Roles of Coarse Graining in Hydrodynamic Modeling and Variational Principle

Rafael Derradi de Souza
Tomoi Koide
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A Tribute to Laszlo Csernai

And 40 years of Anniversary of Hydrodynamic Description of Relativistic Heavy Ion Collisions

Great Success of Relativistic Hydro Approaches in Relativistic Heavy Ion Collisions

…

- Flow Observables (V1, V2, V3..Vn) vs. Global Parameters(pT, Centrality, y..) in AA Collisions
- Ridge, and possbile initial geometric nature

:

- Big Expectations for Extracting QGP properties and Initial Condition !...

- But in pA also ?

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- But in pA also ? YES - WHY NOT - NO!

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- Big Expectations for Extracting QGP properties and Initial Condition !

- But in pA also ?   YES - WHY NOT - NO!   - Depends....

What is Hydrodynamics?

Let me start with **trivialities**

What is Hydrodynamics?

Let me start with **trivialities** (maybe till the end..)

What is a True Hydrodynamics?

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What is a True Hydrodynamics?

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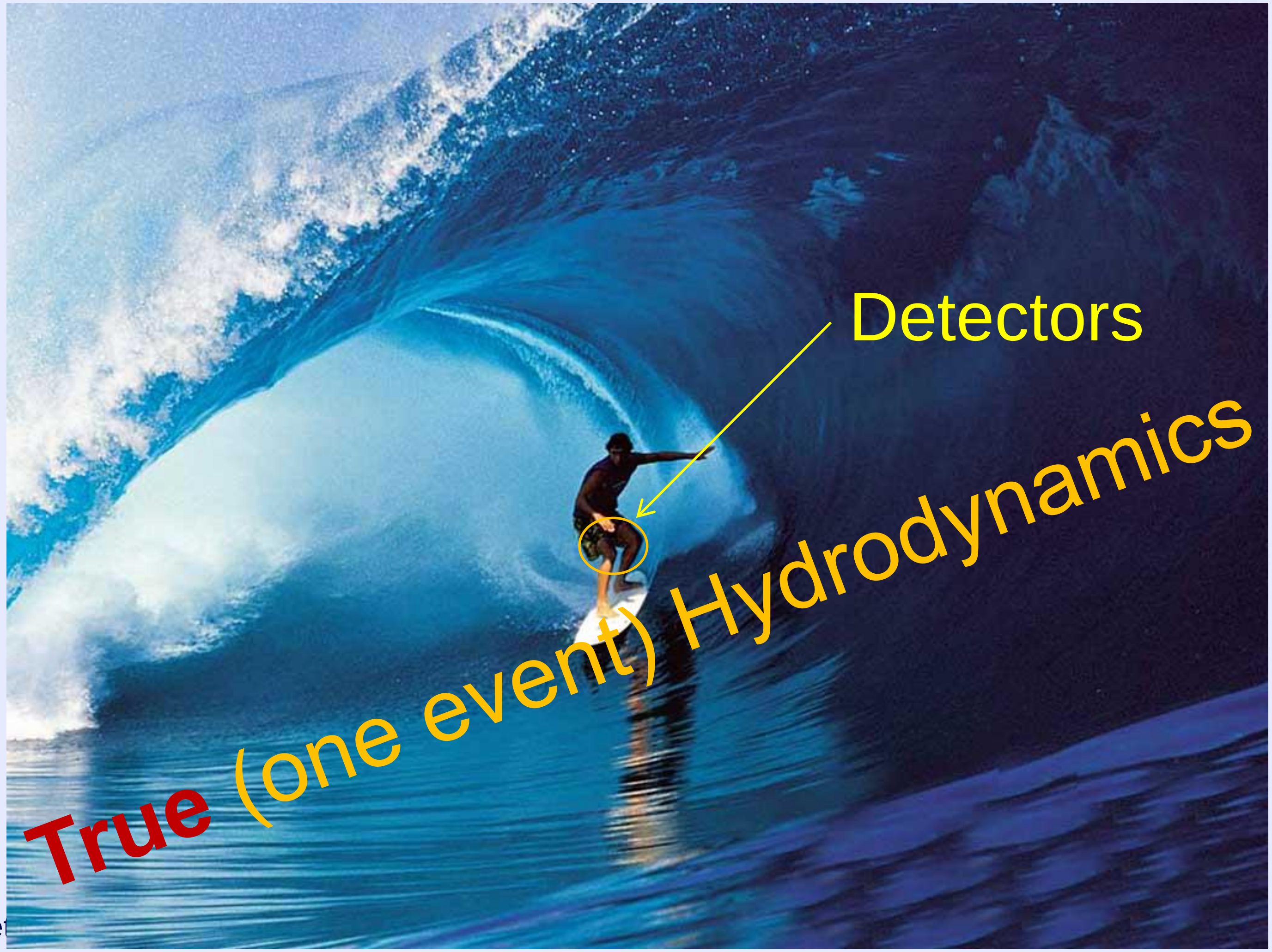

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But we only know later the effects.. (Larissa)

What is a "TRUE" Hydrodynamics?

A closed system of equations for local macroscopic variables, based of conservation Laws for the Energy and Momentum Tensor, and conserved currents,

(ideal)

$$\partial_{\mu} T^{\mu\nu}(\vec{x},t) = 0,$$

$$\partial_{\mu} j^{\mu}(\vec{x},t) = 0,$$

with additional equations to describe the **well-defined thermodynamical properties** of the matter in question (EoS, Relaxation equations with transport coefficients, if necessarly) in **LOCAL THERMAL EQUILIBRIUM (sufficient condition ! – Alina's talk)**

What is a "TRUE" Hydrodynamics?

- A **precise analysis** of hydro-profile (dynamics) would furnish the information of properties of the matter and initial condition..... but
- Question is: How precise we are looking the profile ?
- Are we really observong a true hydro in Relativistic Heavy Ion collisions?

What is a “TRUE” Hydrodynamics?

- A **precise analysis** of hydro-profile (dynamics) would furnish the information of properties of the matter and initial condition.....
- Question is: How precise we are looking the profile ?
- If we don’t have (or don’t care) a good resolution of macroscopic dynamics, the properties of the matter that we deduce may not be precise also.

What is a “TRUE” Hydrodynamics?

- A **precise analysis** of hydro-profile (dynamics) would furnish the information of properties of the matter and initial condition.....

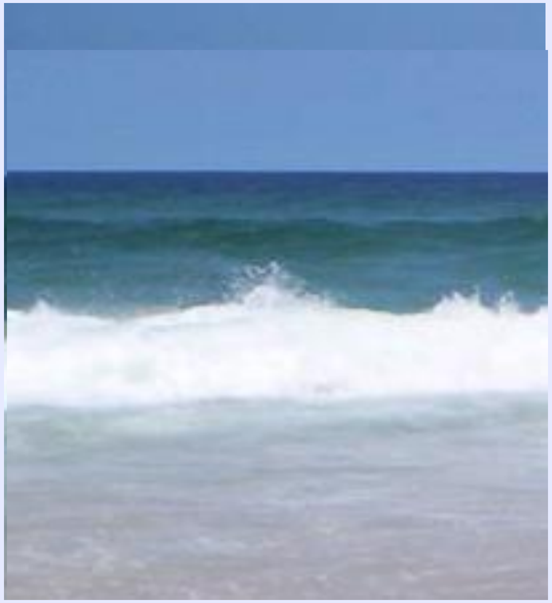
- Relativistic Heavy Ion Case

1. Violent process (short time scale) in a Small system, large fluctuation
2. What is the resolution? Effective EoS ? (Conservation law, continuum effective variables)

P. Mota et al, The European Physical Journal A 48 (11), 1-12, 2012

Waves on the beach

Waves on the beach



Waves on the beach

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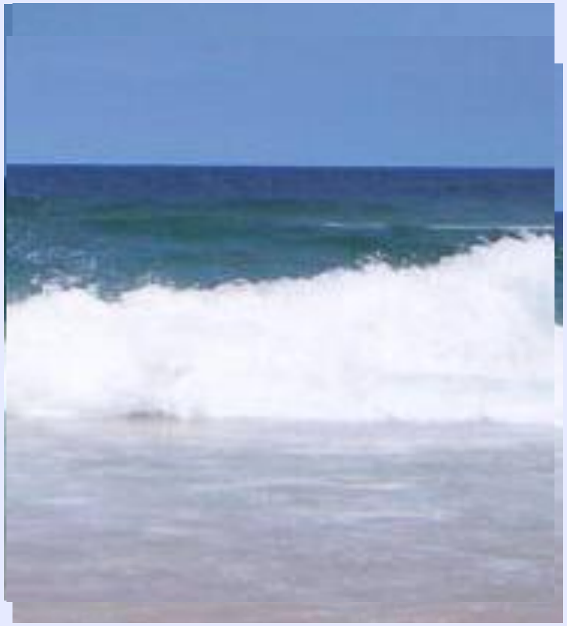
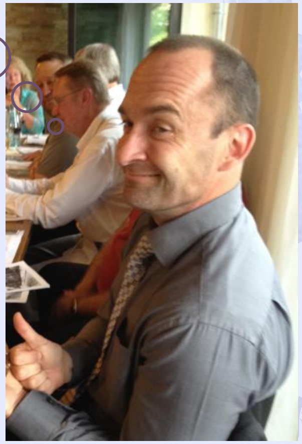


Waves on the beach

They are different, nice to look, but
We usually don't care much

Waves on the beach



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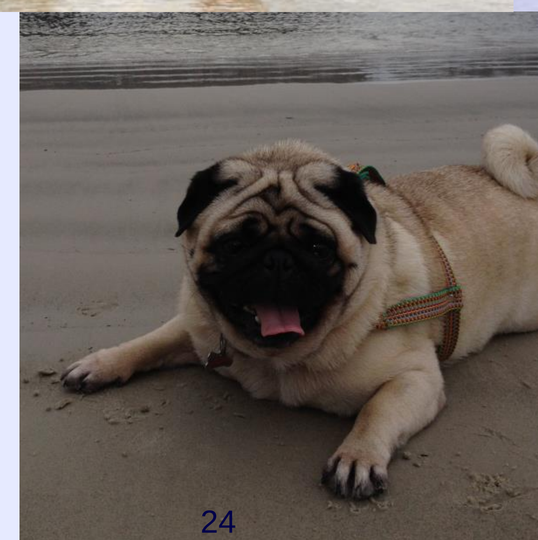


But some do care!

Or, most people think all the same,..

But there is always someone who knows the important differences in event-by-event basis

What are "hydro" Observables?

Dependence on parameters associated with the (loosely guessed) initial geometry of final state observables (spectra, particle correlations).

Usually not determined for ONE event..

Hydro in Relativistic Heavy Ion Collision
How Quantitatively precise ?

Uncertainties associated

- EoS, Transport Coefficients (?)
- Freezeout Mechanism (Tough)
- Initial Condition (Challenging)
- Event-by-Event vs. Ensemble Average? (To be clarified)

We need to keep some care,..

- In a Japanese popular- saying, “Typically in the following three conditions,

- in a twilight,   - from far,   - half-hidden by a hat”

We need to keep some care,..

- In a Japanese popular- saying, “Typically in the following three conditions,

- in a twilight, - from far, - half-hidden by a hat”
make the man(woman) looking nice,…. “

(We usually see what we WANT to see)

Counter-examples of Real Hydro (“Pseudo Hydro”)

- Schrödinger Equation – Quantum Hydro
- Isotropic massless gas – Non Equilibrium
- Initial state correlation in free streaming case
- Event average -> Effective EoS

:

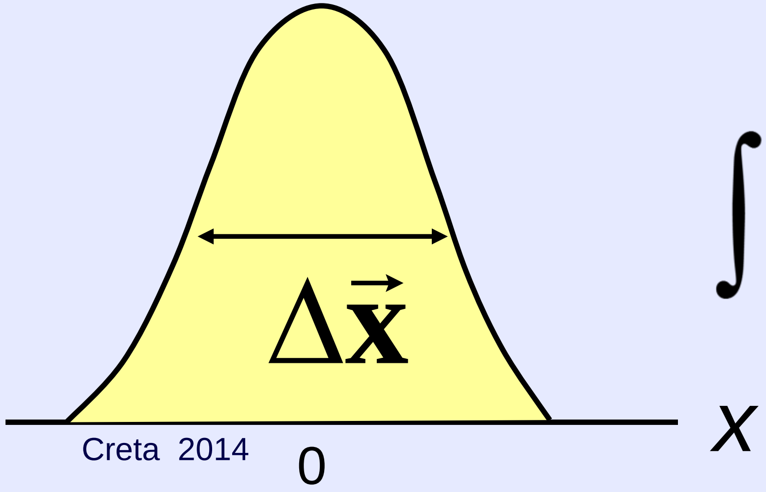
Hydrodynamical Representation of microscopic dynamics

- Starting from a microscopic dynamical model which contains many-body interactions (compression effects), define the energy momentum tensor and current and compare with the hydrodynamics
- and reproduce the Collective flows…
- For example, PHSD as one available microscopic model for this purpose   -> Next talk by Rafael D. De Souza

Define the Energy-Momentum Tensor by smoothing function

$$T^{\mu\nu}(\vec{x},t) = \sum_i \frac{p^{\mu}{}_i(t)\, p^{\nu}{}_i(t)}{p^{0}{}_i(t)} W(\vec{x} - \vec{x}_i(t); \Delta\vec{x})$$

$$W(\vec{\mathbf{x}};\Delta\vec{\mathbf{x}}) \leftrightarrow \text{smoothing kernel}$$



$$\int W(\vec{\mathbf{x}};\Delta\vec{\mathbf{x}})\, d^3\vec{\mathbf{x}} = 1$$

Diagonalization of $T^{\mu\nu}(\vec{x},t)$

$$T^{\mu}{}_{\nu}(\vec{x},t) \rightarrow \left(T_L\right)^{\mu}{}_{\nu} = \begin{pmatrix} \varepsilon(\vec{x},t) & 0 \\ 0 & \vec{p}(\vec{x},t) \end{pmatrix}$$

by a Lorentz Boost $\Lambda(\boldsymbol{\beta})$ + Spatial Rotation

$$\left(T_L\right)^{\mu}{}_{\nu} = \begin{pmatrix} \varepsilon(\vec{x},t) & 0 \\ 0 & \vec{p}(\vec{x},t) \end{pmatrix}$$

Local Rest Frame (Landau)

Diagonalization of $T^{\mu\nu}(\vec{x},t)$

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by a Lorentz Boost $\Lambda(\boldsymbol{\beta})$ + Spatial Rotation

$T^{\mu}{}_{\nu}(\vec{x},t)$ depends on the coarse-graining size, so do

$\vec{\beta},\ \varepsilon,\ \vec{p}$ (the flow profile)

Hydrodynamic description

We want to close a system of equations for

$$T^{\mu}{}_{\nu}(\vec{x},t) \rightarrow \left(T_L\right)^{\mu}{}_{\nu} = \begin{pmatrix} \varepsilon(\vec{x},t) & 0 \\ 0 & \vec{p}(\vec{x},t) \end{pmatrix}$$

together with the Lorentz transformation $\Lambda(\boldsymbol{\beta},\mathbf{R})$

Hydrodynamic description

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together with the Lorentz transformation $\Lambda(\beta,\mathbf{R})$

Can we close the system? Usually not.

How to close the system of Coarse Grained Variables?

From the microscopic theory, derive the equations for macroscopic variables by

- Trancation of Some expansion,
- Method of Projection,
- Effective Theory through Variational method
- :

Variacional Method

Variacional Method

Consider Physical Process as Optimization procedure of a Scalar Quantity

Variacional Method

Consider Physical
Process as Optimization
procedure of
a Scalar Quantity

- Useful for physical insight
- Formal development of the theory
- Approximation Methods

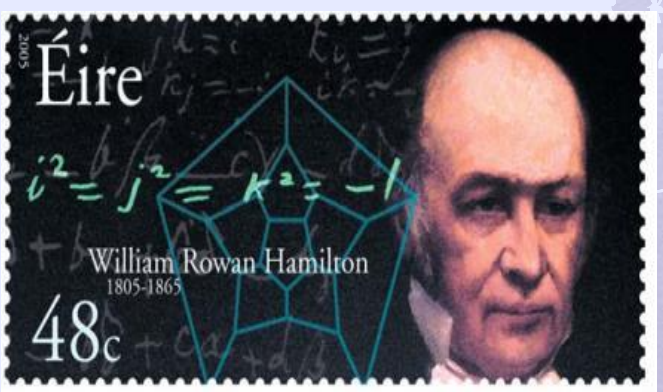
Variational Formlation of Classical Mechanics

$$I[q(t)] = \int dt\, L\left(q, \frac{dq}{dt}\right) \longrightarrow \delta I[q(t)] = 0, \quad \forall \delta q(t)$$

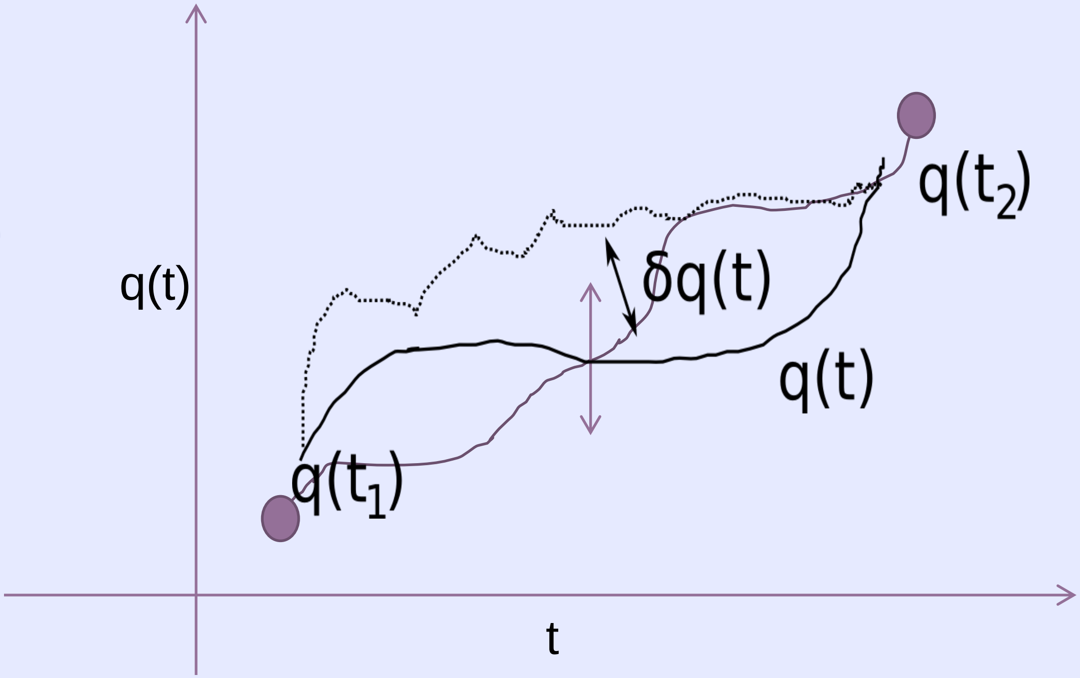
ou

$$I[q(t), p(t)] = \int dt \left\{ p\frac{dq}{dt} - H(q,p) \right\}$$

$$\longrightarrow \quad \delta I[q(t), p(t)] = 0, \quad \forall \delta q(t),\ \delta p(t)$$

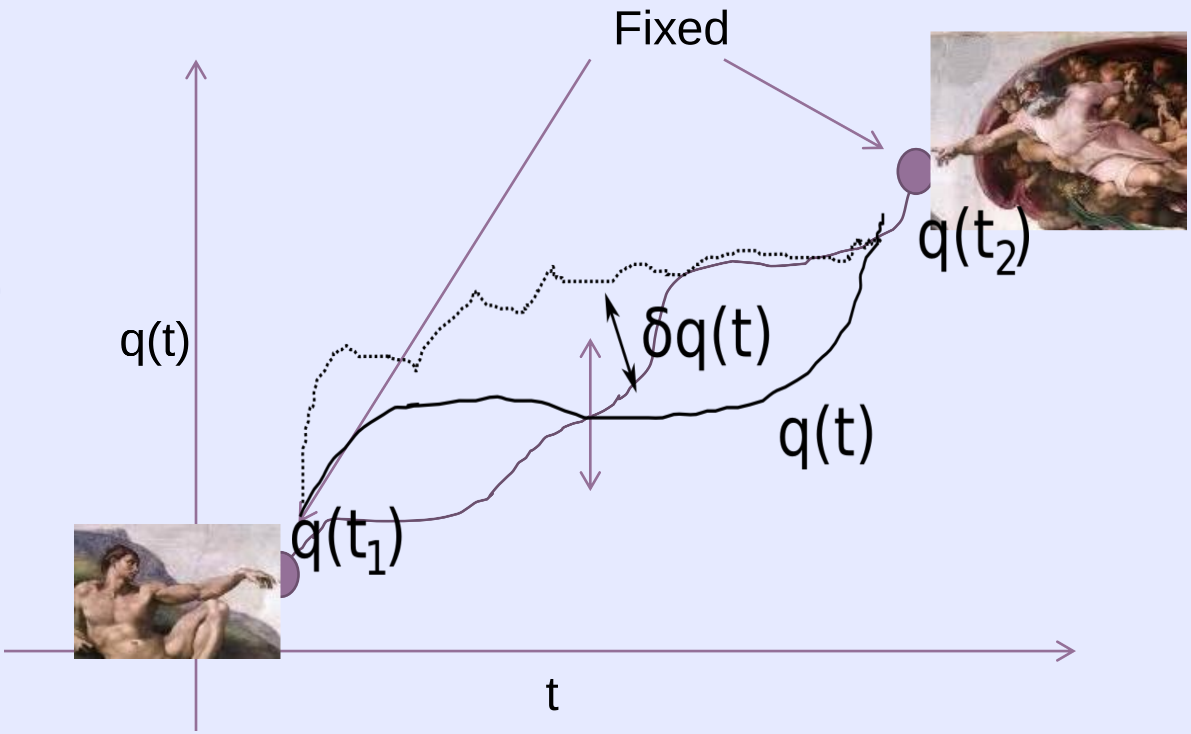


$$\delta I[q(t)] = 0, \quad \forall \delta q(t)$$



Important !

$$\delta I[q(t)] = 0, \quad \forall \delta q(t)$$



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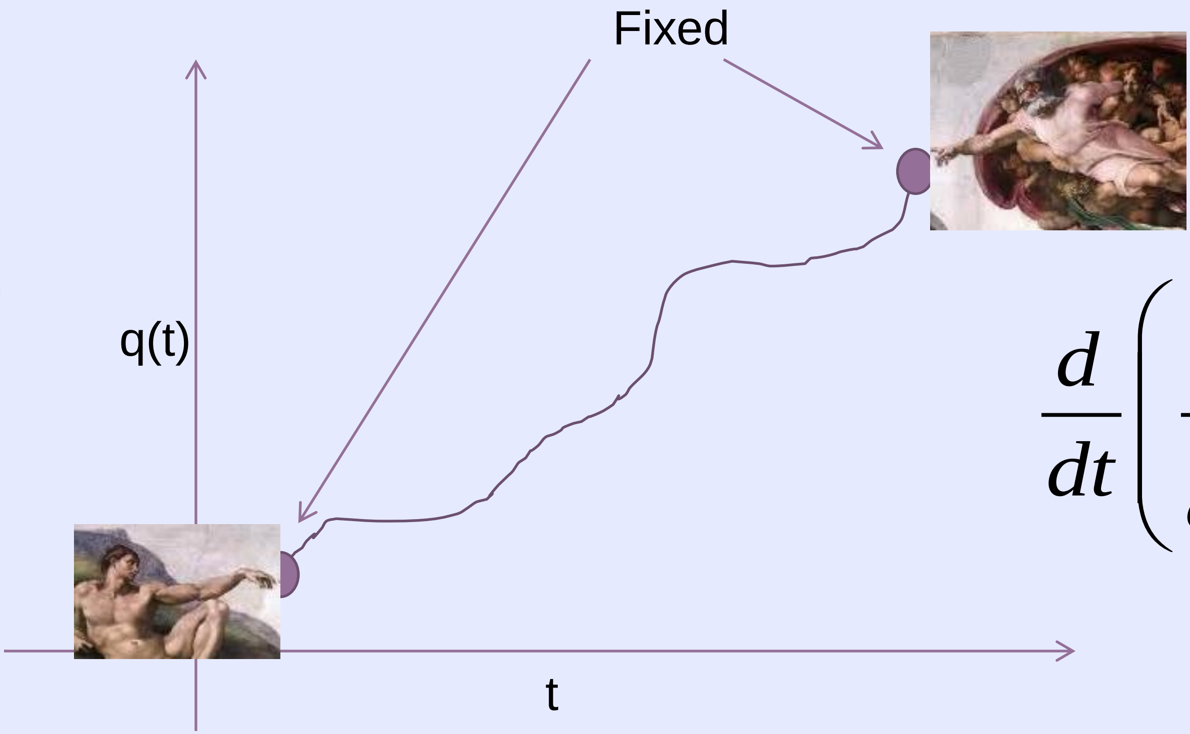


$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Although the variational approach assumes the future information is fixed but the resultant equation reduces to the problem of initial condition !!

Another Aspects of Variational Approach

Symmetry and Conservation Laws

In Quantum Mechanics, this role of Variational Approach is replaced by the representation of operators in Hilbert space of physical states.

$$I_{QM}\left[\psi\right]=\int dt\left\langle \psi\left(t\right)\middle| i\hbar\partial_t - H\middle|\psi\left(t\right)\right\rangle$$

Once the variational approach is established for a problem.....

$$I_{True} = I_{True}\left[\left\{\vec{q}(t)\right\}\right],$$

$$\delta I_{True} = 0,$$

Use as Approximation method

$$\vec{q}(t) \cong \sum_{i=1}^{N} C_i(t)\vec{n}_{\ddot{\imath}}$$

$$I_{True} \Rightarrow I_{App}\left[\left\{C_i(t), i = 1,.., N\right\}\right],$$

$$\delta I_{App} = 0,$$

$$for \quad \left\{C_i(t), i = 1,.., N\right\}$$

Once the variational approach is established for a problem.....

$$I_{True} = I_{True}\left[\left\{\vec{q}(t)\right\}\right],$$

$$\delta I_{True} = 0,$$

Use as Approximation method or Model Construction

$$\vec{q}(t) \cong \sum_{i=1}^{N} C_i(t)\vec{n}_{\ddot{\imath}}$$

$$I_{True} \Rightarrow I_{App}\left[\left\{C_i(t), i = 1,.., N\right\}\right],$$

$$\delta I_{App} = 0,$$

$$for \quad \left\{C_i(t), i = 1,.., N\right\}$$

$$\vec{q}(t) \Rightarrow \xi_M$$

$$I_{True} \Rightarrow I_{Model}\left[\xi_M\right],$$

$$\delta I_{Model} = 0,$$

$$for \quad \xi_M$$

Example: Relativistic Hydrodynamics

$$L = -\varepsilon(n,s) + \lambda \partial_\mu (nu^\mu) + \xi \partial_\mu (su^\mu) + \frac{1}{2}\zeta (u_\mu u^\mu - 1)$$

$$\delta I = \delta \int d^4x \, L(n,s,u^\mu,\lambda,\xi,\zeta) = 0,$$

$$\Downarrow$$

$$\boxed{\partial_\nu \left[(\varepsilon + p) u^\mu u^\nu - p g^{\nu\mu} \right] = 0}$$

H-T Elze, Y. Hama, T. Kodama, M. Makler, J. Rafelski,
J. Phys. G: Nucl. Part. Phys. **25** 1935 (1999)

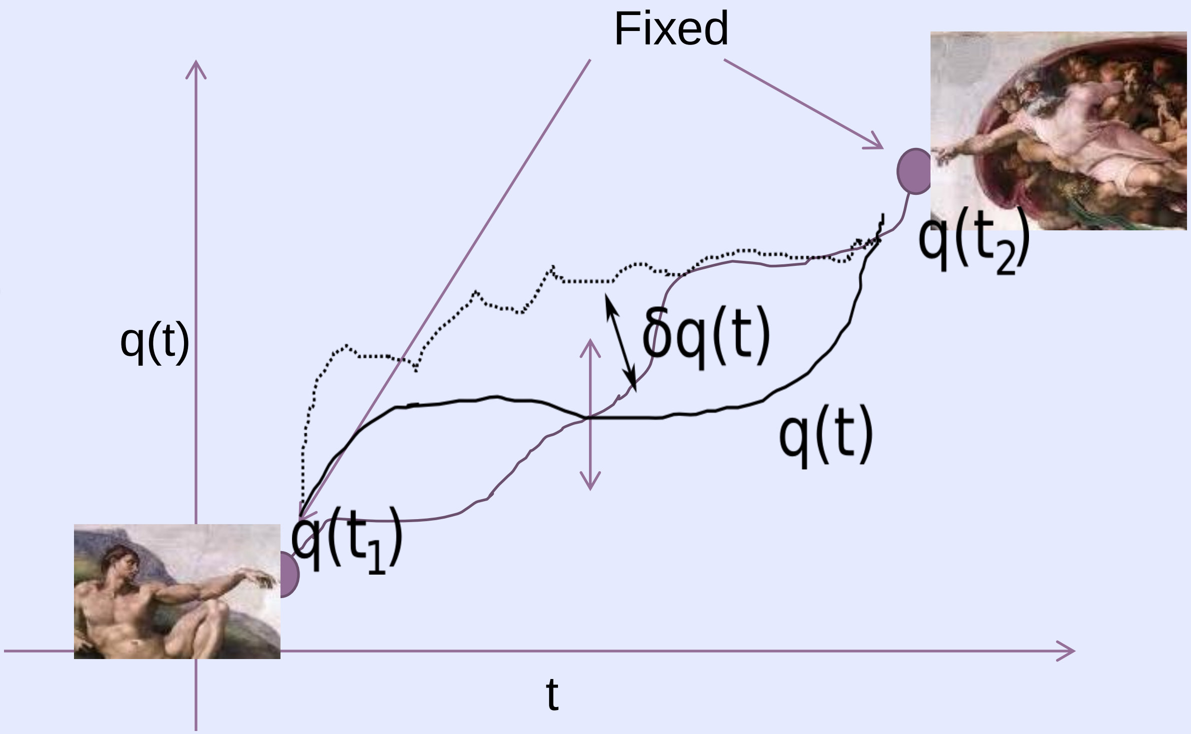
Examples in QM

$$I_{QM}[\psi] = \int dt \left\langle \psi(t) \middle| i\hbar\partial_t - H \middle| \psi(t) \right\rangle$$

- Hartree-Fock Approx.
- QMD Model
- …

How to formulate a dissipative process in terms of Variational Principle?

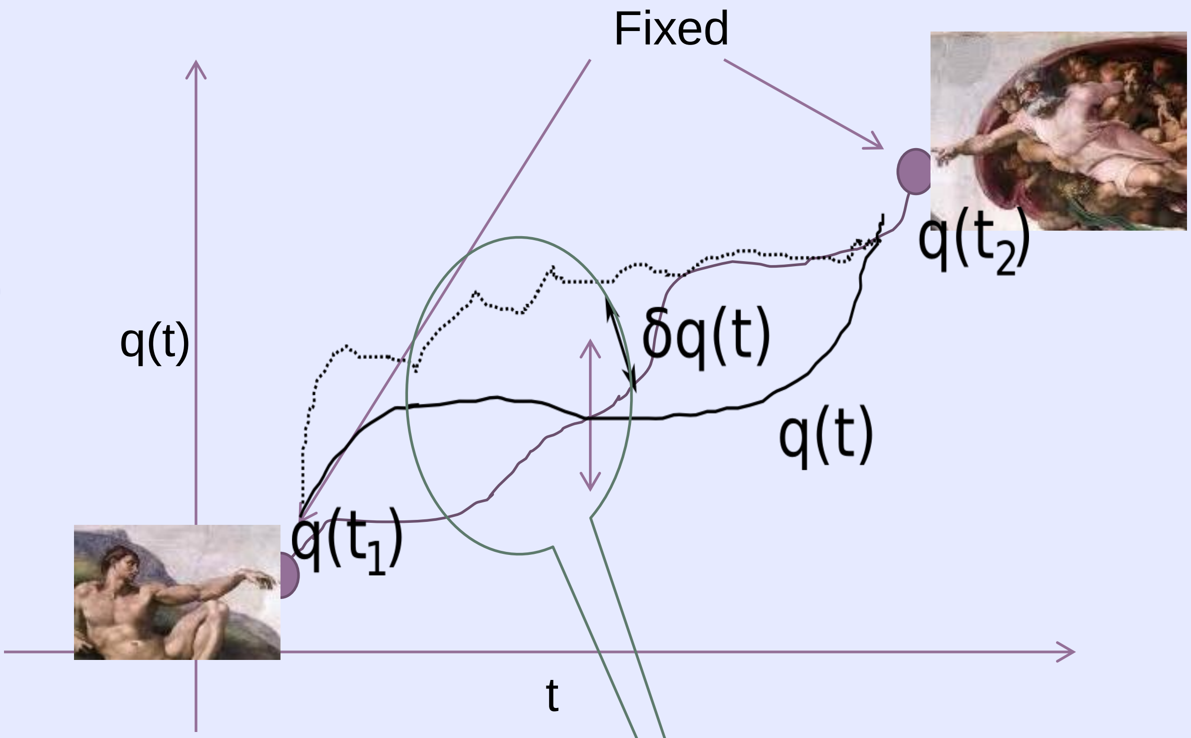
How to Do?



INSTEAD OF CHOOSING ONE OPTIMAL PATH,

$$\delta I[q(t)] = 0, \quad \forall \delta q(t)$$

How to Do?



INSTEAD OF CHOOSING ONE OPTIMAL PATH,

$$\delta I[q(t)] = 0, \quad \forall \delta q(t)$$

Determine the optimal **distribution of paths** under the influence of random noise….

Essential Difference between the ideal case

THIS IS AN IDEAL CASE

Essential Difference between the ideal case

And Presence of a Noise

THIS IS NOT AN IDEAL CASE

Stochastic Process

The effect of microscopic degrees of freedom can be treated as noise, with Stochastic Differential Equation (SDE)

Classical case

$$\frac{d}{dt}X(t) = V(t)$$

Stochastic

$$\frac{d}{dt}X(t) = V(t) + \xi(t)$$

Noise

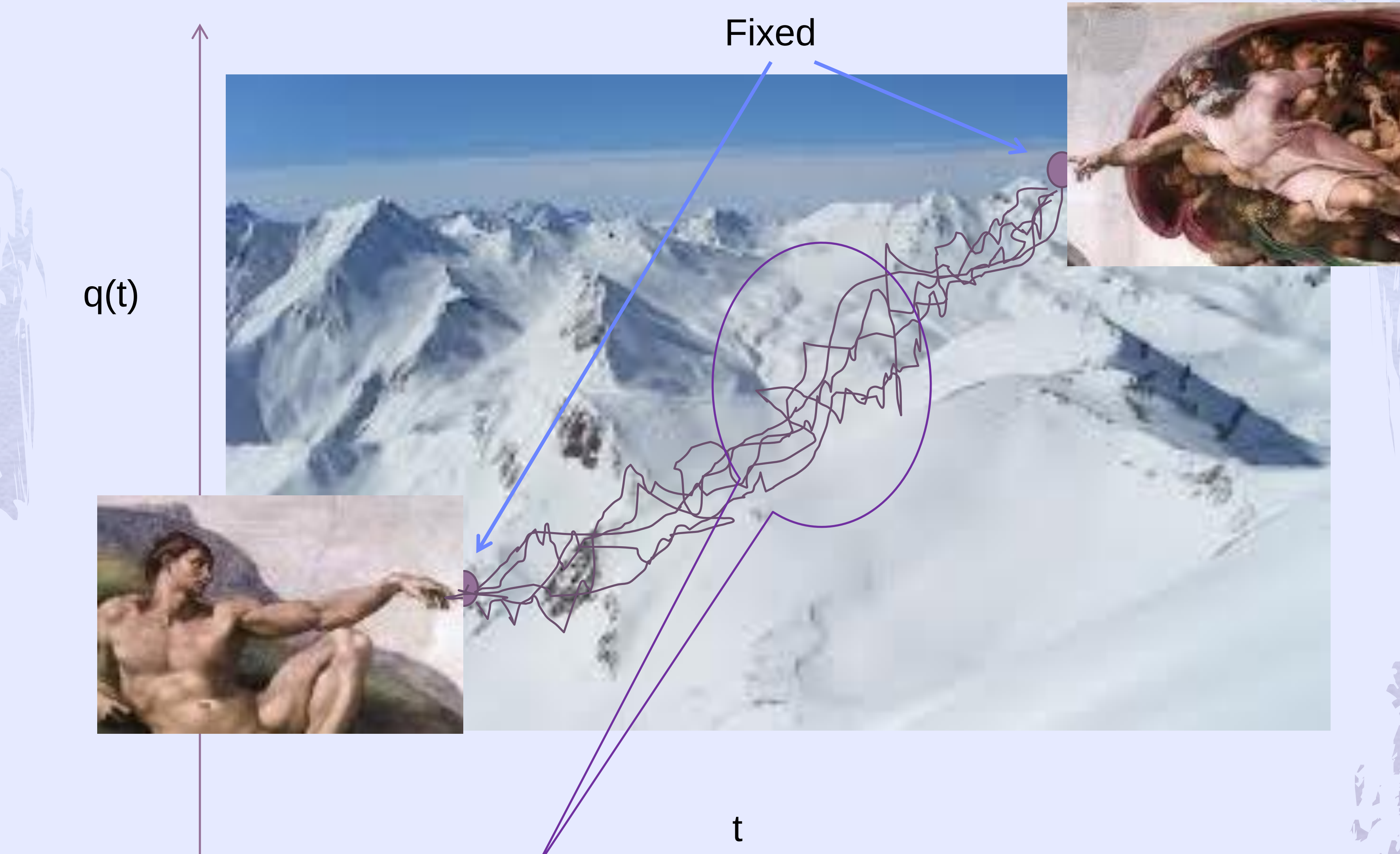
$$\frac{d}{dt}X(t) = V(t)$$

$$\frac{d}{dt}X(t) = V(t) + \xi(t)$$

Variational Approach with Noises?

Determine the optimal distribution of paths under the influence of random noise….

Variational Approach with Noises?



Determine the optimal distribution of paths under the influence of random noise….

For a given stochastic motion under the influence of noise, we can write Fokker-Plank equation,

For the Probability Density as

$$\rho(\vec{x},t) = \left\langle \delta(\vec{x} - \vec{x}(t)) \right\rangle$$

Average over all events.

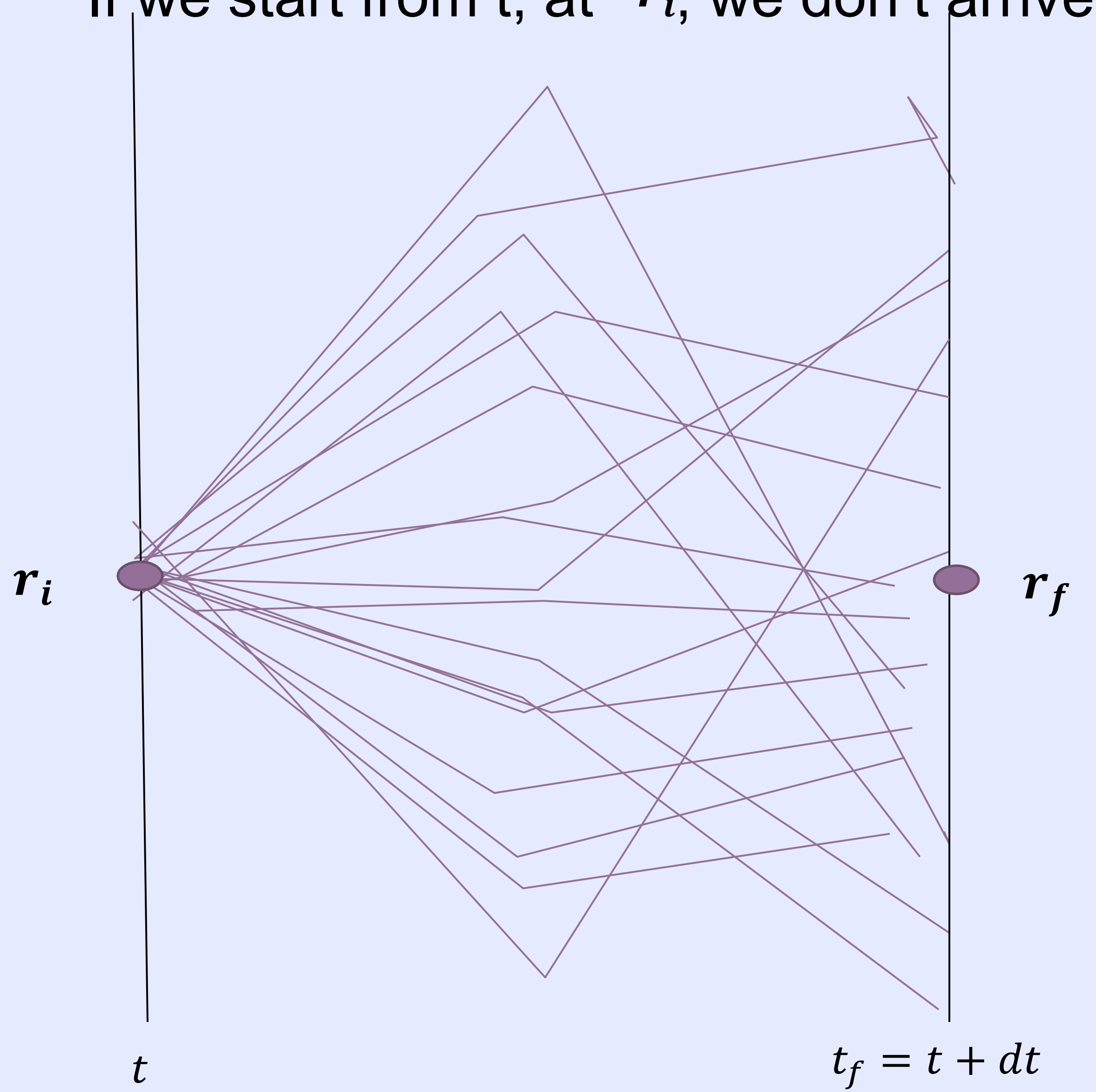
One event given by a SDE

$$\partial_t \rho = -\nabla \left(\vec{u} - \nu \nabla \right) \rho$$

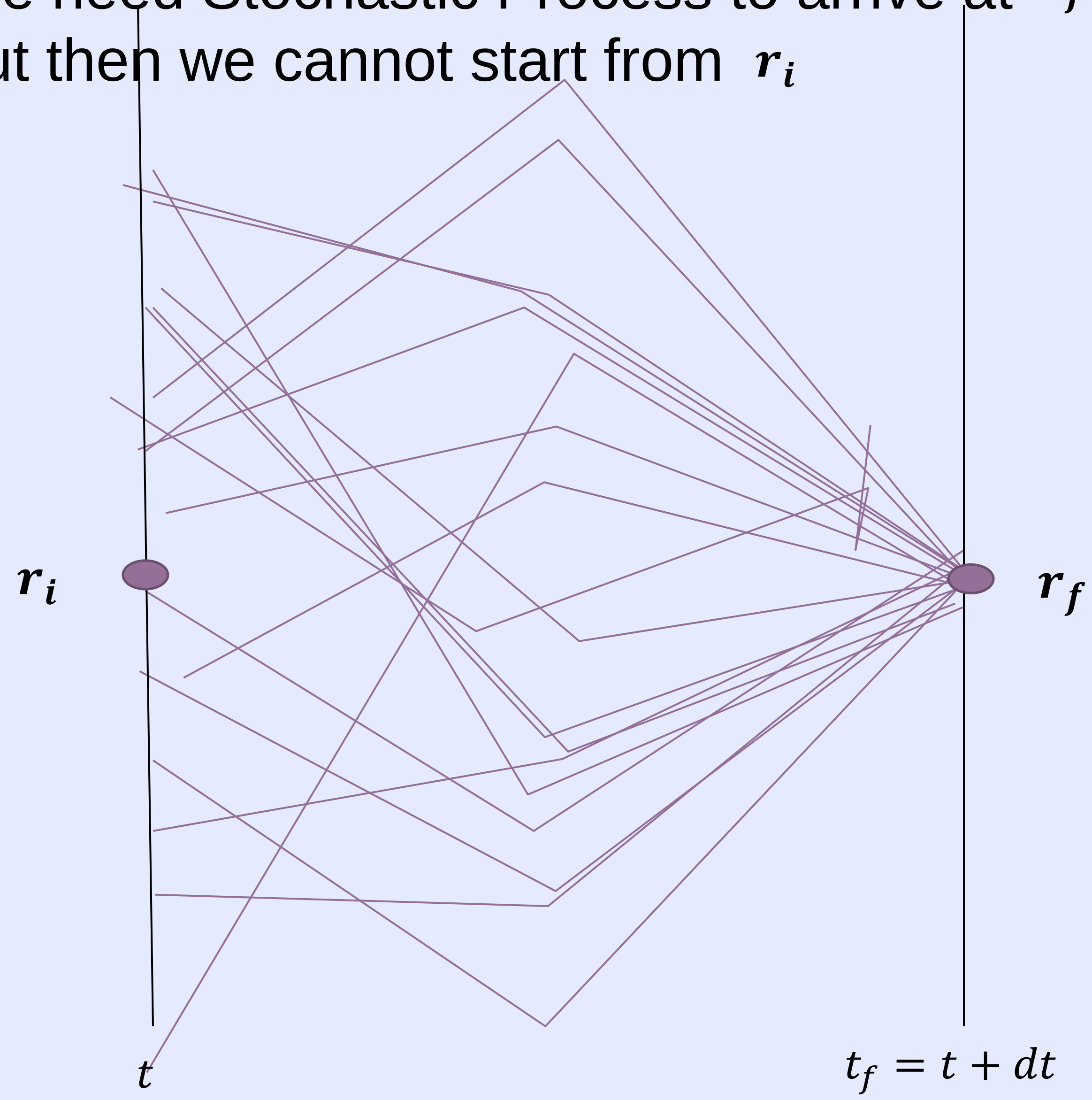
once the velocity field $\vec{u}$ is known.

Stochastic Process
vs.
Boundary Condition

If we start from t, at $\boldsymbol{r_i}$, we don't arrive at $\boldsymbol{r_f}$



We need Stochastic Process to arrive at $\boldsymbol{r_f}$
but then we cannot start from $\boldsymbol{r_i}$



Forward SDE

$$d\vec{r} = \overrightarrow{u_F}(\vec{r},t)dt + \sqrt{2\nu}\cdot d\overrightarrow{W_F}(t) \qquad \left(dt > 0\right)$$

Backward SDE

$$d\vec{r} = \vec{u}_B(\vec{r},t)dt + \sqrt{2\nu}\cdot d\vec{W}_B(t) \qquad \left(dt < 0\right)$$

with white noise

$$\left\langle d\overrightarrow{W_{F,B}}(t)\right\rangle = 0 \quad \left\langle dW_L{}^i(t)dW_M{}^j(t)\right\rangle = \delta^{ij}\delta_{LM}dt,$$

Corresponding Distribution

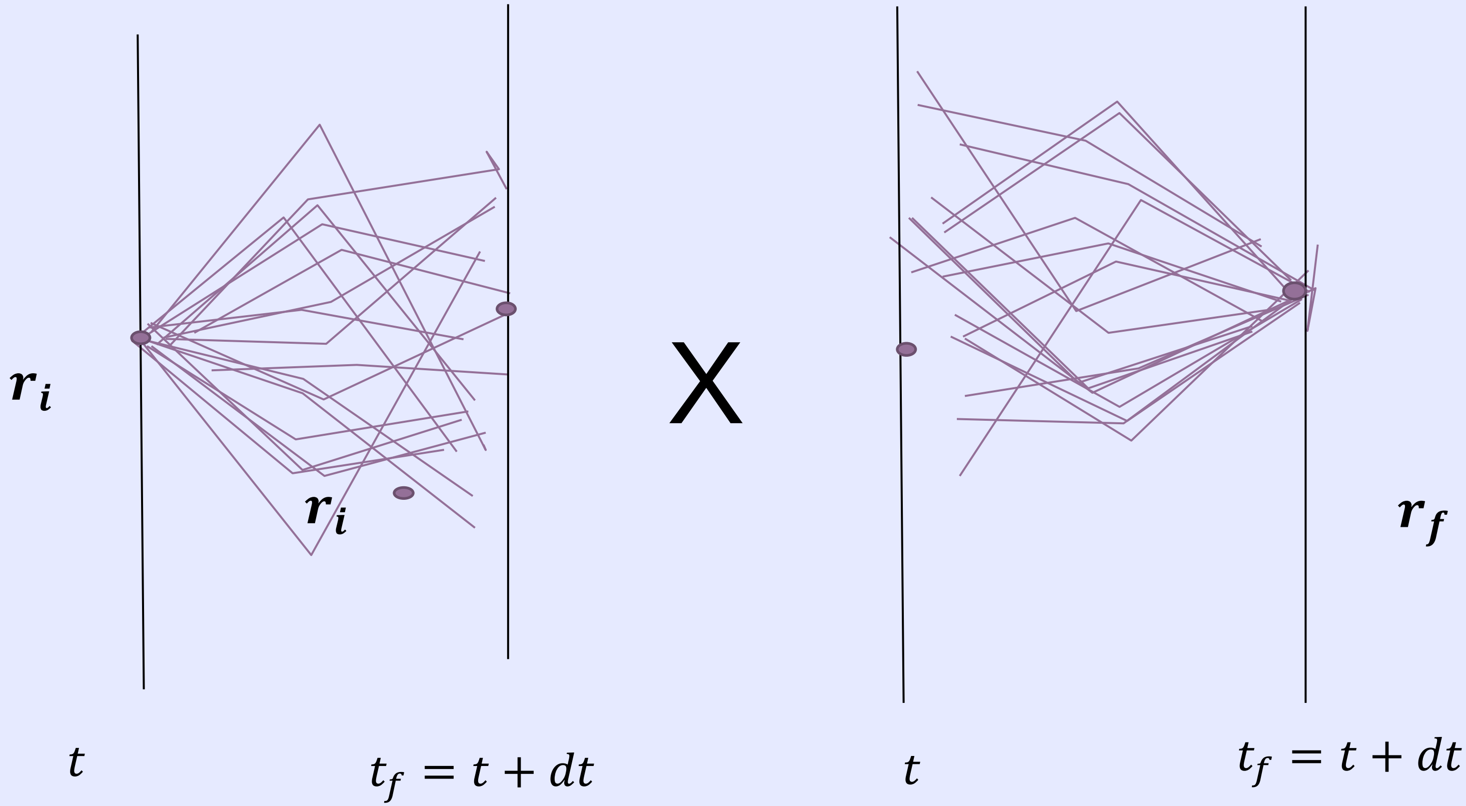
Forward Fokker-Planck

$$\partial_t \rho_F = -\nabla\left(\vec{u}_F - \nu\nabla\right)\rho_F$$

Backward Fokker-Planck

$$\partial_t \rho_B = -\nabla\left(\vec{u}_B + \nu\nabla\right)\rho_B$$

Real World



With <u>maximum probability</u>

Real distribution

For given $\rho_F(r,t), \ \ \rho_B(r,t)$

assume $\rho(r,t) \propto \sqrt{\rho_F(r,t) \times \rho_B(r,t)}$

and require $Max \int d^3\vec{r} \ \rho_F(r,t)\rho_B(r,t)$

with $\int d^3\vec{r} \ \rho_F(r,t) = 1, \quad \int d^3\vec{r} \ \rho_B(r,t) = 1,$

$$\longrightarrow \quad \rho(r,t) = \rho_F(r,t) = \rho_B(r,t)$$

Then,

$$\partial_t \rho_F = -\nabla\left(\vec{u}_F - \nu\nabla\right)\rho_F$$

$$\partial_t \rho_B = -\nabla\left(\vec{u}_B + \nu\nabla\right)\rho_B$$

$$\partial_t \rho + \nabla\left(\vec{u}_M \rho\right) = 0,$$

with $\vec{u}_M = \left(\vec{u}_F + \vec{u}_B\right)/2,$

and $\vec{u}_B = \vec{u}_F + 2\nu\nabla \ln \rho$

Consistency condition…

How to determine $\vec{u}_M, \rho$?

For an Ideal Classical Dust under the potential,

$$I(\rho,\vec{u}) = \int_a^b dt \int d^3\vec{r}\,\rho(r,t)\left(\frac{m}{2}\vec{u}^2 - V\right) \text{with} \quad \partial_t\rho + \nabla\cdot\left(\rho\vec{u}_m\right) = 0,$$

This is equivalent to

$$I(\rho,\vec{u},\lambda) = \int_a^b dt \int d^3\vec{r}\,\rho(r,t)\left(\frac{m}{2}\vec{u}^2 - V - \nu\dot{\lambda} - \nu\nabla\lambda\cdot\vec{u}\right)$$

How to determine $\vec{u}_M , \rho$?

For an Ideal Classical Dust under the potential,

$$I(\rho,\vec{u},\lambda) = \int_a^b dt \int d^3\vec{r}\, \rho(r,t)\left(\frac{m}{2}\vec{u}^2 - V - \nu\dot{\lambda} - \nu\nabla\lambda\cdot\vec{u} \right)$$

With noise, but time-reversal symmetric case (additive generator)

$$I(\rho,\vec{u},\lambda) = \frac{1}{2}\left\{ I(\rho_F,\vec{u}_F,\lambda_F) + I(\rho_B,\vec{u}_B,\lambda_B) \right\}$$

$$= \int_a^b dt \int d^3\vec{r}\, \rho(r,t)\left(\frac{m}{2}{\vec{u}_M}^2 - V - \kappa^2\left(\nabla\ln\rho\right)^2 - \nu\dot{\lambda} - \nu\nabla\lambda\cdot\vec{u}_M \right)$$

$$I=\int_a^b dt\int d^3\vec{r}\,\rho(r,t)\left(\frac{m}{2}\vec{u}_M{}^2-V-\kappa^2\left(\nabla\ln\rho\right)^2-\nu\dot{\lambda}-\nu\nabla\lambda\cdot\vec{u}_M\right)$$

In terms of a complex variable,

$$\psi=\sqrt{\rho}\;e^{i\lambda},$$

the above Action is rewritten as

$$I=\int_a^b dt\int d^3\vec{r}\;\psi^*(\vec{r},t)\left(i\kappa\frac{\partial}{\partial t}-\frac{\kappa^2}{2m}\nabla^2+V\right)\psi(\vec{r},t)$$

If we set the noise intensity $\kappa=\hbar,$

The Action for Schroedingrer Equation !!!!

Summary I

- Hydrodynamics can be regarded as an effective theory based on variational principle so that there still is place that the transport coefficients etc do not ncecessarily the same as that of real local thermal equilibrium.
- Variational Principle can be extended to the stochastic (conservative and non-conservative cases)

Summary II

- Quantum Mechanical Variational Action can be identified with the action of a classical fluid with noises, consistent to the boundary condition (initial and final – Nelson/Yasue) with maximum probability of the product set of the systems with forward noise and backward noise.

- This approach opens a possible generalization of the Quantum Mechanics in the smaller scale than that of Planck, when the space-time fluctuation is relevant.

- In Rio de Janeiro

Dear Laszlo,

Happy Birthday!

- In Birmingham

Dear Laszlo,

Happy Birthday!

- In Rio de Janeiro, Parati

Dear Laszlo,

Happy Birthday!

- In Rio de Janeiro, Parati

Dear Laszlo,

Happy Birthday!

- In Poland, Academia Europaea

Dear Laszlo,

Happy Birthday!

- Greetings From Japan

Horst Oda

Dear Laszlo,

Happy Birthday!

- Greetings From Japan

Dear Laszlo,

Happy Birthday!

Dear Laszlo,

Happy Birthday!

20 Years invariant!

(even younger)

- 1994

- 2013

CONCLUSION

$$Laszlo\left(t\right) = Const.$$

Dear Laszlo,

Happy Birthday!

And Toast with the
Perfect liquid
for another 20 years !