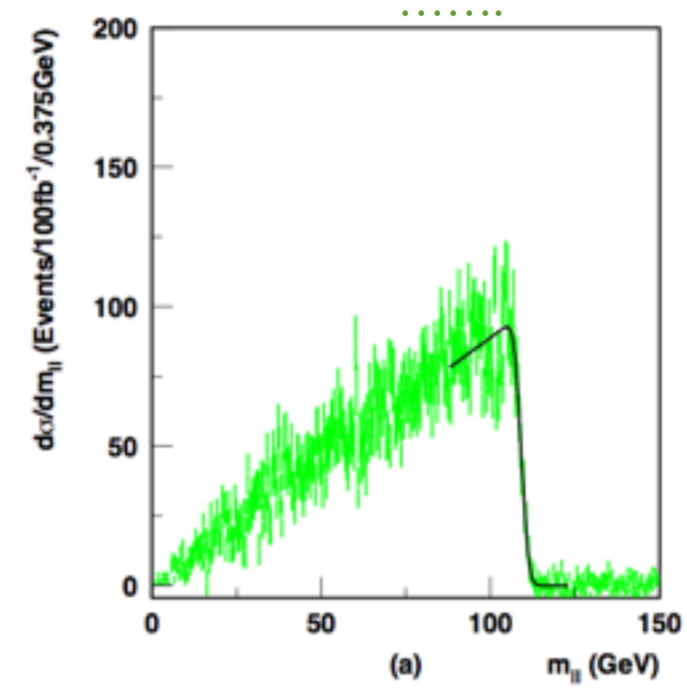




Hinchliffe, Paige, et.al. 1997
Allanach, Lester, et.al. 2000
....

Edge



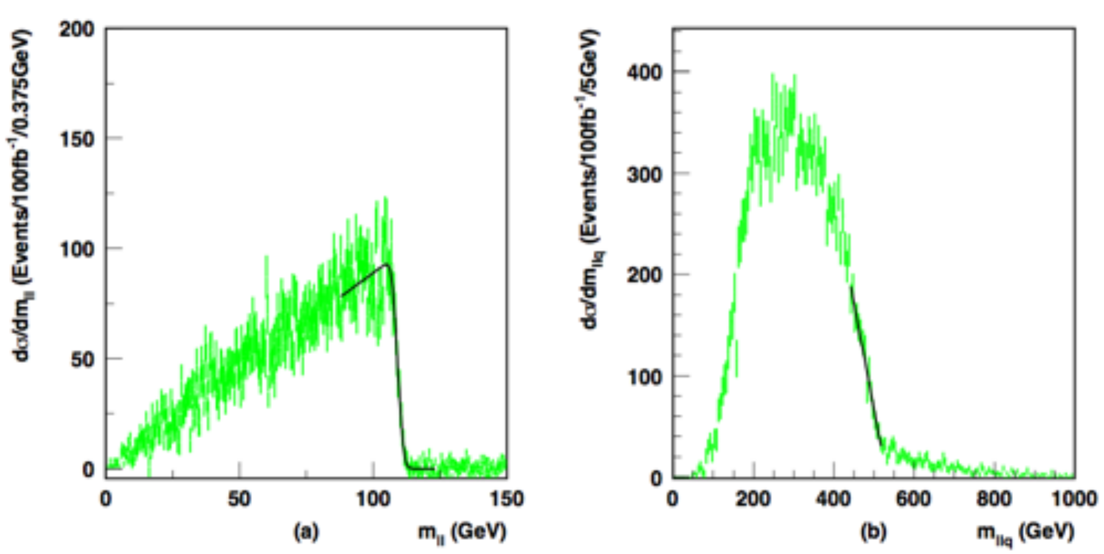


Beyond 3-body Phase Space

- Basically all known (and hypothesized) particles will decay either to two or three final state particles
- Any longer decay chain can factorized as subsequent 2 or 3 body decays

$$dPS_n(P; p_1, \ldots, p_n) = dPS_{n-1}(P; p_1, \ldots, p_{n-1,n}) dPS_2(p_{n-1,n}; p_{n-1}, \ldots, p_n) \frac{dm^2_{n-1,n}}{2\pi}$$

- Isn't it good enough to analyze the cascade step by step, looking for 1D edges and endpoints in each step?



Hinchliffe, Paige, et.al. 1997
Allanach, Lester, et.al. 2000
....

1D projection does not include the full phase space correlations

- Consider the full phase space instead of 1D distributions

4-body Phase Space

- Simplest case $X \to 1 + 2 + 3 + \color{red}{4}$

particle 4 is invisible

- Phase space density in terms of invariant masses (Generalized Dalitz phase space)

$$d\Pi_4 = \left(\prod_{i<j} dm_{ij}^2\right) \frac{\mathrm{C}}{M_X^2 \Delta_4^{1/2}}\, \delta\left(\sum_{i<j} m_{ij}^2 - \mathrm{K}\right),$$

$$\Delta_4 = \text{sum of minor of Det} \begin{bmatrix} p_1^2 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot p_4 \\ p_2 \cdot p_1 & p_2^2 & p_2 \cdot p_3 & p_2 \cdot p_4 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3^2 & p_3 \cdot p_4 \\ p_4 \cdot p_1 & p_4 \cdot p_2 & p_4 \cdot p_3 & p_4^2 \end{bmatrix}$$

- The boundary of phase space density

$$\Delta_4 = 0$$

Phase space density is enhanced near the boundary

- Five independent variables, but only three are measurable

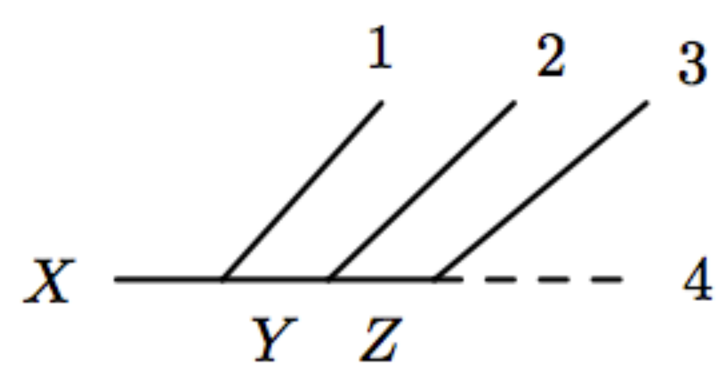
3n-4(E/p conserv.)-3(rotation inv.) = 3n-7

$$m_{12}^2,\ m_{13}^2,\ m_{23}^2$$

Classification of 4-body Decay

- Full 4-body decay (no example)

$$X \to 1 + 2 + 3 + 4$$

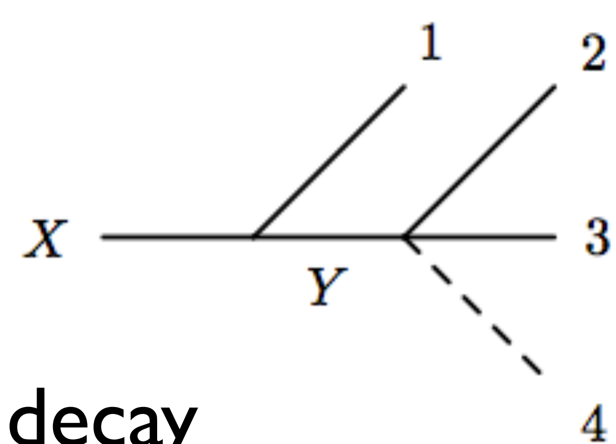


- 2 + 2 + 2 Decay

$$X \to 1 + Y,\ Y \to 2 + Z,\ Z \to 3 + 4$$

- 2 + 3 Decay

$$X \to 1 + Y,\ Y \to 2 + 3 + 4$$



- 3 + 2 Decay (same as 2 + 3)

$$X \to 1 + 2 + Y,\ Y \to 3 + 4$$

- Others could reduce to 3 body or 2 body decay

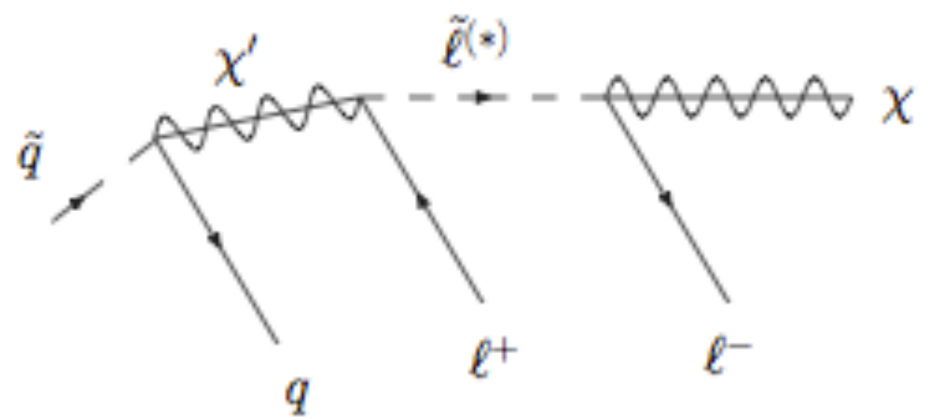
$$X \to Y + Z,\ Y \to 1 + 2,\ Z \to 3 + 4$$

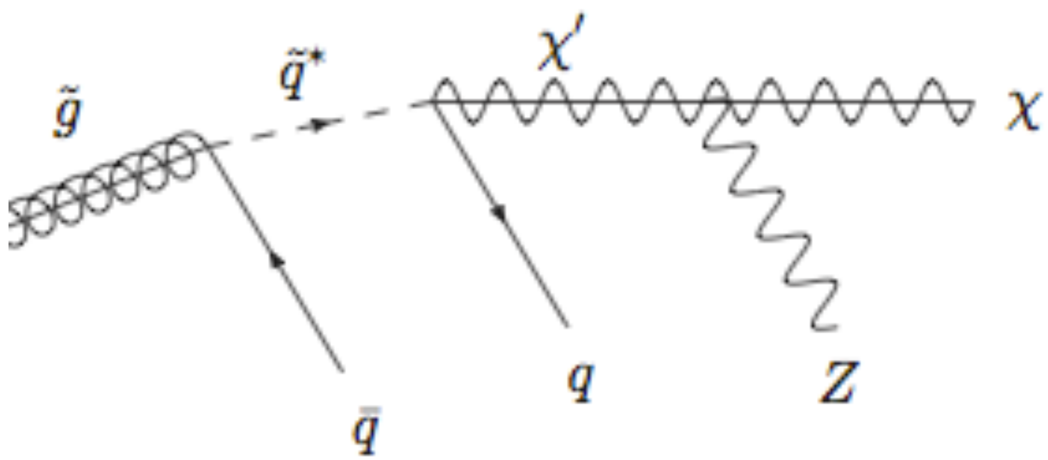
- The task is

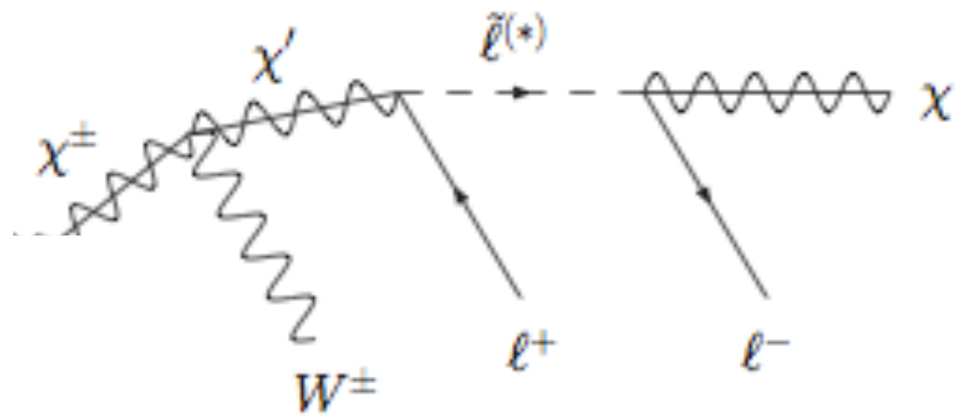
Given the data	$m_{12}^2,\ m_{13}^2,\ m_{23}^2$
falsify mass hypothesis	$m_X,\ m_Y,\ (m_Z,)\ m_4$

Examples in SUSY

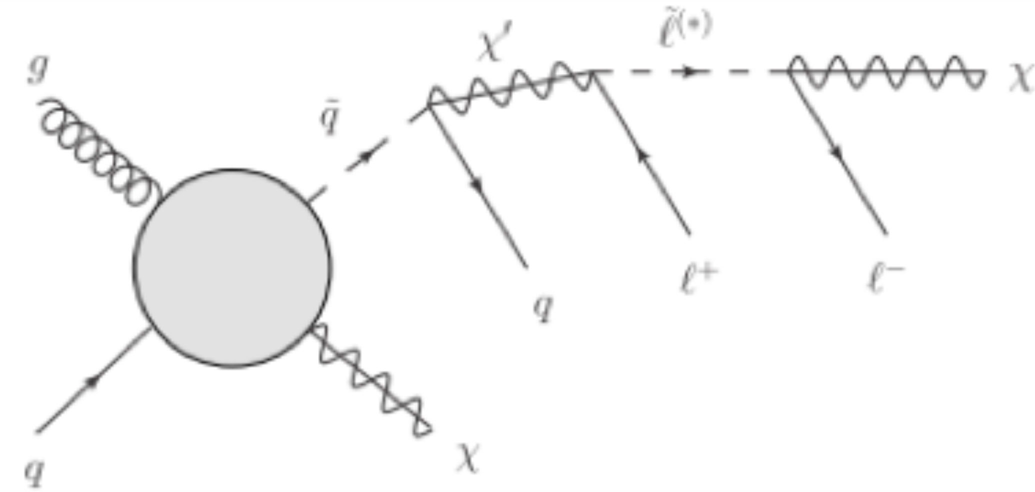
- 4-body Decay



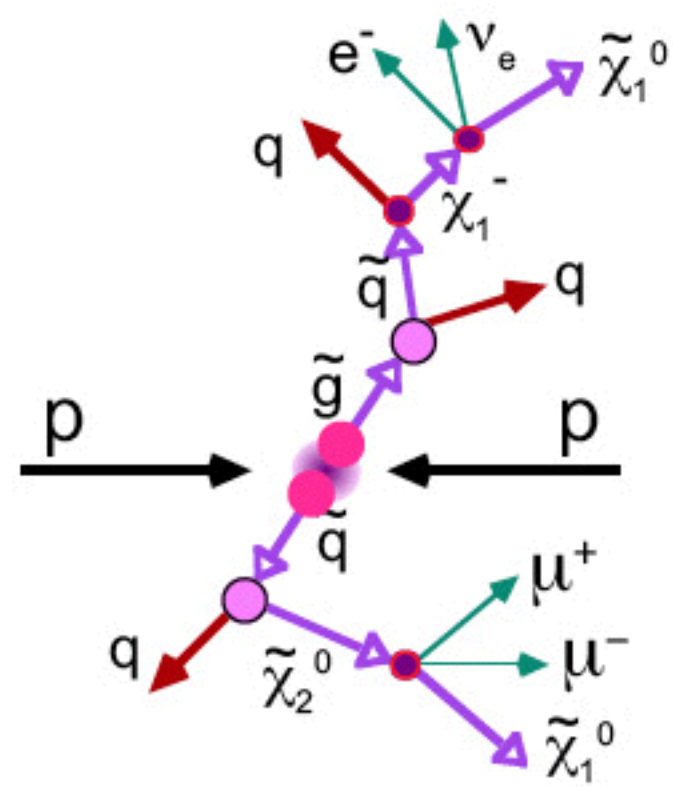




- Single chain production



- double chain production (focus on one chain)



Neglect combinatorial effects

How to Use the Phase Space Boundary

- In endpoint/edge method

mass hypothesis ⟷ endpoint/edge

Use data to falsify the mass hypothesis

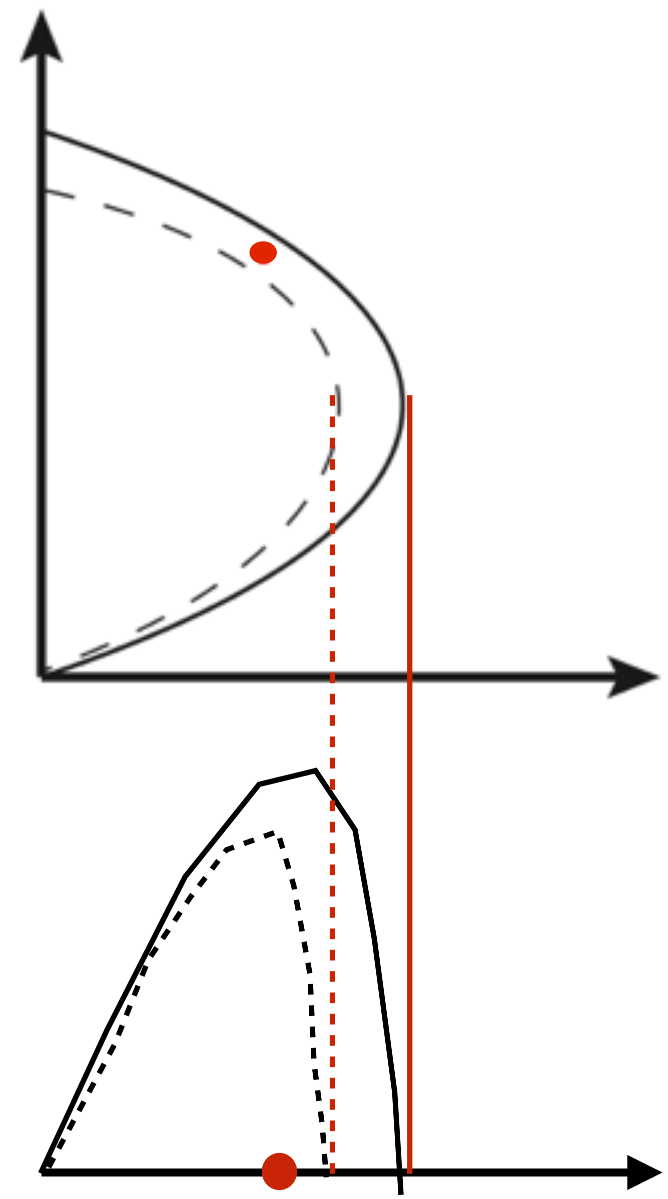
- How to assign a quality-of-fit to multi-D method?

mass hypothesis ⟷ phase space boundary

Use data to falsify the mass hypothesis

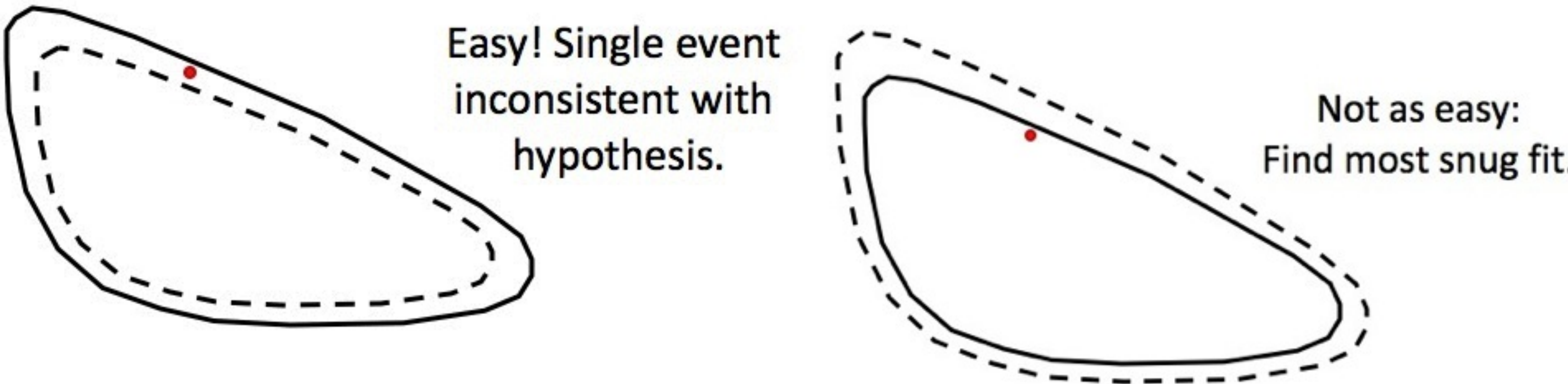
- If we only have limited statistics, it is better to use all available data, not just edge/endpoint

In 1D method, only the data near the true endpoint/edge is useful.



Phase Space Density

- Two kinds of false mass hypothesis



Phase space density is enhanced near the boundary

For a event, choose the contour which gives larger phase space density

- The criteria is (for each mass hypothesis)

$$\mathcal{L}(\tilde{m}_X, \tilde{m}_R) = \begin{cases} 0 & \text{if some data outside the boundary}\,, \\ \sum_{data} \frac{\pi^2}{2^5 \tilde{m}_X^2} (\Delta_4(\tilde{m}_X, \tilde{m}_R))^{-\frac{1}{2}} & \text{if all data inside the boundary}\,. \end{cases}$$

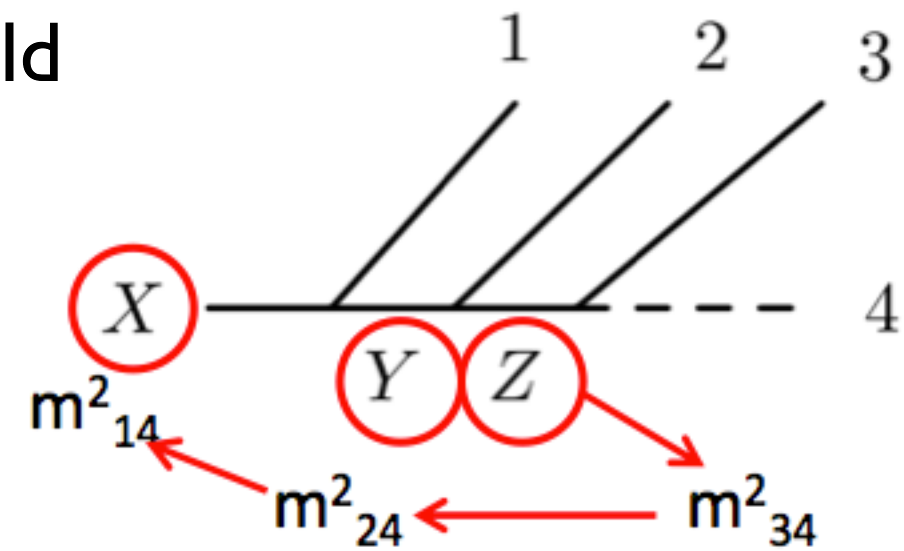
Pick up the hypothesis gives the largest value L

2+2+2 Cascade Decay

- Given the mass hypothesis, one could reconstruct the invariant masses

$$m_X, \, m_Y, \, m_Z, \, m_4$$

$$m_{12}^2, \, m_{23}^2, \, m_{13}^2$$



- The phase space density is obtained

$$\boxed{\mathcal{L}(\tilde{m}_\sigma, m_{ij}^2) \simeq \frac{1}{4\pi\tilde{m}_X^2}\left(1-\frac{\tilde{m}_Y^2}{\tilde{m}_X^2}\right)^{-1}\left(1-\frac{\tilde{m}_Z^2}{\tilde{m}_Y^2}\right)^{-1}\left(1-\frac{\tilde{m}_4^2}{\tilde{m}_Z^2}\right)^{-1}\Theta(\Delta_4)\frac{1}{\Delta_4^{1/2}}}$$

- Benchmark point

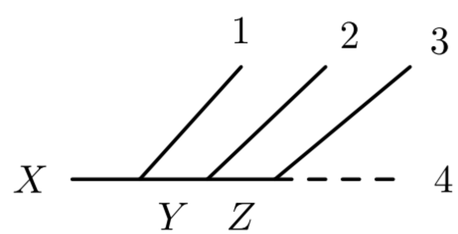
$$M_X = 500 \text{ GeV}, \qquad M_Y = 350 \text{ GeV},$$
$$M_Z = 200 \text{ GeV}, \qquad M_4 = 100 \text{ GeV},$$
$$m_1 = m_2 = m_3 = 5 \text{ GeV},$$

For both 1D endpoint/edge and multi-D phase space method

No spin correlation considered
No background added
No smearing/detector simulation
Only consider limited statistics

Edges and Endpoints

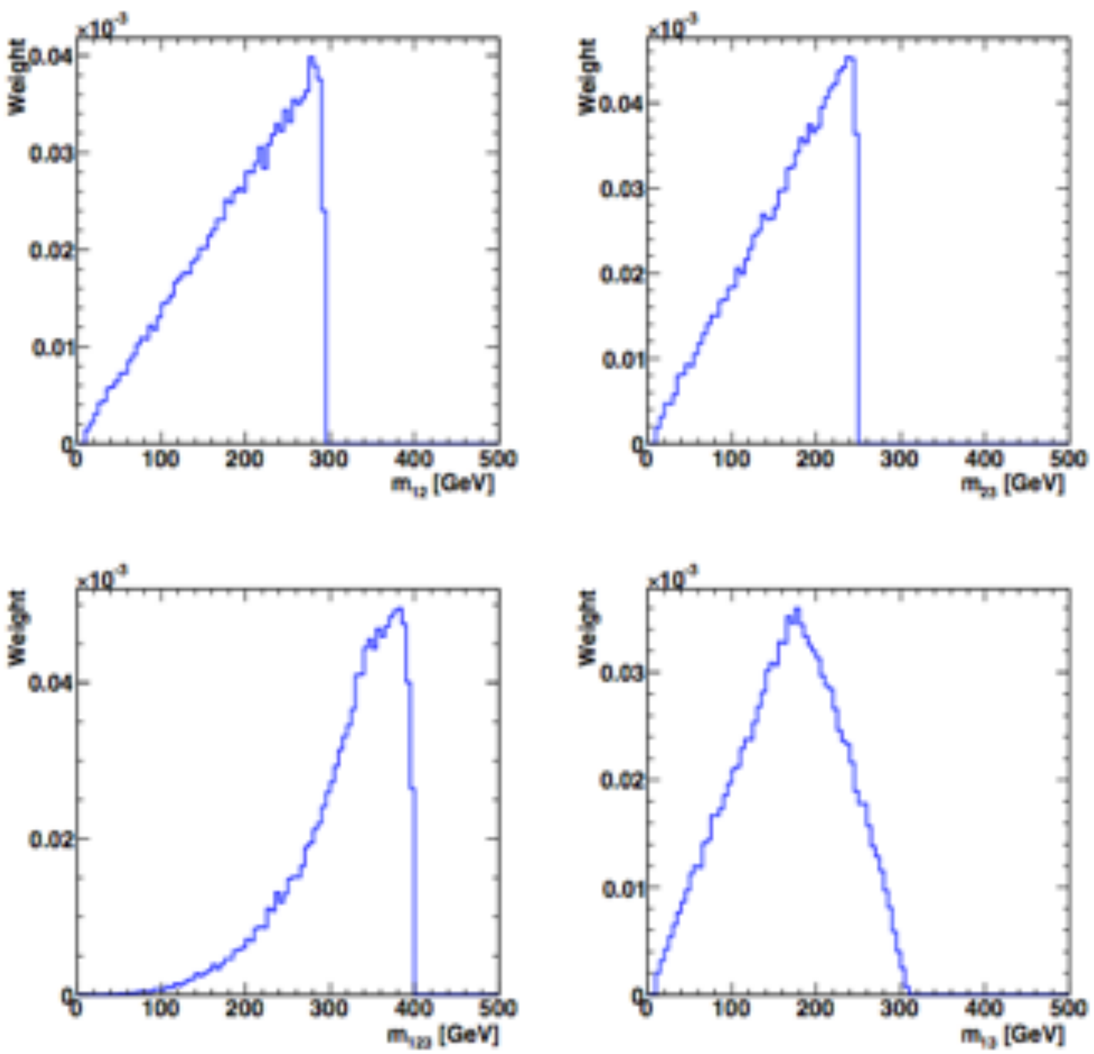
$$(m^2_{123})_{max} = \begin{cases} \frac{(m_X^2-m_Y^2)(m_Y^2-m_4^2)}{m_Y^2} & \frac{m_X}{m_Y} > \frac{m_Y}{m_Z}\frac{m_Z}{m_4} \\ \frac{(m_X^2 m_Z^2-m_Y^2 m_4^2)(m_Y^2-m_Z^2)}{m_Y^2 m_Z^2} & \frac{m_Y}{m_Z} > \frac{m_Z}{m_4}\frac{m_X}{m_Y} \\ \frac{(m_X^2-m_Z^2)(m_Z^2-m_4^2)}{m_Z^2} & \frac{m_Z}{m_4} > \frac{m_X}{m_Y}\frac{m_Y}{m_Z} \\ (m_X-m_4)^2 & \text{otherwise} \end{cases}$$



$$(m^2_{23})_{max} = (m_Y^2 - m_Z^2)(m_Z^2 - m_4^2)/m_Z^2,$$

$$(m^2_{12})_{max} = (m_X^2 - m_Y^2)(m_Y^2 - m_Z^2)/m_Y^2,$$

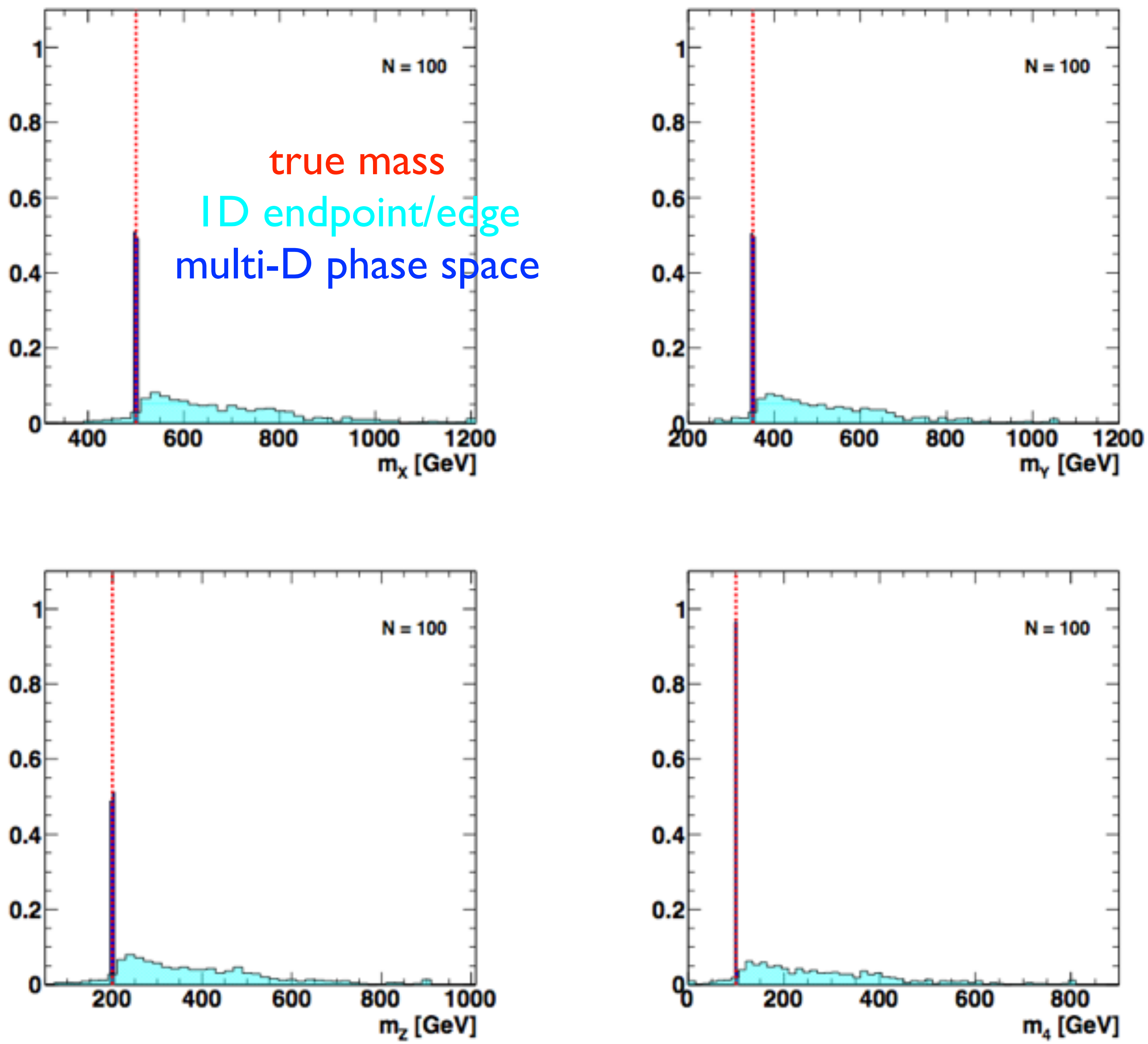
$$(m^2_{13})_{max} = (m_X^2 - m_Y^2)(m_Z^2 - m_4^2)/m_Z^2\,.$$



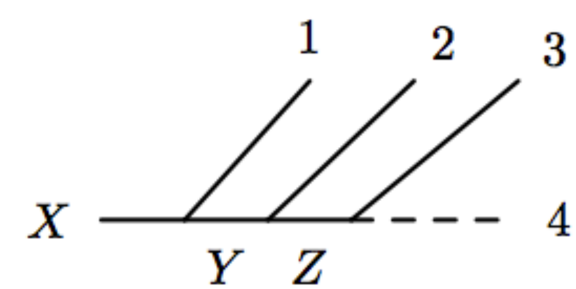
$$Q = \left(\sum_{i=\text{endpts.}} \left(\frac{\mathcal{O}_{i,predicted} - \mathcal{O}_{i,measured}}{\mathcal{O}_{i,measured}} \right)^2 \right)$$

$$\mathcal{O}_i = \{(m^2_{123})_{max}, (m^2_{12})_{max}, (m^2_{23})_{max}, (m^2_{13})_{max}\}$$

Histograms in 2+2+2



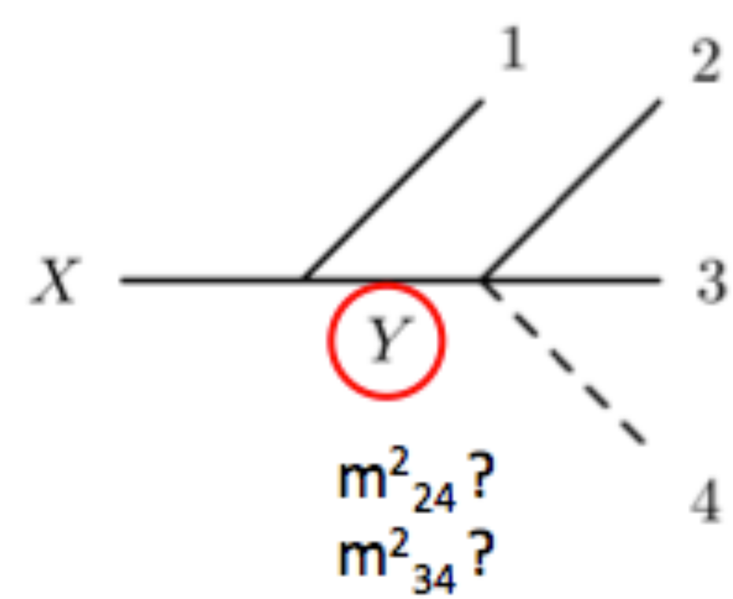
Limited statistics
N = 100 events



1D method, there is a flat direction due to lack of the full phase space correlations

2+3 Cascade Decay

- Not enough to determine all invariant masses involved in missing particle 4



2+2+2: two additional on-shell conditions
2+3: only one additional on-shell condition

Always one unknown (m34 or m24)

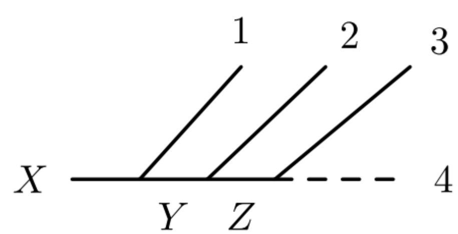
- The phase space density could be obtained by integrating over m34

$$\mathcal{L}(\tilde{m}_\sigma, m_{ij}^2) \simeq \frac{1}{2\tilde{m}_X^2}\left(1-\frac{\tilde{m}_Y^2}{\tilde{m}_X^2}\right)^{-1}\frac{1}{\tilde{m}_Y^2}\left(1-\frac{\tilde{m}_4^4}{\tilde{m}_Y^4}-2\frac{\tilde{m}_4^2}{\tilde{m}_Y^2}\log\left(\frac{\tilde{m}_Y^2}{\tilde{m}_4^2}\right)\right)^{-1}\Theta(-G1)\Theta(-G2)\frac{1}{\sqrt{\lambda_0}}.$$

- Benchmark point

$$m_X = 500\text{ GeV}, \qquad m_Y = 350\text{ GeV}, \qquad m_4 = 100\text{ GeV},$$
$$m_1 = m_2 = m_3 = 5\text{ GeV}.$$

Edges and Endpoints

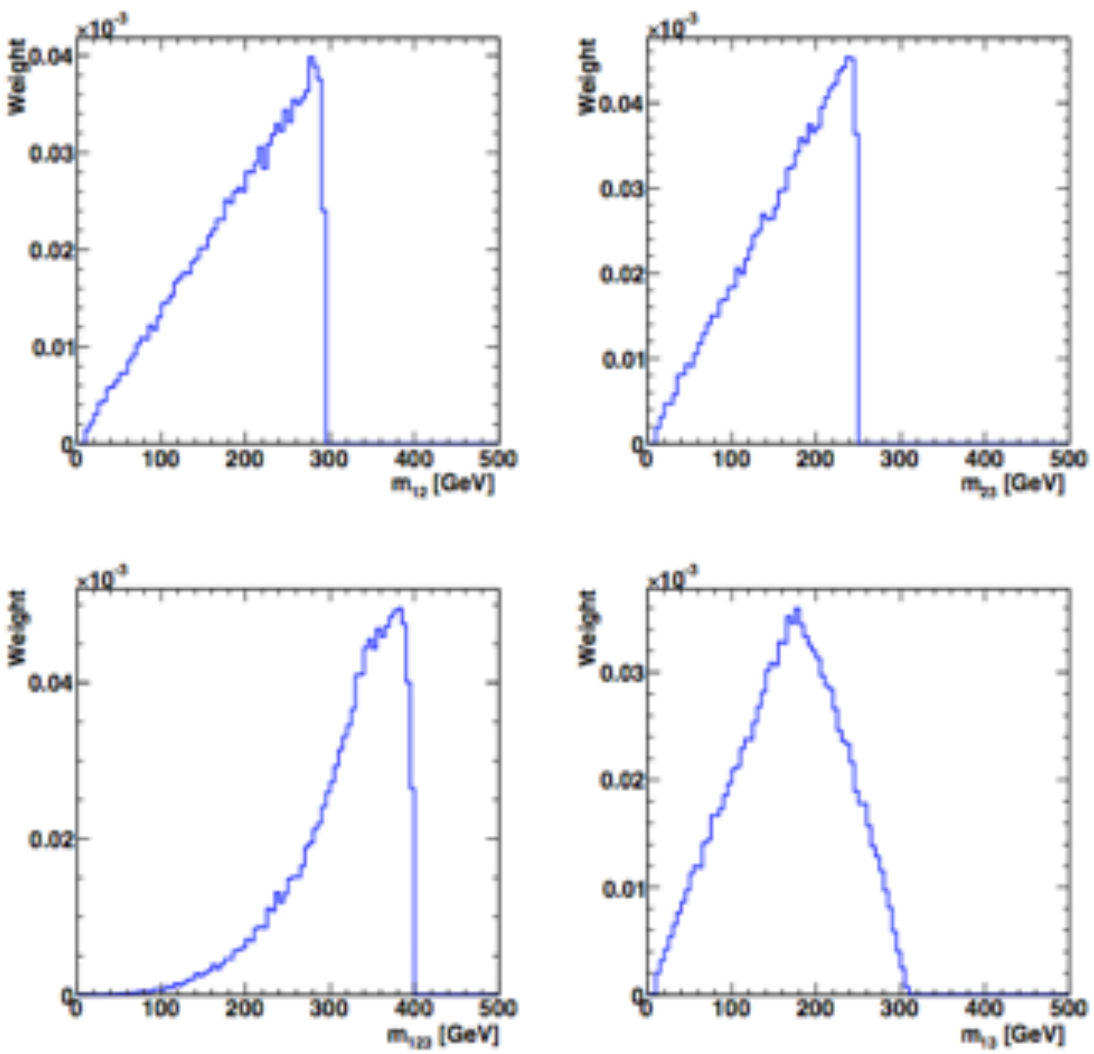


$$(m_{123}^2)_{max} = \begin{cases} \frac{(m_X^2 - m_Y^2)(m_Y^2 - m_4^2)}{m_Y^2} & \frac{m_X}{m_Y} > \frac{m_Y}{m_Z}\frac{m_Z}{m_4} \\ \frac{(m_X^2 m_Z^2 - m_Y^2 m_4^2)(m_Y^2 - m_Z^2)}{m_Y^2 m_Z^2} & \frac{m_Y}{m_Z} > \frac{m_Z}{m_4}\frac{m_X}{m_Y} \\ \frac{(m_X^2 - m_Z^2)(m_Z^2 - m_4^2)}{m_Z^2} & \frac{m_Z}{m_4} > \frac{m_X}{m_Y}\frac{m_Y}{m_Z} \\ (m_X - m_4)^2 & \text{otherwise} \end{cases}$$

$$(m_{23}^2)_{max} = (m_Y^2 - m_Z^2)(m_Z^2 - m_4^2)/m_Z^2,$$

$$(m_{12}^2)_{max} = (m_X^2 - m_Y^2)(m_Y^2 - m_Z^2)/m_Y^2,$$

$$(m_{13}^2)_{max} = (m_X^2 - m_Y^2)(m_Z^2 - m_4^2)/m_Z^2\,.$$

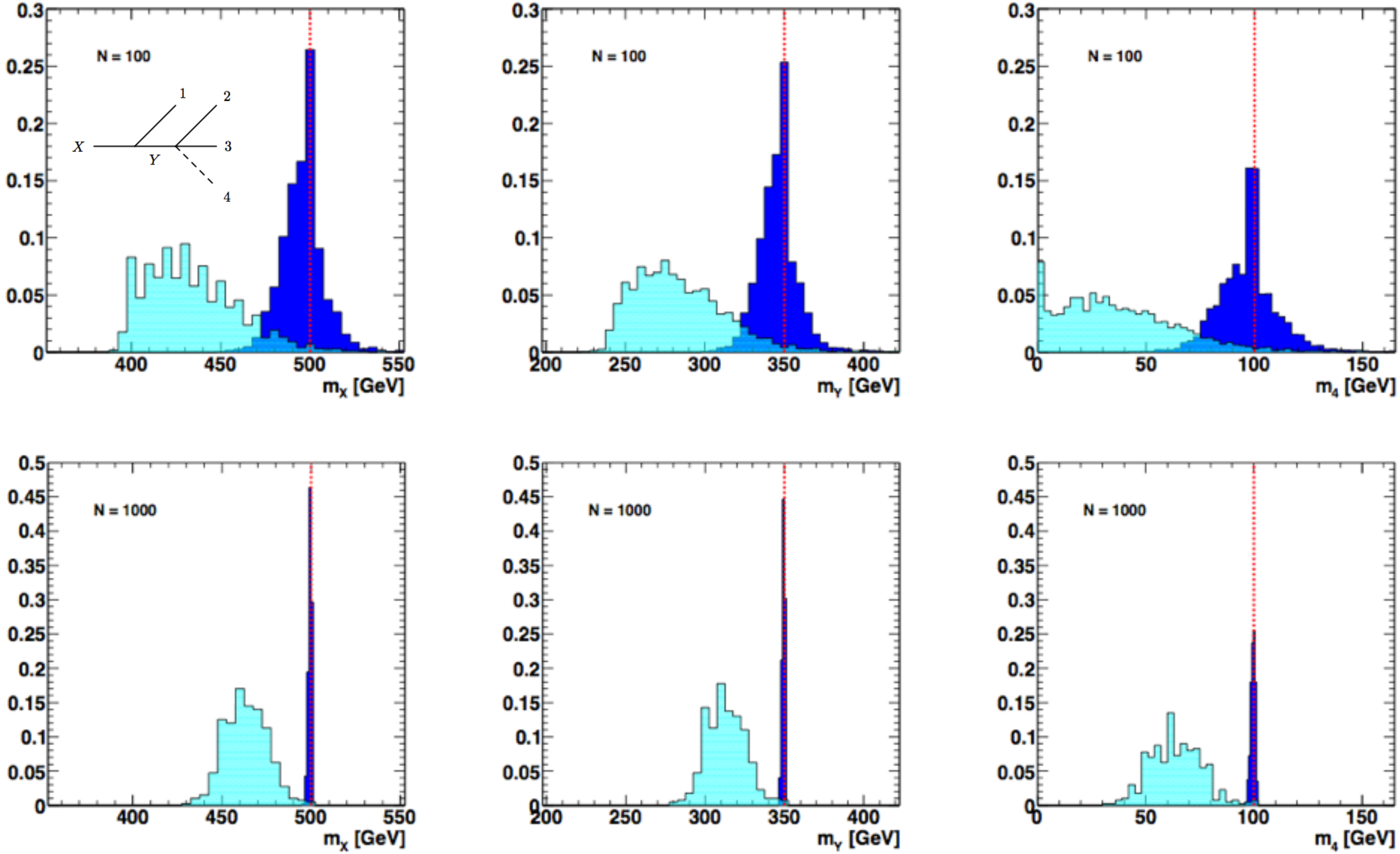


$$Q = \left(\sum_{i=\text{endpts.}} \left(\frac{\mathcal{O}_{i,predicted} - \mathcal{O}_{i,measured}}{\mathcal{O}_{i,measured}} \right)^2 \right)$$

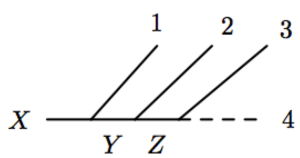
$$\mathcal{O}_i = \{(m_{123}^2)_{max}, (m_{12}^2)_{max}, (m_{23}^2)_{max}, (m_{13}^2)_{max}\}$$

Histograms in 2+3 Decay

true mass
1D endpoint/edge
multi-D phase space

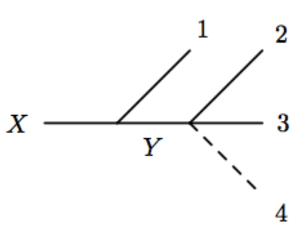


Results



$$M_X = 500 \text{ GeV}, \quad M_Y = 350 \text{ GeV},$$
$$M_Z = 200 \text{ GeV}, \quad M_4 = 100 \text{ GeV},$$
$$m_1 = m_2 = m_3 = 5 \text{ GeV},$$

Mass (GeV)	Phase space	End-points
m_X	499.89 ± 0.60	677.41 ± 157.47
m_Y	349.90 ± 0.59	527.19 ± 155.96
m_Z	199.92 ± 0.59	380.11 ± 160.57
m_4	99.93 ± 0.65	277.87 ± 156.42



$$m_X = 500 \text{ GeV}, \quad m_Y = 350 \text{ GeV}, \quad m_4 = 100 \text{ GeV},$$
$$m_1 = m_2 = m_3 = 5 \text{ GeV}.$$

Mass (GeV)	$N_{events} = 100$		$N_{events} = 1000$	
	Phase space	Endpoints	Phase space	Endpoints
m_X	495.84 ± 11.95	434.32 ± 25.93	499.40 ± 0.96	463.32 ± 11.66
m_Y	345.69 ± 12.13	284.11 ± 28.48	349.39 ± 0.97	312.94 ± 12.08
m_4	96.86 ± 13.97	37.61 ± 27.45	99.56 ± 1.08	63.83 ± 11.91

Flat direction

- We observe there is a flat direction, where all hypothesis masses are raised or lowered together
- It is easy to see the improvement in this flat direction

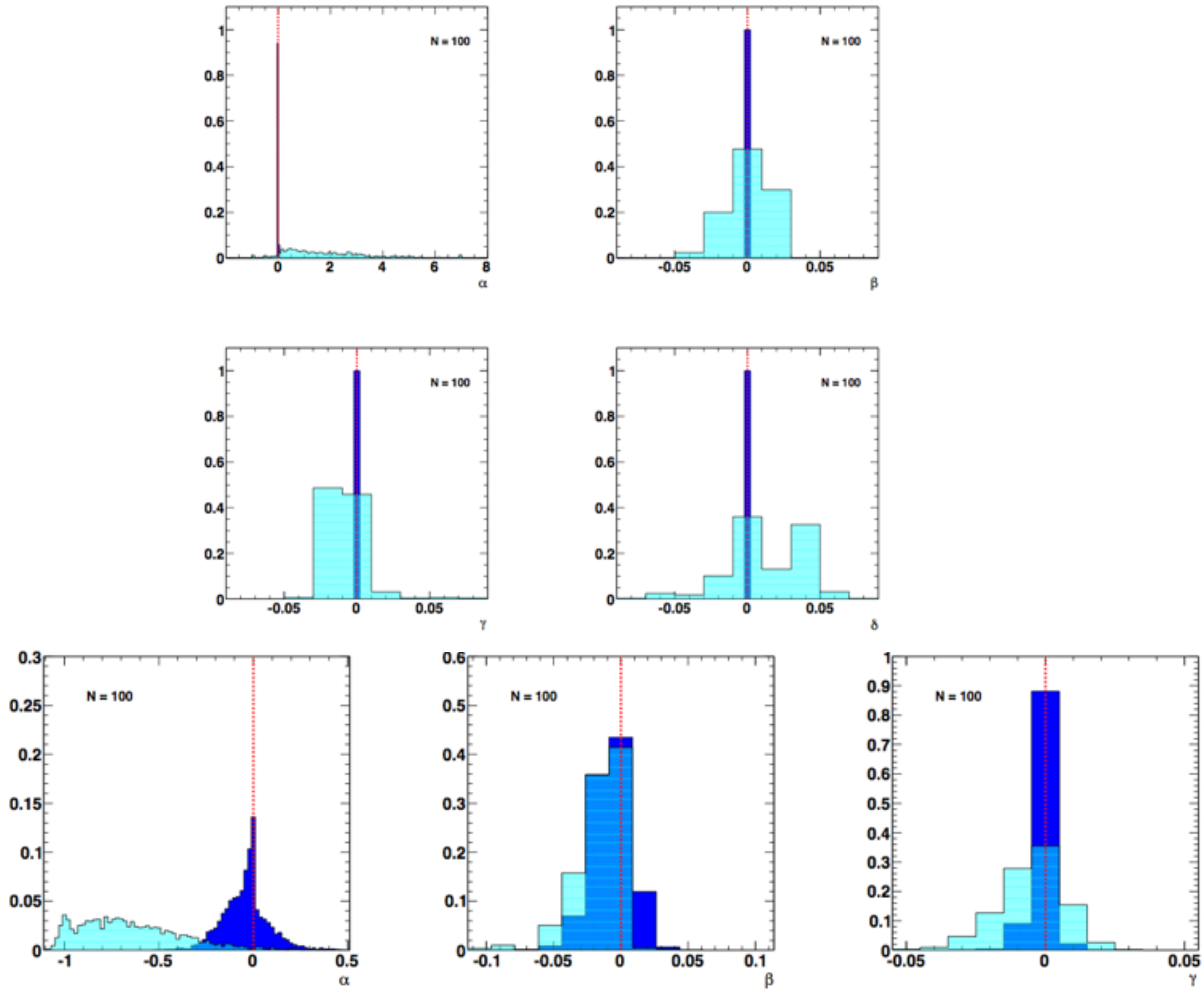
$$\tilde{m}_\sigma = M_\sigma + (100 \text{ GeV}) \left(\alpha V_\sigma^{(1)} + \beta V_\sigma^{(2)} + \gamma V_\sigma^{(3)} + \delta V_\sigma^{(4)}\right) \qquad \sigma = \{X, Y, Z, 4\}\,,$$

$$\begin{aligned} V_\sigma^{(1)} &= \{1, 1, 1, 1\}, \\ V_\sigma^{(2)} &= \{1, -1, 0, 0\}, \\ V_\sigma^{(3)} &= \{1, 1, -1, -1\}, \\ V_\sigma^{(4)} &= \{0, 0, 1, -1\}\,. \end{aligned}$$

$$\tilde{m}_\sigma = M_\sigma + (100 \text{ GeV}) \left(\alpha V_\sigma^{(1)} + \beta V_\sigma^{(2)} + \gamma V_\sigma^{(3)}\right) \qquad \sigma = \{X, Y, 4\}\,,$$

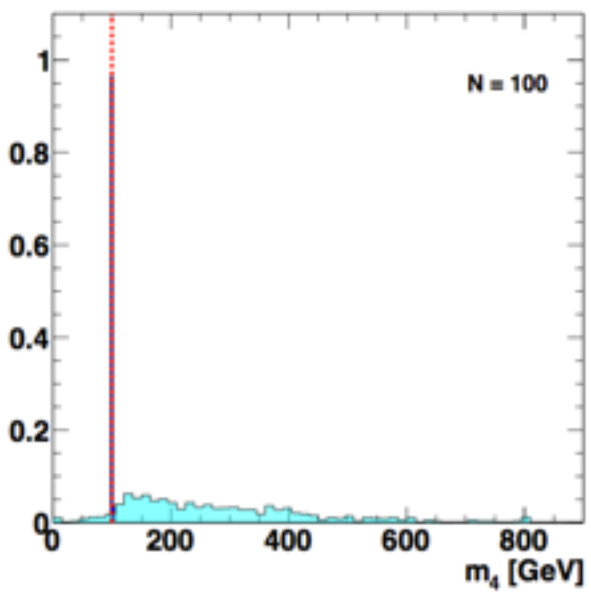
$$\begin{aligned} V_\sigma^{(1)} &= \{1, 1, 1\} \\ V_\sigma^{(2)} &= \{0, 1, -1\} \\ V_\sigma^{(3)} &= \{2, -1, -1\}\,. \end{aligned}$$

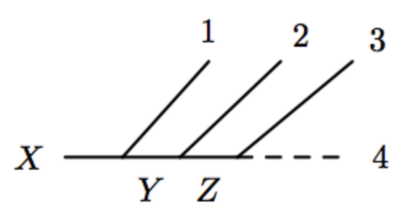
Flat direction

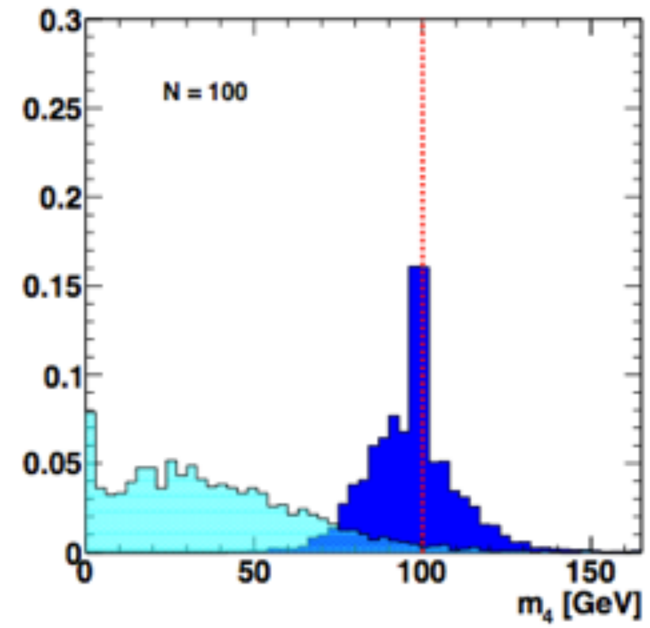


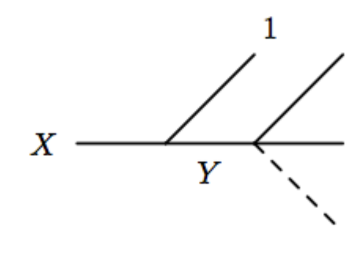
Why does 1D fail?

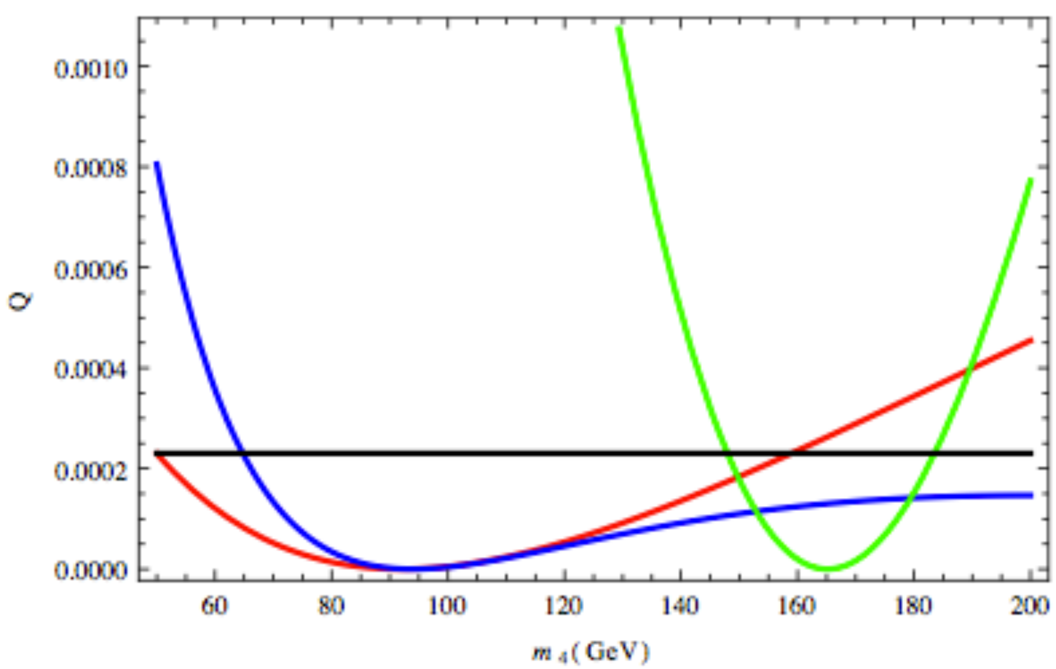
Look at the minimal chi square for each mij in 1D method

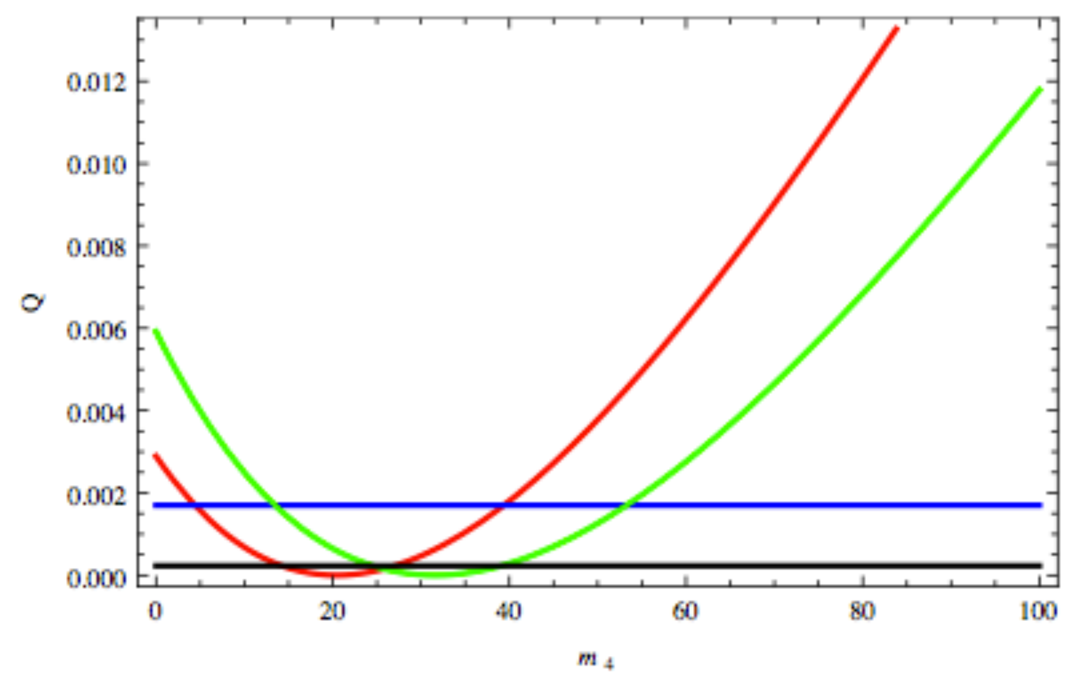










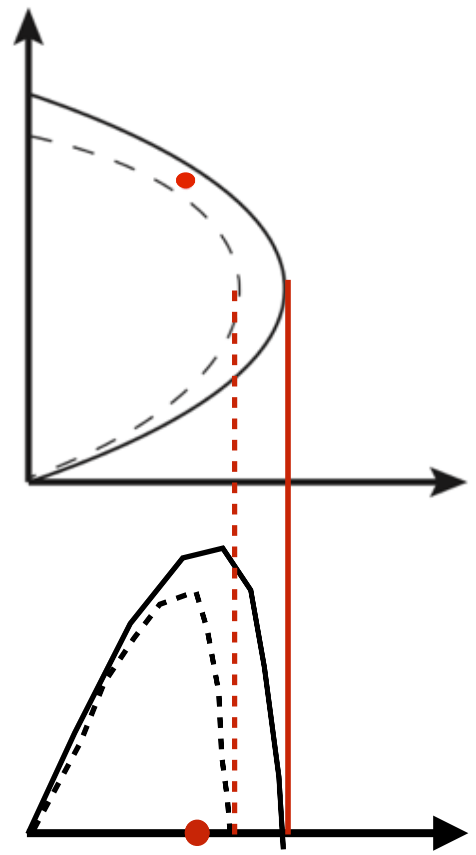


m13 (green) drives the best fit towards higher value at low statistics

m13 (green) m12(red) drives the best fit towards lower value at low statistics

Conclusion

- The endpoint/edge is just the 1D projection of the full phase space boundary, lack of full phase space correlation



- Our multi-D method using the full phase space boundary greatly improves the efficiency for mass measurement at low statistics

- Toy simulations show there is no flat direction in multi-D method

- More realistic considerations (smearing, backgrounds, combinatorial effects) are on the way

- Extend the multi-D analysis to the double chain case

Thank you!

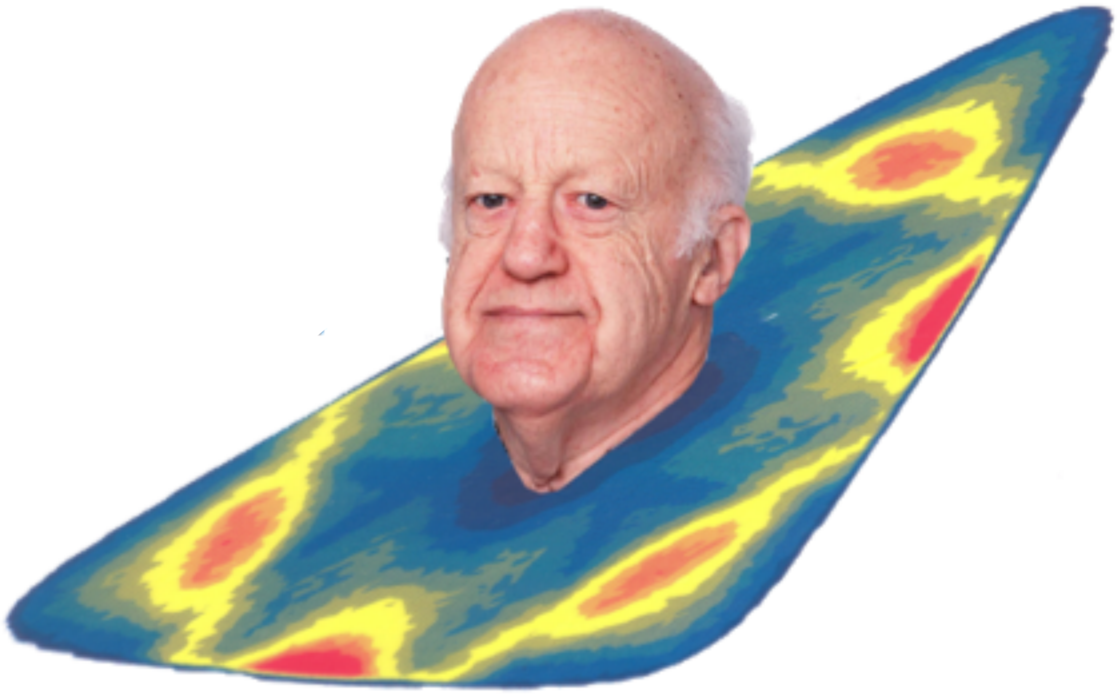


Image credit: Mike Pennington

Back up

Towards a Realistic Analysis

- Detector resolution, smearing
- Combinatorial effects for single chain
- Effect of spin in the matrix element
- Including backgrounds
- Extend to symmetric double chain

Longer Cascade Decays

- N-body phase space

Is the enhancement in n= 4 a generic feature?

$$d\Pi_n \propto \left(\sum_{i<j} dm_{ij}^2\right) (\sqrt{\Delta_4})^{n-5}\ \delta(\Delta_5)\cdots\delta(\Delta_n)\ \delta\left(\sum_{i<j} dm_{ij}^2 - C\right)$$

Naively, the enhancement goes away

- Factorization and Jacobian

Factorize to 4-body and/or less

For example, X -> 1 + Y, Y -> (2 + 3) + Z, Z -> 4 + 5

Jacobian factors appear if there are on-shell resonances

(1) X -> 1 + Y, Y -> 2 + Z, Z -> 3 + K, K -> 4 + 5

(2) X -> 1 + Y, Y -> 2 + Z, Z -> 3 + 4 + 5

(3) X -> 1 + Y, Y -> 2 + 3 + Z, Z -> 4 + 5

Some Details

Label by Δ_i the coefficients of the characteristic polynomial of $\mathcal{Z}$, namely the equation

$$\begin{aligned} 0 &= \mathrm{Det}\left[\lambda \mathrm{I}_{n\times n} - \mathcal{Z}\right] \\ &= \lambda^n - \left(\sum_{i=1}^{n} \Delta_i \lambda^{n-i}\right). \end{aligned} \tag{3}$$

For example, $\Delta_1 = \mathrm{Tr}[\mathcal{Z}] = \sum_{i=1}^{n} m_i^2$ for any n, and $\Delta_4 = -\mathrm{Det}[\mathcal{Z}]$ for $n = 4$. It is

$$\lambda_0 = \lambda\left(m_1^2, m_{23}^2, m_{123}^2\right), \tag{17}$$

$$G_1 = G\left(m_{12}^2, m_{23}^2, m_{123}^2, m_2^2, m_1^2, m_3^2\right), \tag{18}$$

$$G_2 = G\left(m_{123}^2, \tilde{m}_Y^2, \tilde{m}_X^2, m_{23}^2, m_1^2, \tilde{m}_4^2\right), \tag{19}$$

and the kinematic functions λ and G are defined as [85].

$$\lambda(X,Y,Z) = X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX,$$

$$\begin{aligned} G(X,Y,Z,U,V,W) = {} & XY(X+Y-Z-U-V-W) + ZU(Z+U-X-Y-V-W) \\ & + VW(V+W-X-Y-Z-U) + XZW + XUV + YZV + YUW. \end{aligned}$$

The end