Dispersive approach to hadronic light-by-light scattering: Reconstructing $\gamma^*\gamma^* \to \pi\pi$

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based in parts on arXiv:1402.7081 and arXiv:1309.6877

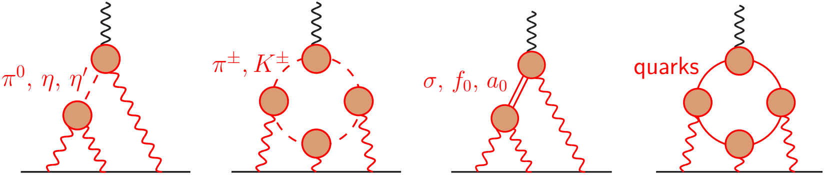
Mainz, April 10, 2014

Outline

1. Dispersive approach to HLbL
   - Master formula for one- and two-pion intermediate states
   - Experimental input

2. Dispersion relations for $\gamma^*\gamma^* \to \pi\pi$
   - Roy–Steiner equations
   - Generalization to virtual photons
   - Anomalous thresholds

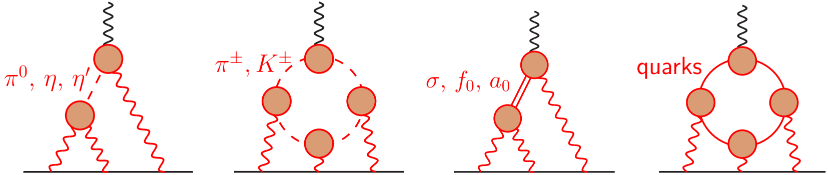
Anatomy of HLbL scattering



- Large **model dependence**

  $\hookrightarrow$ Can one find a **data-driven** approach also for **light-by-light scattering**?

Anatomy of HLbL scattering



- Large **model dependence**

  $\hookrightarrow$ Can one find a **data-driven** approach also for **light-by-light scattering**?

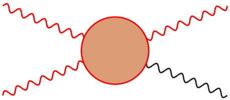
- **Dispersive point of view**
  - Analytic structure: poles and cuts

    $\hookrightarrow$ **Residues** and **imaginary parts** $\Rightarrow$ by definition **on-shell** quantities

    $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment
  - Out of the above only **pion pole** model independent
  - Expansion: mass of intermediate states, partial waves

- **Light-by-light tensor**

$\gamma^*(q_1,\mu)\gamma^*(q_2,\nu) \to \gamma^*(-q_3,\lambda)\gamma(k,\sigma)$ $\qquad i\Pi^{\mu\nu\lambda\sigma} =$



  - **Gauge invariance**: 29 independent gauge-invariant structures cf. Bijnens et al. 1995

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{29} A_i^{\mu\nu\lambda\sigma} \Pi_i$$

$\hookrightarrow$ in practice use 45 (redundant) structures

  - 5 kinematic variables: $s = (q_1+q_2)^2$, $t = (q_1+q_3)^2$, q_1^2, q_2^2, q_3^2

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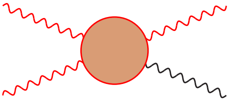

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  - 5 kinematic variables: $s = (q_1+q_2)^2$, $t = (q_1+q_3)^2$, q_1^2, q_2^2, q_3^2

- Decompose the tensor according to

$$\Pi_{\mu\nu\lambda\sigma} = \Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}} + \Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

$\hookrightarrow$ accounts for **one-** and **two-pion** intermediate states

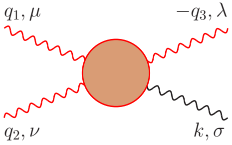
Pion pole: pion transition form factor

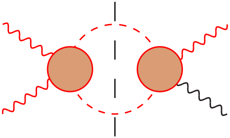
Knecht, Nyffeler 2001

Master formula for pion-pole contribution

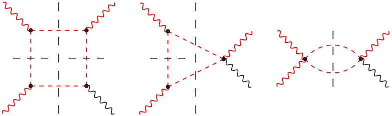
$$a_\mu^{\pi^0\text{-pole}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s \left((p+q_1)^2 - m^2\right)\left((p-q_2)^2 - m^2\right)}$$
$$\times \left\{ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) F_{\pi^0\gamma^*\gamma^*}(s,0)}{s - M_\pi^2} T_1(q_1,q_2;p) + \frac{F_{\pi^0\gamma^*\gamma^*}(s,q_1^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2 - M_\pi^2} T_2(q_1,q_2;p) \right\}$$

- **Wick rotation**: only **space-like** s, q_1^2, q_2^2 contribute
- Crucial ingredient: **pion transition form factor** $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$
- **Dispersive approach**
  - Fix parameters wherever data are available
  - Use **analyticity** to go to the space-like region





- **Unitarity**: restrict to $\pi\pi$ (and $K\bar{K}$) intermediate states
  $\hookrightarrow$ **two-meson reducible** contributions
- Separate terms with **simultaneous cuts**

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2) \times \Bigg[$$  $$\Bigg]$$

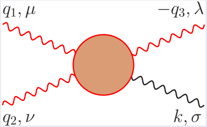
- Multiplication of sQED diagrams with F^V_π gives correct q^2-dependence
  $\hookrightarrow$ **not an approximation**
- Remaining $\pi\pi$ contribution included in $\bar{\Pi}_{\mu\nu\lambda\sigma}$ has cuts only in one channel
  $\hookrightarrow$ dispersion relations for this part Talk by G. Colangelo at MITP work shop

Master formula for $\pi\pi$ intermediate states

Colangelo, MH, Procura, Stoffer arXiv:1402.7081

Master formula for $\pi\pi$ intermediate states

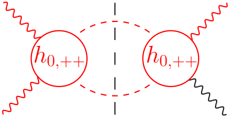
$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i T_i(q_1,q_2;p) I_i\left(s,q_1^2,q_2^2\right)}{q_1^2 q_2^2 s\left((p+q_1)^2-m^2\right)\left((p-q_2)^2-m^2\right)}$$



- **Partial waves** up to $L=2$, manifest **crossing symmetry**
- $I_i\left(s,q_1^2,q_2^2\right)$: dispersive integrals over $\gamma^*\gamma^* \to \pi\pi$ **helicity partial waves**, e.g.

$$I_1\left(s,q_1^2,q_2^2\right) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'-s} \left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda\left(s',q_1^2,q_2^2\right)}\right) \operatorname{Im} h^0_{++,++}\left(s';q_1^2,q_2^2;s,0\right)$$

$$\operatorname{Im} h^0_{++,++}\left(s';q_1^2,q_2^2;s,0\right) = \frac{\sqrt{1-\frac{4M_\pi^2}{s'}}}{16\pi} h_{0,++}\left(s';q_1^2,q_2^2\right) h_{0,++}\left(s';s,0\right)$$



$$\lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

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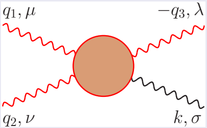
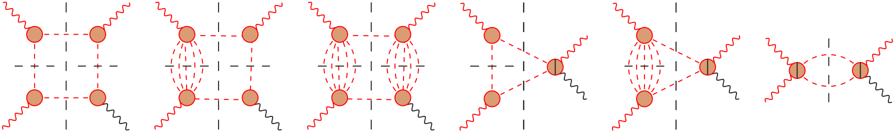
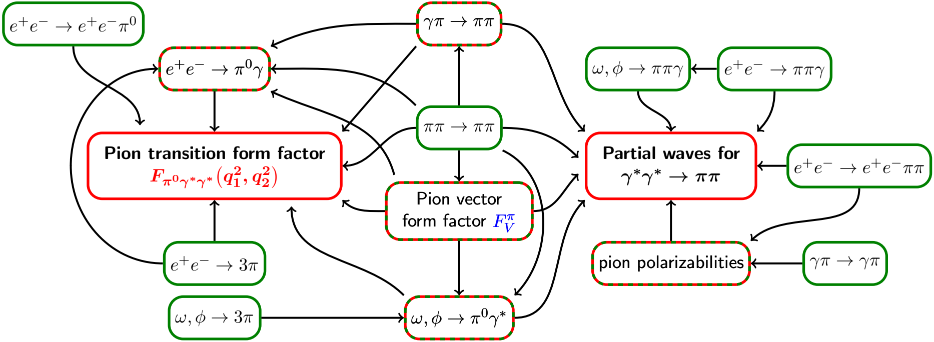


- **Partial waves** up to $L = 2$, manifest **crossing symmetry**
- What is included? How?



$\hookrightarrow$ sorted by analytic structure in the crossed channel

Experimental input: summary



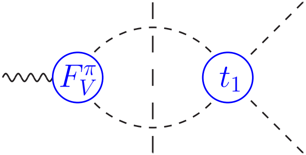
- Pion transition form factor: Talk by B. Kubis at MITP work shop
- Ideal world: measure space-like, doubly-virtual $\gamma^*\gamma^* \to \pi\pi$ for arbitrary virtualities

  $\hookrightarrow$ partial-wave analysis
- This talk: how to realistically **reconstruct** $\gamma^*\gamma^* \to \pi\pi$

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\text{Im}\, F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big) F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

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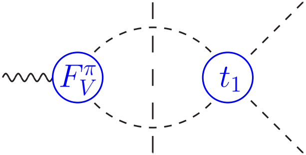

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- Solution in terms of **Omnès function** Omnès 1958

$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow P(s) = 1$

$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion

$+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds**

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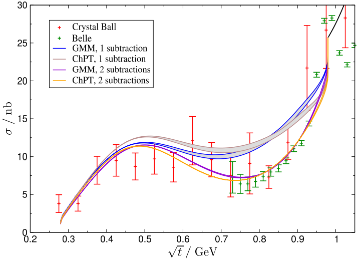

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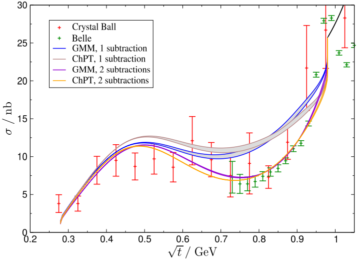
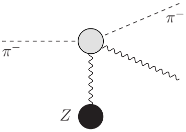
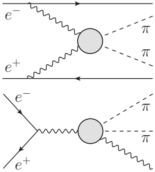


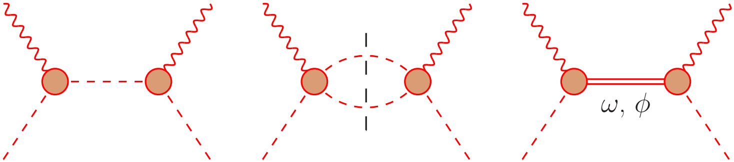
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$





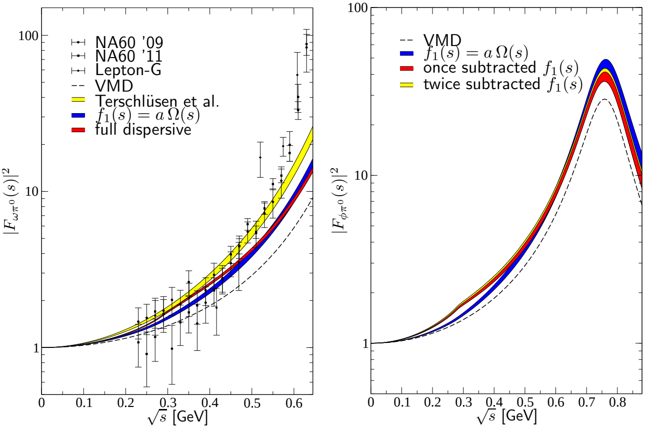
$\hookrightarrow$ discuss these constraints in the following

Left-hand cut



- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude
    $\hookrightarrow$ see pion transition form factor
  - $3\pi \sim \omega, \phi$: narrow-width approximation
    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$
  - Higher intermediate states also potentially relevant: **axials**, **tensors**
    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

$\omega, \phi \to \pi^0\gamma^*$ transition form factor



Schneider, Kubis, Niecknig 2012

- Puzzle of steep rise in $F_{\omega\pi^0}$

  $\hookrightarrow$ Measurement of $F_{\phi\pi^0}$ would be extremely valuable

- Clarification important for pion transition form factor, but also $\gamma^*\gamma^* \to \pi\pi$

Subtraction functions

Omnès representation for S-wave

$$h_{0,++}(s) = \Delta_{0,++}(s) + \Omega_0(s)\bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2) + \frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2) + q_1^2q_2^2 b(q_1^2,q_2^2)$$

$$+\frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|} + \frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+\frac{2q_1^2q_2^2 s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\bigg] \qquad s_\pm = q_1^2 + q_2^2 \pm 2\sqrt{q_1^2q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s), \Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2) - a_-(q_1^2,q_2^2)$ multiply $q_1^2q_2^2$ and $\sqrt{q_1^2q_2^2}$
    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2) = (a_+(q_1^2,q_2^2) + a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements
    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

Subtraction functions: chiral constraints

- 1-loop result for arbitrary q_i^2, e.g.

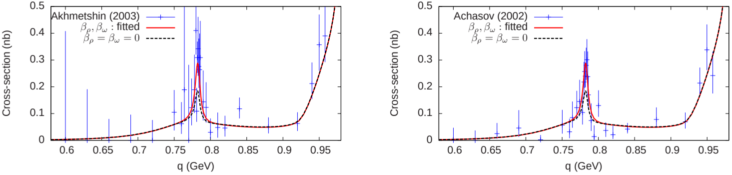
$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2F_\pi^2(q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2q_2^2\Big)C_0(q_1^2,q_2^2)$$
$$+q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

- Special case: $q_1^2=q_2^2=0$

$$a^{\pi^\pm}(0,0) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$
$$a^{\pi^0}(0,0) = -\frac{1}{96\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0) = -\frac{1}{1440\pi^2F_\pi^2M_\pi^2}+\cdots$$

$\hookrightarrow$ resum higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation



Moussallam 2013

- Singly-virtual case: phenomenological representation with chiral constraints

  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013

- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$

  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics

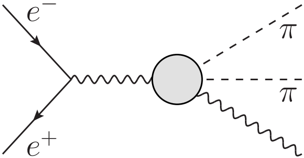
- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

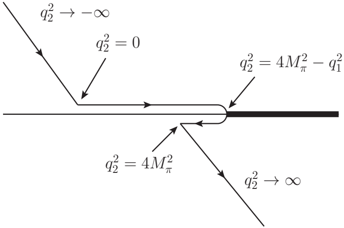
$$a^2(q_1^2, q_2^2) = \alpha_0 \Big[\alpha^2 + \alpha\Big(q_1^2 \mathscr{F}^\rho(q_1^2) + q_2^2 \mathscr{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2 \mathscr{F}^\rho(q_1^2)\mathscr{F}^\rho(q_2^2)\Big]$$

$$\mathscr{F}^\rho(q^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^*\Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2, 0)$ alone!

Anomalous thresholds





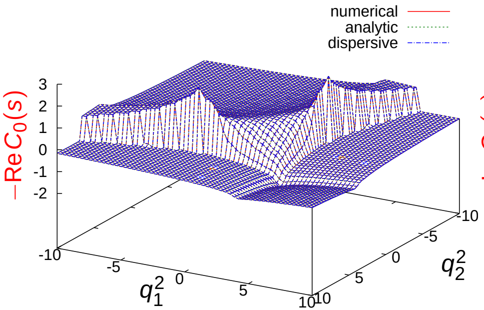
- **Analytic continuation** in q_i^2 in time-like region non-trivial in doubly-virtual case
- Singularities from second sheet move onto first one

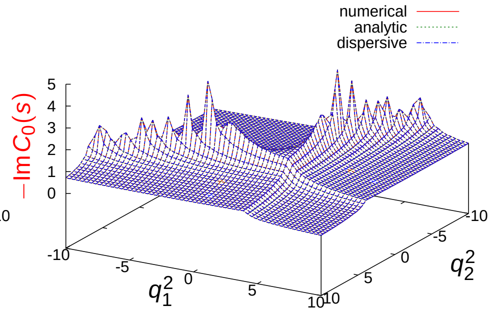
  $\hookrightarrow$ need to **deform** the **integration contour**
- Problem already occurs for a simple triangle loop function $C_0(s)$

  $\hookrightarrow$ extra factor $t_\ell(s)/\Omega_\ell(s)$ is well defined in the whole complex plane

  $\hookrightarrow$ remedy in case of $C_0(s)$ can be taken over to full Omnès solution
- Becomes relevant for $e^+e^+ \to e^+e^-\pi\pi$ in time-like kinematics

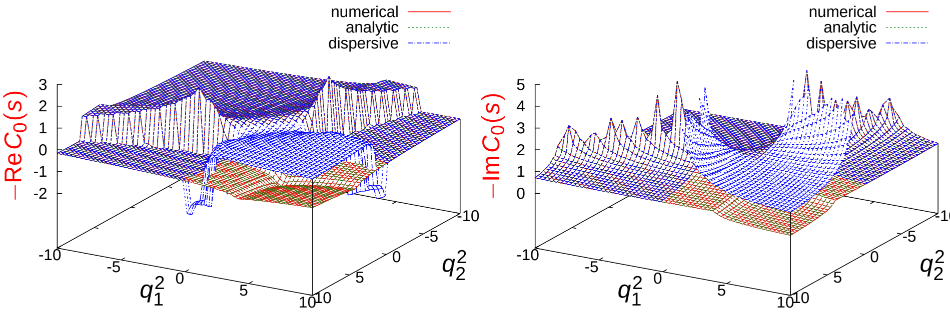
Numerical check of anomalous thresholds





- Comparison for $s = 5$, $M_\pi = 1$

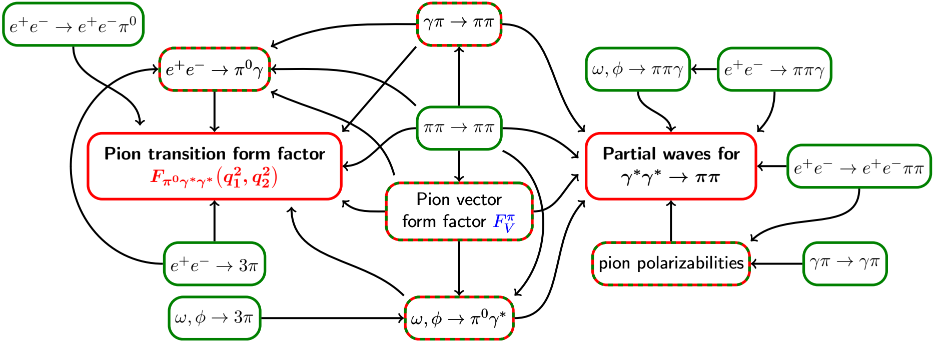
  $\hookrightarrow$ **Dispersive reconstruction** of $C_0(s)$ works!



- Ignore anomalous piece

  $\hookrightarrow$ Substantial deviations for **large virtualities**!

Experimental input: summary



- (Transition) Form factors to fix **left-hand cut**
- $e^+e^- \to e^+e^-\pi\pi$ **singly-virtual** / $e^+e^- \to \pi\pi\gamma$ + ChPT + pion polarizabilities to fix **subtraction constants**

Summary

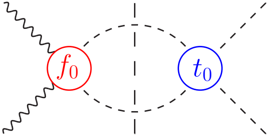
- **Road map** towards a representation for HLbL scattering **connected to data** as closely as possible
  - **Pion pole**: pion transition form factor
  - $\pi\pi$ **intermediate states**: helicity partial waves for $\gamma^*\gamma^* \to \pi\pi$
- **Roy–Steiner equations** for $\gamma^*\gamma^* \to \pi\pi$
  - **Left-hand cut**: (transition) form factors, $\gamma^* \to 3\pi$
  - **Subtraction constants**: chiral constraints, pion polarizabilities, **singly-virtual data** already help a lot!
  - **Anomalous thresholds** for **time-like** $e^+e^- \to e^+e^-\pi\pi$

$\gamma^*\gamma^* \to \pi\pi$

- **Left-hand cut** approximated by **pion pole** + **resonances**
- **Unitarity** for $\gamma^*\gamma^* \to \pi\pi$ system: Watson's theorem

$$\text{disc}\, f_0(s;q_1^2,q_2^2) = 2i\sigma_s f_0(s;q_1^2,q_2^2)\, t_0^*(s)$$

$$t_0(s) = \frac{1}{\sigma_s} e^{i\delta_0(s)} \sin\delta_0(s) \qquad \sigma_s = \sqrt{1-\frac{4M_\pi^2}{s}}$$



$\hookrightarrow$ solution in terms of **Omnès function**, e.g. for pion pole only

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2) \sin\delta_0(s')}{(s'-s)|\Omega_0(s')|} \qquad N_0(s;q_1^2,q_2^2) = \frac{2L}{\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$$L = \log \frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \qquad \lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

- **Analytic continuation** in q_i^2?

$\gamma^*\gamma^* \to \pi\pi$: analytic continuation

$$L = \log \frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \xrightarrow{q_2^2\to 0} \pm\log\frac{1+\sigma_s}{1-\sigma_s}$$

- Singularities of the log: **anomalous thresholds**

$$s_\pm = q_1^2+q_2^2-\frac{q_1^2q_2^2}{2M_\pi^2}\pm\frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2-4M_\pi^2)q_2^2(q_2^2-4M_\pi^2)}$$

$\hookrightarrow$ usual Omnès derivation breaks down

- Idea: consider first the **scalar loop function**

$$C_0(s)\equiv C_0\big((q_1+q_2)^2;q_1^2,q_2^2\big)=\frac{1}{i\pi^2}\int\frac{\mathrm{d}^4k}{(k^2-M_\pi^2)\big((k+q_1)^2-M_\pi^2\big)\big((k-q_2)^2-M_\pi^2\big)}$$

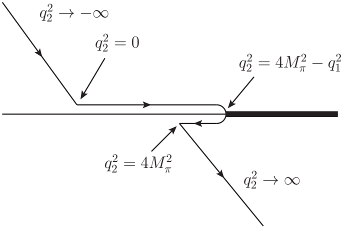
$$\operatorname{disc} C_0(s) = -\frac{2\pi i}{\sqrt{\lambda(s,q_1^2,q_2^2)}}L = -\pi i\,\sigma_s N_0(s;q_1^2,q_2^2)$$

$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2)q_2^2(q_2^2 - 4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

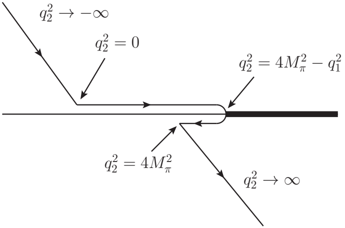
  $\hookrightarrow$ moves through unitarity cut onto first sheet



$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2-4M_\pi^2)q_2^2(q_2^2-4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$
  $\hookrightarrow$ moves through unitarity cut onto first sheet
- Need to deform the contour



$$C_0(s) = \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s'-s}$$

$$+\,\theta\left(q_1^2+q_2^2-4M_\pi^2\right)\frac{1}{2\pi i}\int_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x-s}$$

$$s_x = x4M_\pi^2 + (1-x)s_+ \qquad \text{disc}_{\text{an}}\, C_0(s) = \frac{4\pi^2}{\sqrt{\lambda\left(s,q_1^2,q_2^2\right)}}$$

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\,C_0(s)}{\pi i\,\sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\,C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ Additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

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- Integrand similar to the scalar-loop example

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$\hookrightarrow$ Additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

Omnès representation for $\gamma^*\gamma^* \to \pi\pi$

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

$$+\,\theta(q_1^2+q_2^2-4M_\pi^2)\frac{\Omega_0(s)}{2\pi i}\int_0^1 dx\,\frac{\partial s_x}{\partial x}\frac{\mathrm{disc}_{\mathrm{an}} f_0(s_x;q_1^2,q_2^2)}{s_x-s}$$

$$s_x = x4M_\pi^2 + (1-x)s_+ \qquad \mathrm{disc}_{\mathrm{an}} f_0(s;q_1^2,q_2^2) = -\frac{8\pi}{\sqrt{\lambda(s,q_1^2,q_2^2)}}\frac{t_0(s)}{\Omega_0(s)}$$