

Gluon Collective Modes in Anisotropic Quark-Gluon Plasma

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in collaboration with

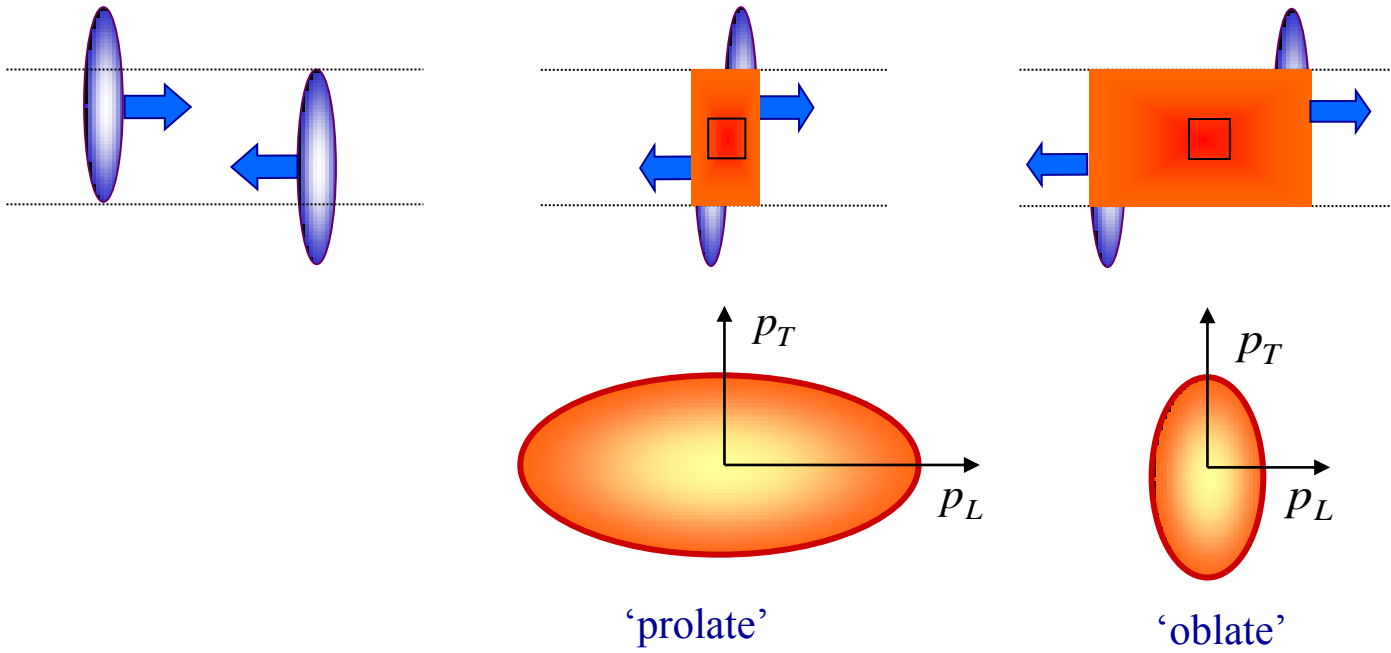
Margaret Carrington & Katarzyna Deja

Motivation

- ▶ Spectrum of collective excitations is an important characteristics of any many body system.
- ▶ Anisotropic plasma is qualitatively different than the isotropic one.

Motivation cont.

- ▶ QGP from relativistic heavy-ion collisions is anisotropic.



Motivation cont.

► Existing analyses of collective excitations

St. Mrówczyński, Phys. Lett. B **314**, 118 (1993), Phys. Rev. C **49**, 2191 (1994)

P. Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003), *ibid* D **70**, 116006 (2004)

P.B. Arnold, J.Lenaghan and G.D. Moore, JHEP **0308**, 002 (2003)

.....

are not complete.

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon polarization tensor

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0 \quad k^\mu \equiv (\omega, \mathbf{k})$$

Gluon collective modes – solutions $\omega(\mathbf{k})$

Instabilities – solutions with $\text{Im}\omega > 0$ $\Rightarrow A^\mu(x) \sim e^{\text{Im}\omega t}$

Dynamical information is hidden in the polarization tensor $\Pi^{\mu\nu}(k)$

Polarization tensor

Diagrammatic hard-loop
approach

Linear response analysis
within kinetic theory

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E} \left[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+} \right] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

$f(\mathbf{p})$ - momentum distribution of plasma constituents

Momentum distribution

Isotropic momentum distribution rescaled in the direction of \mathbf{n}

$$f_{\xi}(\mathbf{p}) = C_{\xi} f_{\text{iso}} \left(\sqrt{\mathbf{p}^2 + \xi (\mathbf{p} \cdot \mathbf{n})^2} \right)$$

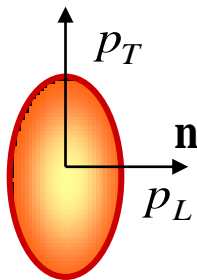
$$\xi \in (-1, \infty)$$

$$f_{\xi}(\mathbf{p}) \xrightarrow{\xi \rightarrow \infty} \delta(p_L)$$

$$f_{\sigma}(\mathbf{p}) = C_{\sigma} f_{\text{iso}} \left(\sqrt{(\sigma+1)\mathbf{p}^2 - \sigma (\mathbf{p} \cdot \mathbf{n})^2} \right)$$

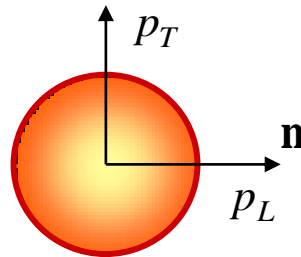
$$\sigma \in (-1, \infty)$$

$$f_{\sigma}(\mathbf{p}) \xrightarrow{\sigma \rightarrow \infty} \delta(p_T)$$



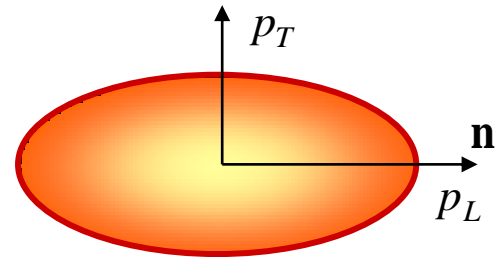
oblate

$$\xi > 0, \quad \sigma < 0$$



isotropic

$$\xi = \sigma = 0$$



prolate

$$\xi < 0, \quad \sigma > 0$$

Choosing normalization constants

$$f_\xi(\mathbf{p}) = C_\xi f_{\text{iso}}\left(\sqrt{\mathbf{p}^2 + \xi(\mathbf{p} \cdot \mathbf{n})^2}\right)$$

$$f_\sigma(\mathbf{p}) = C_\sigma f_{\text{iso}}\left(\sqrt{(\sigma+1)\mathbf{p}^2 - \sigma(\mathbf{p} \cdot \mathbf{n})^2}\right)$$

Parton number normalization

$$\blacktriangleright \int \frac{d^3 p}{(2\pi)^3} f_\xi(\mathbf{p}) = \int \frac{d^3 p}{(2\pi)^3} f_{\text{iso}}(|\mathbf{p}|) = \int \frac{d^3 p}{(2\pi)^3} f_\sigma(\mathbf{p}) \quad \longrightarrow \quad \begin{cases} C_\xi = \sqrt{1+\xi} \\ C_\sigma = \sigma+1 \end{cases}$$

Debye mass normalization

$$\blacktriangleright m^2 \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{f_\xi(\mathbf{p})}{|\mathbf{p}|} = \int \frac{d^3 p}{(2\pi)^3} \frac{f_{\text{iso}}(|\mathbf{p}|)}{|\mathbf{p}|} = \int \frac{d^3 p}{(2\pi)^3} \frac{f_\sigma(\mathbf{p})}{|\mathbf{p}|} \quad \longrightarrow \quad \begin{cases} C_\xi = \frac{\sqrt{\xi}}{\text{Arctanh}\sqrt{\xi}} \\ C_\sigma = \frac{\sqrt{\sigma(\sigma+1)}}{\text{Arctan}\sqrt{\frac{\sigma}{\sigma+1}}} \end{cases}$$

Dispersion equation

Dispersion equation: $\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$

4 x 4 matrix

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Dielectric tensor: $\varepsilon^{ij}(\omega, \mathbf{k}) = \delta^{ij} - \frac{1}{\omega^2} \Pi^{ij}(\omega, \mathbf{k})$ 3 x 3 matrix

Dispersion equation:

$$\det[\Sigma(\omega, \mathbf{k})] = 0$$

$$\Sigma^{ij}(\omega, \mathbf{k}) \equiv (\omega^2 - \mathbf{k}^2) \delta^{ij} + k^i k^j - \Pi^{ij}(\omega, \mathbf{k})$$
 inverse gluon propagator in temporal axial gauge

Solution $\omega(\mathbf{k})$ - gluon collective mode

Instead of looking for zeros of $\det \Sigma$, one looks for poles of Σ^{-1}

How to invert matrix Σ ?

Method to inverse the matrix Σ

The symmetric matrix Σ depends on \mathbf{k} and \mathbf{n}

$$\Sigma = \alpha A + \beta B + \gamma C + \delta D$$

basis of 4 projectors

$$\left\{ \begin{array}{l} A^{ij} = \delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}, \quad B^{ij} = \frac{k^i k^j}{\mathbf{k}^2} \\ C^{ij} = \frac{n_T^i n_T^j}{\mathbf{n}_T^2}, \quad D^{ij} = n_T^i k^j + k^i n_T^j \end{array} \right. \quad n_T^i \equiv \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) n^j$$

$$k^i \Sigma^{ij} k^j = k^2 \beta,$$

$$n_T^i \Sigma^{ij} k^j = n_T^2 k^2 \delta,$$

$$n_T^i \Sigma^{ij} n_T^j = n_T^2 (\alpha + \gamma),$$

$$\text{Tr} \Sigma = 2\alpha + \beta + \gamma,$$



$$\alpha, \beta, \gamma, \delta$$

$$\Sigma^{-1} = \bar{\alpha} A + \bar{\beta} B + \bar{\gamma} C + \bar{\delta} D$$

$$\Sigma \Sigma^{-1} = \mathbf{1} \quad \longrightarrow \quad \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$$

Dispersion equations

$$\Sigma^{-1} = \frac{A - C}{\omega^2 - k^2 - \alpha} + \frac{(\omega^2 - k^2 - \alpha - \gamma)B + (\omega^2 - \beta)C + \delta D}{(\omega^2 - \beta)(\omega^2 - k^2 - \alpha - \gamma) - k^2 n_T^2 \delta^2}$$

Dispersion equations (poles of Σ^{-1})

- 1) $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$
- 2) $(\omega^2 - \beta(\omega, \mathbf{k}))(\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k})) - k^2 n_T^2 \delta^2(\omega, \mathbf{k}) = 0$

A modes - solutions of the first equation

G modes - solutions of the second equation

Collective modes in isotropic QGP

$$\Sigma = (\omega^2 - k^2 - \alpha_{\text{iso}})A + (\omega^2 - \beta_{\text{iso}})B \quad (\gamma_{\text{iso}} = \delta_{\text{iso}} = 0)$$

$$\left\{ \begin{array}{l} \alpha_{\text{iso}}(\omega, \mathbf{k}) = \omega^2 - k^2 - \frac{m^2 \omega^2}{2k^2} \left[1 - \left(\frac{\omega}{2k} - \frac{k}{2\omega} \right) \ln \left(\frac{\omega+k}{\omega-k} \right) \right] \\ \beta_{\text{iso}}(\omega, \mathbf{k}) = \omega^2 + \frac{m^2 \omega^2}{k^2} \left[1 - \frac{\omega}{2k} \ln \left(\frac{\omega+k}{\omega-k} \right) \right] \end{array} \right.$$

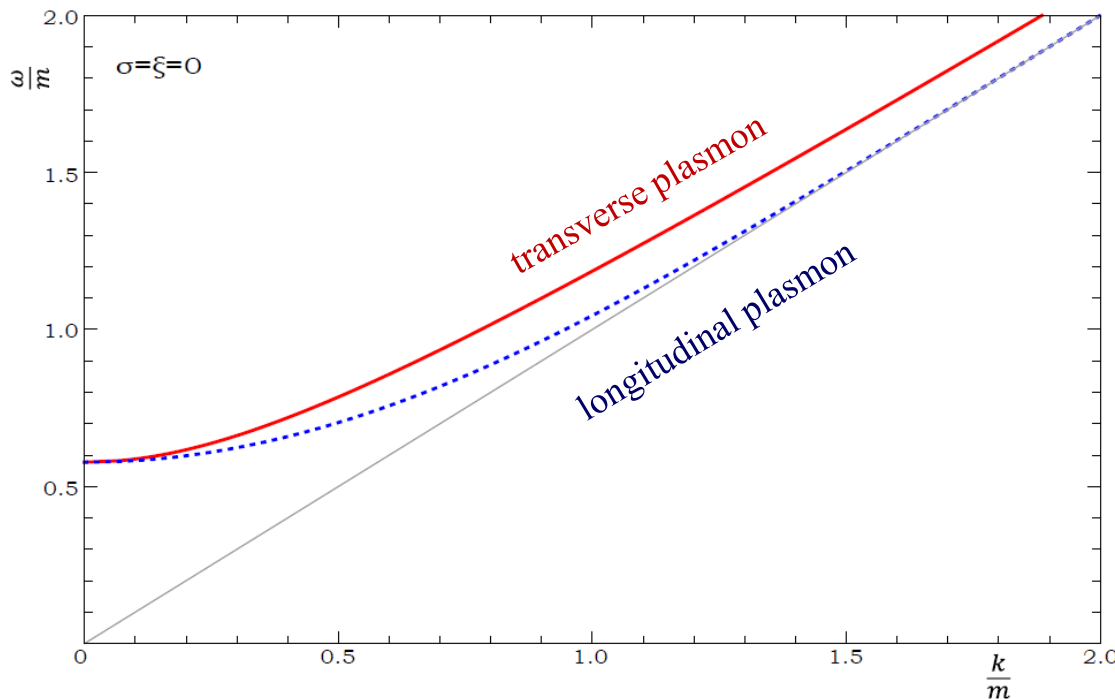
Dispersion equations

$$\left\{ \begin{array}{l} \omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0 \\ \omega^2 - \beta(\omega, \mathbf{k}) = 0 \end{array} \right.$$

4 real solutions
(2 positive & 2 negative)

Debye mass

$$m^2 = g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{f_\xi(\mathbf{p})}{|\mathbf{p}|}$$



Transverse and longitudinal modes

▶ longitudinal modes – $\mathbf{k} \parallel \mathbf{E}$, $\delta\rho \sim e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}$

▶ transverse modes – $\mathbf{k} \perp \mathbf{E}$, $\delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})}$

\mathbf{E} – electric field, \mathbf{k} – wave vector, ρ – charge density, \mathbf{j} – current

Weakly anisotropic system

$$|\xi| \ll 1$$

$$f_{\xi}(\mathbf{p}) \approx \left(1 + \frac{\xi}{3}\right) f_{\text{iso}}(p) + \frac{\xi}{2} \frac{d f_{\text{iso}}(p)}{d p} p(\mathbf{v} \cdot \mathbf{n})^2$$

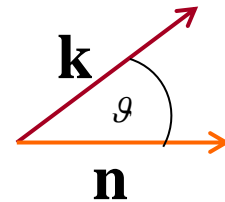
$$\alpha(\omega, \mathbf{k}) = \left(1 + \frac{\xi}{3}\right) \alpha_{\text{iso}}(\omega, \mathbf{k}) + \xi \frac{m^2}{8} \left\{ \frac{8}{3} \cos^2 \vartheta + \frac{2}{3} (5 - 19 \cos^2 \vartheta) \frac{\omega^2}{k^2} - 2(1 - 5 \cos^2 \vartheta) \frac{\omega^4}{k^4} \right. \\ \left. + \left[1 - 3 \cos^2 \vartheta - (2 - 8 \cos^2 \vartheta) \frac{\omega^2}{k^2} + (1 - 5 \cos^2 \vartheta) \frac{\omega^4}{k^4} \right] \frac{\omega}{k} \ln \left(\frac{\omega + k}{\omega - k} \right) \right\},$$

$$\beta(\omega, \mathbf{k}) = \dots,$$

$$\gamma(\omega, \mathbf{k}) = \dots,$$

$$\delta(\omega, \mathbf{k}) = \dots$$

analytic expressions



Weakly anisotropic system

Dispersion equations

$$1) \quad \omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$$

$$2) \quad (\omega^2 - \beta(\omega, \mathbf{k}))(\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k})) - k^2 n_T^2 \delta^2(\omega, \mathbf{k}) = 0$$



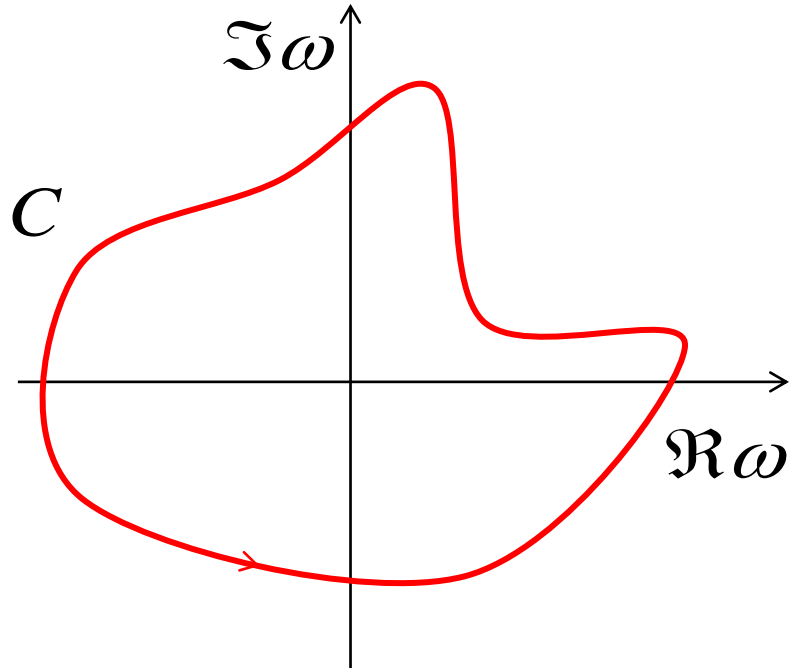
$$\delta^2 = O(\xi^2)$$

$$1) \quad \omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$$

$$2) \quad \omega^2 - \beta(\omega, \mathbf{k}) = 0$$

$$3) \quad \omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k}) = 0$$

Nyquist analysis



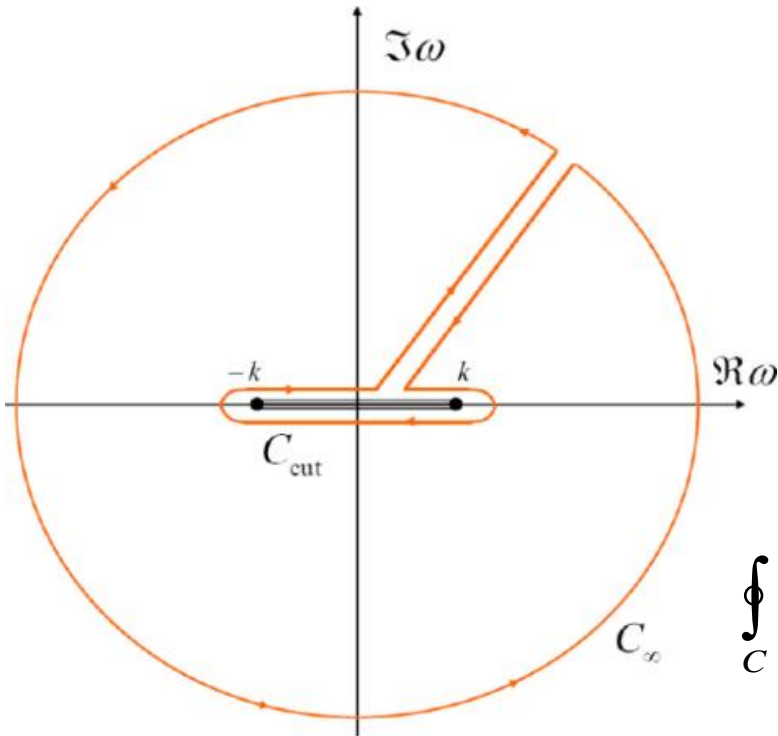
$$f(\omega) = 0$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = n_Z - n_P$$

n_Z - number of zeros of $f(\omega)$ inside C

n_P - number of poles of $f(\omega)$ inside C

Nyquist analysis



$$f(\omega) = 0$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = n_Z - n_P$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = \oint_{C_\infty} \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} + \oint_{C_{cut}} \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)}$$

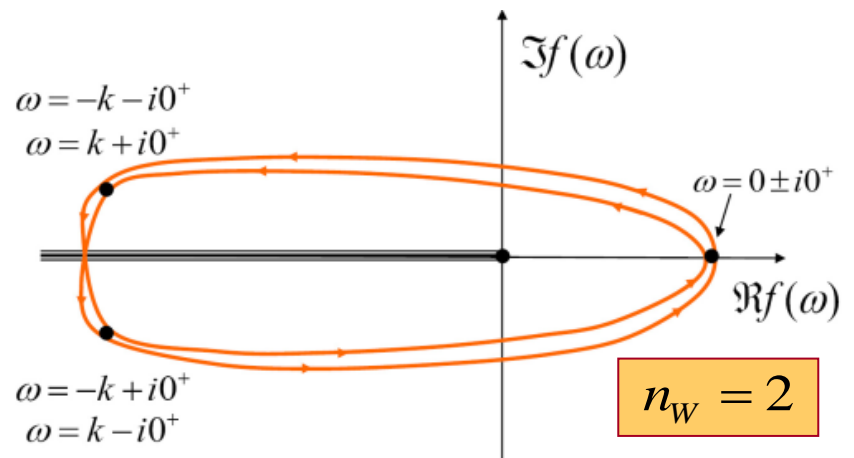
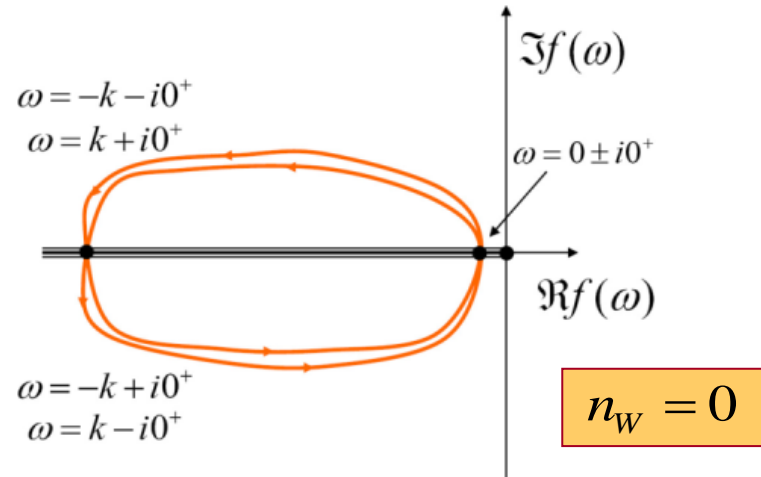
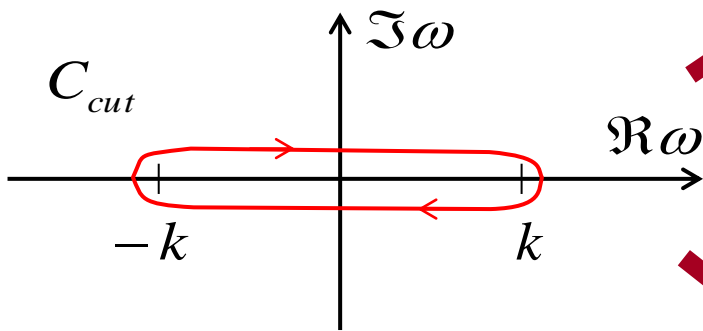
$$C_\infty \quad \oint_{C_\infty} \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = \lim_{|\omega| \rightarrow \infty} \omega \frac{f'(\omega)}{f(\omega)} \equiv n_\infty$$

n_W - winding number

$$C_{cut} \quad \oint_{C_{cut}} \frac{d\omega}{2\pi i} \frac{f'(\omega)}{f(\omega)} = \frac{1}{2\pi i} \oint_{C_{cut}} \frac{d}{d\omega} \ln f(\omega) = \frac{1}{2\pi i} (\ln f(\omega_e) - \ln f(\omega_s)) \equiv n_W$$

Nyquist analysis cont.

Computation of the winding number



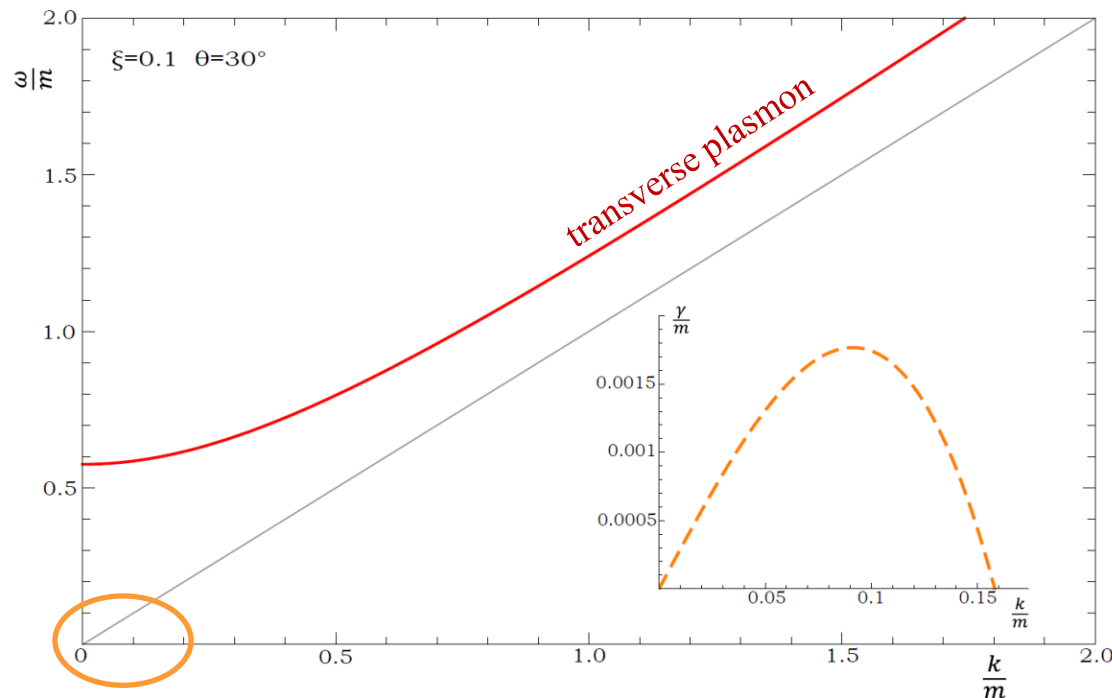
$$n_Z = n_P + n_\infty + n_W$$

Equation $\omega^2 - k^2 - \alpha(\omega, \mathbf{k}) = 0$

$$k^2 - \xi \frac{m^2}{3} \cos^2 \vartheta \geq 0 \quad 2 \text{ solutions}$$

$$k^2 - \xi \frac{m^2}{3} \cos^2 \vartheta < 0 \quad 4 \text{ solutions}$$

$$\omega^2(\mathbf{k}) \underset{k \ll m}{\approx} \frac{m^2}{3} \left(1 - \frac{\xi}{15}\right) + \frac{6}{5} \left[1 + \frac{\xi}{14} \left(\frac{4}{15} + \cos^2 \vartheta\right)\right] k^2$$



$$\omega(\mathbf{k}) \equiv \pm i\gamma(\mathbf{k}), \quad \gamma \in \mathcal{R}$$

$$\gamma(\mathbf{k}) \underset{k \gg \gamma}{\approx} \frac{1}{2} \left(\sqrt{\frac{\lambda^2}{k^2} + 4(k_A^2 - k^2)} - \frac{\lambda}{k} \right)$$

$$\lambda \equiv \frac{\pi}{4} \left[1 - \frac{\xi}{2} \left(\frac{1}{3} - 3 \cos^2 \vartheta \right) \right] m^2$$

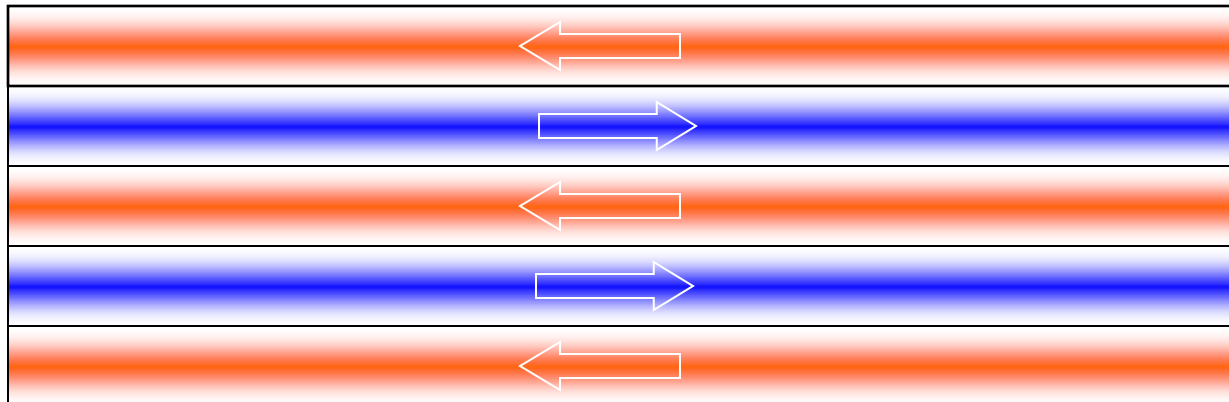
$$k_A \equiv \sqrt{\frac{\xi}{3}} |\cos \vartheta|$$

Seeds of instability

$\langle j_a^\mu(x) \rangle = 0$ but current fluctuations are finite

$$\langle j_a^\mu(x_1) j_b^\nu(x_2) \rangle = \frac{1}{2} \delta^{ab} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0$$

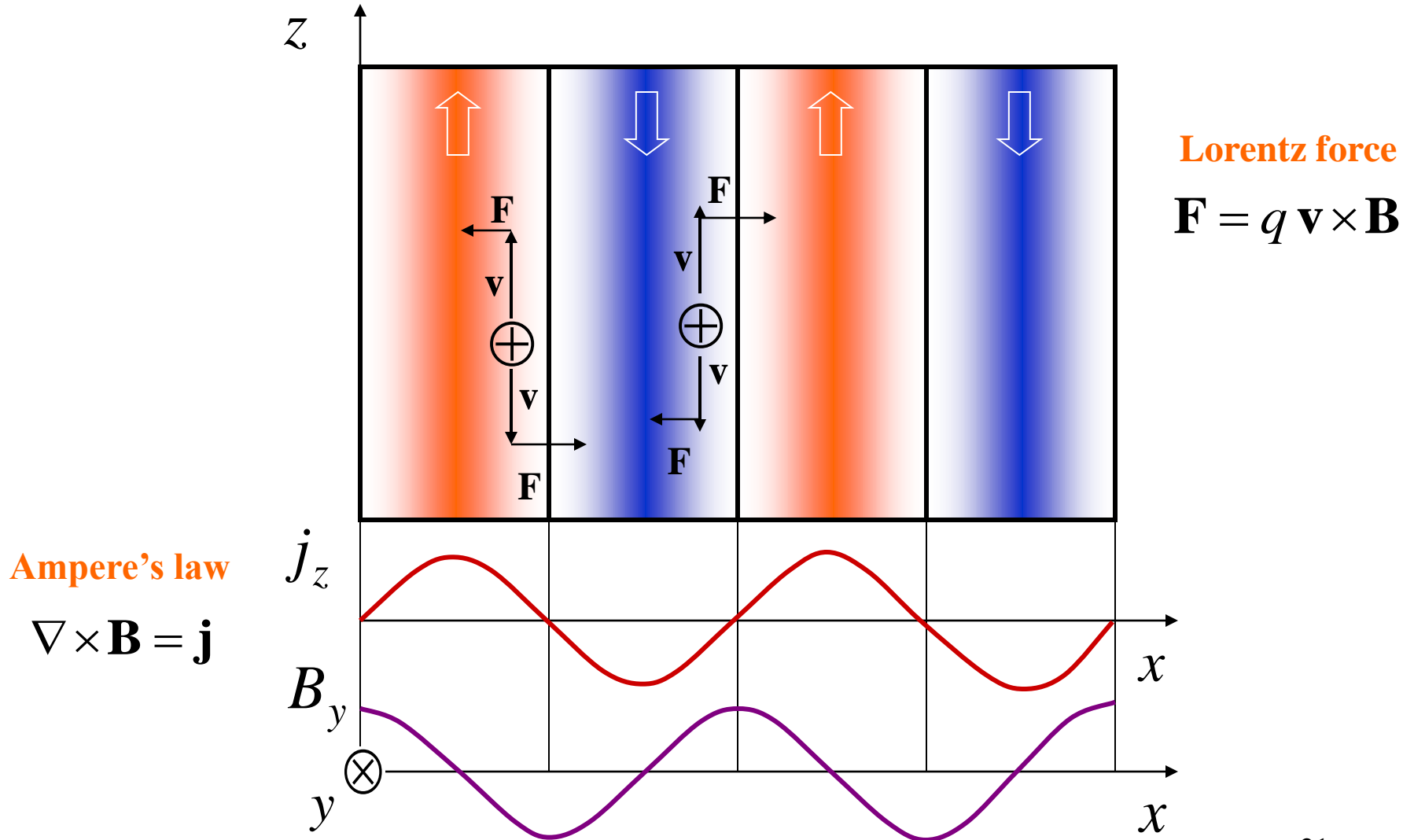
$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus



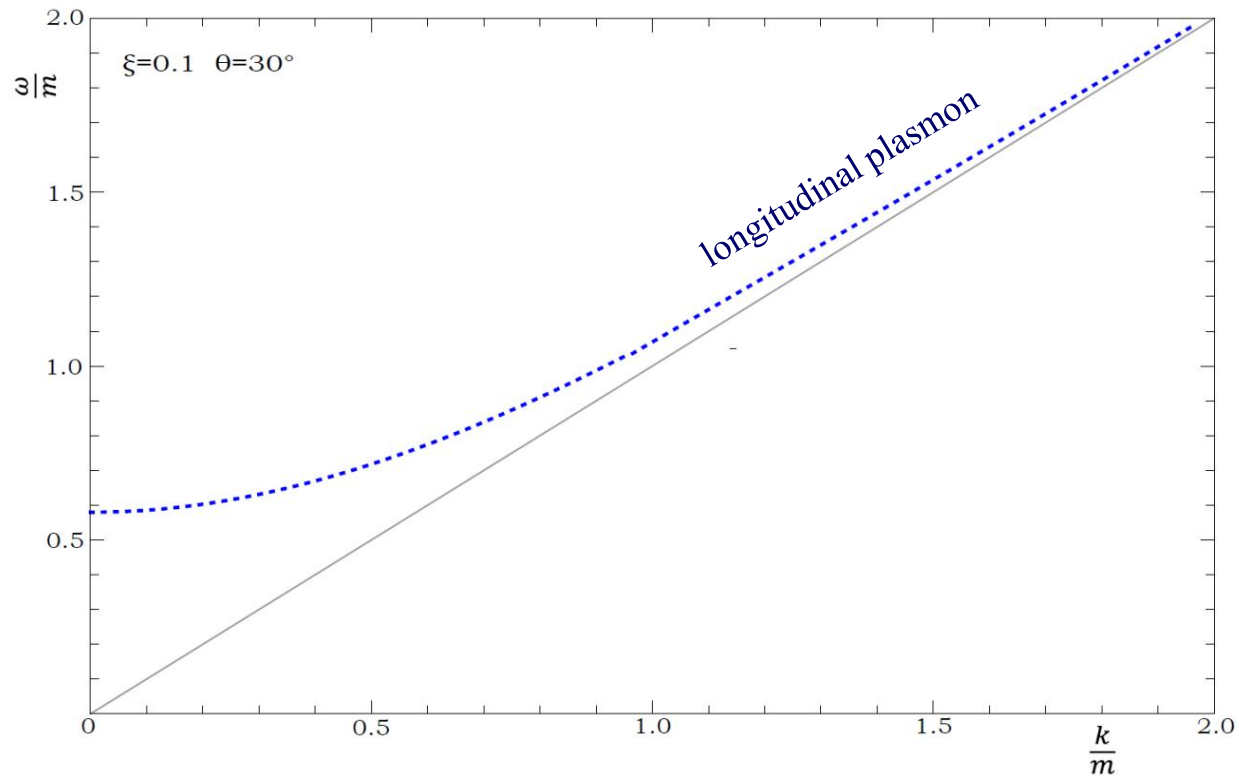
Mechanism of filamentation



Equation $\omega^2 - \beta(\omega, \mathbf{k}) = 0$

2 solutions

$$\omega^2(\mathbf{k}) \underset{k \ll m}{\approx} \frac{m^2}{3} \left(1 + \frac{\xi}{5} \left(-\frac{1}{3} + \cos^2 \vartheta \right) \right) + \frac{3}{5} \left[1 + \frac{4\xi}{35} (1 - 3 \cos^2 \vartheta) \right] k^2$$



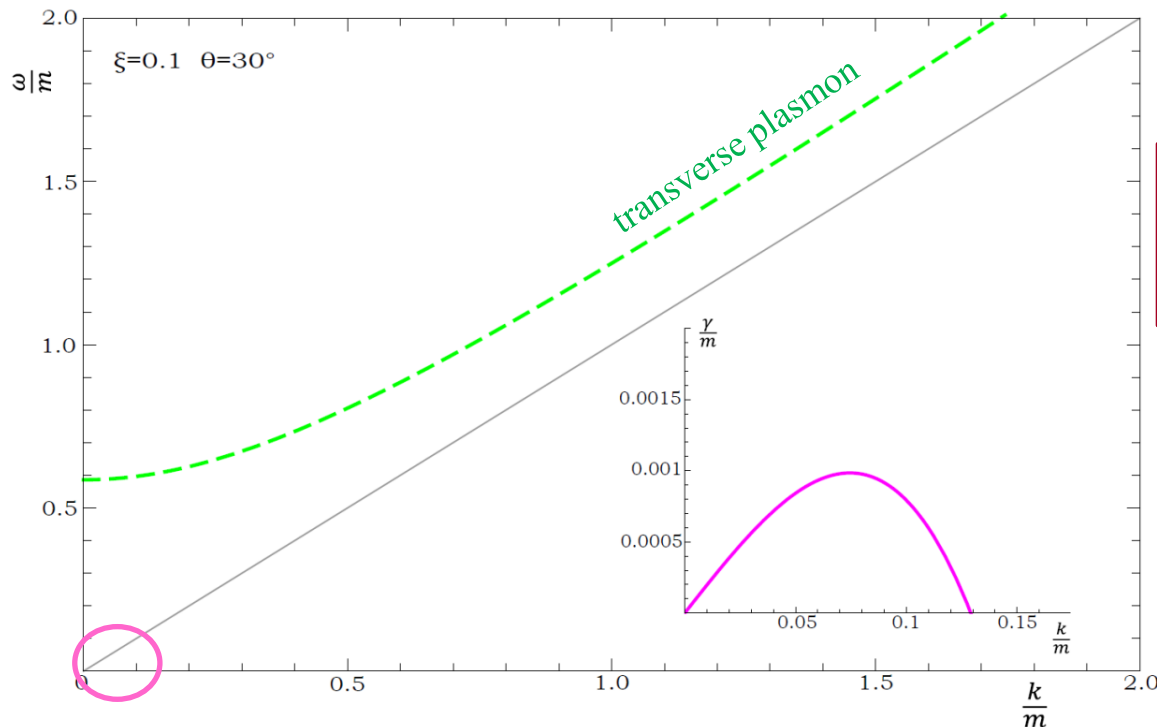
No unstable modes!

$$\text{Equation } \omega^2 - k^2 - \alpha(\omega, \mathbf{k}) - \gamma(\omega, \mathbf{k}) = 0$$

$$k^2 + \xi \frac{m^2}{3} (1 - 2 \cos^2 \vartheta) \geq 0 \quad 2 \text{ solutions}$$

$$k^2 + \xi \frac{m^2}{3} (1 - 2 \cos^2 \vartheta) < 0 \quad 4 \text{ solutions}$$

$$\omega^2(\mathbf{k}) \underset{k \ll m}{\approx} \frac{m^2}{3} \left(1 + \frac{\xi}{5} \left(\frac{2}{3} - \cos^2 \vartheta \right) \right) + \frac{6}{5} \left[1 + \frac{\xi}{5} \left(\frac{23}{42} - \cos^2 \vartheta \right) \right] k^2$$



$$\omega(\mathbf{k}) \equiv \pm i\gamma(\mathbf{k}), \quad \gamma \in \mathbb{R}$$

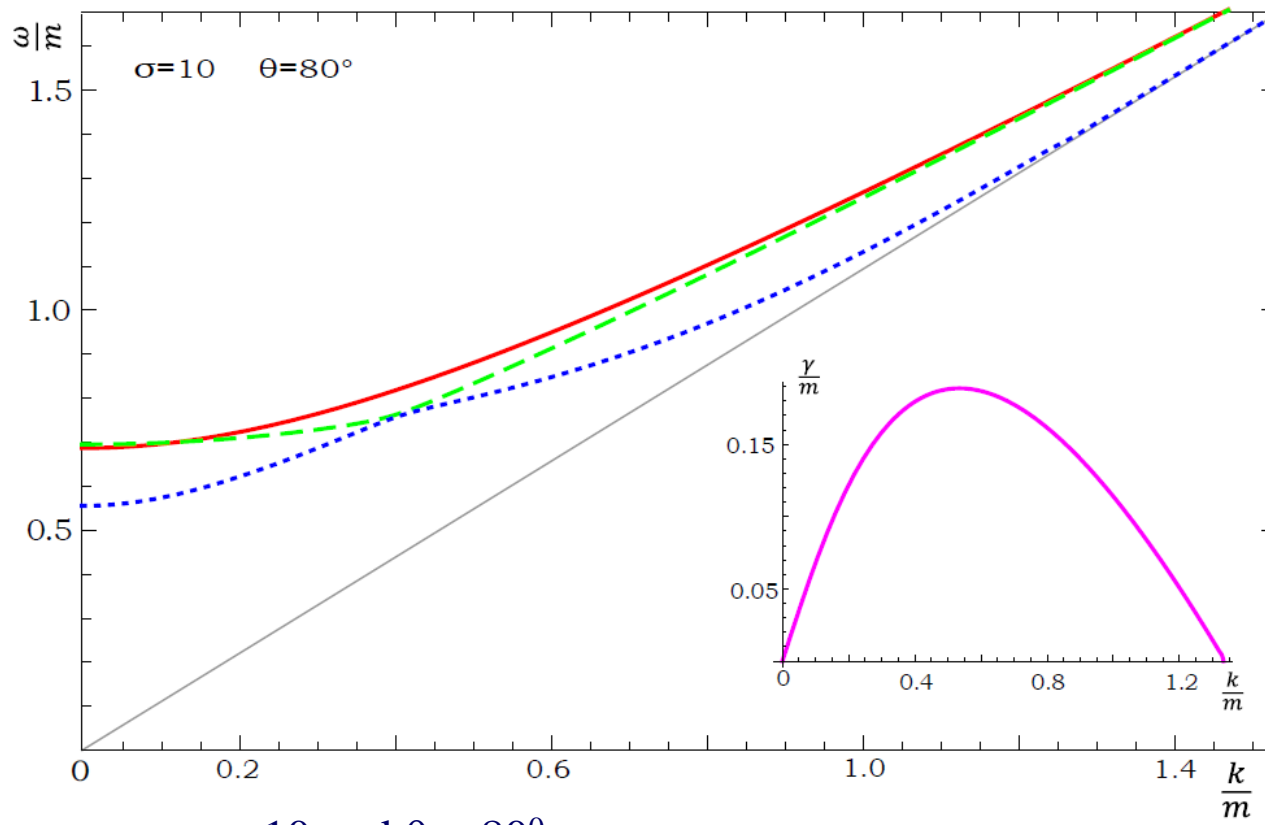
$$\gamma(\mathbf{k}) \underset{k \gg \gamma}{\approx} \frac{1}{2} \left(\sqrt{\frac{\lambda^2}{k^2} + 4(k_A^2 - k^2)} - \frac{\lambda}{k} \right)$$

$$\lambda \equiv \frac{\pi}{4} \left[1 - \frac{\xi}{2} \left(\frac{7}{3} - 5 \cos^2 \vartheta \right) \right] m^2$$

$$k_A \equiv m \Re \sqrt{\frac{\xi}{3} (2 \cos^2 \vartheta - 1)}$$

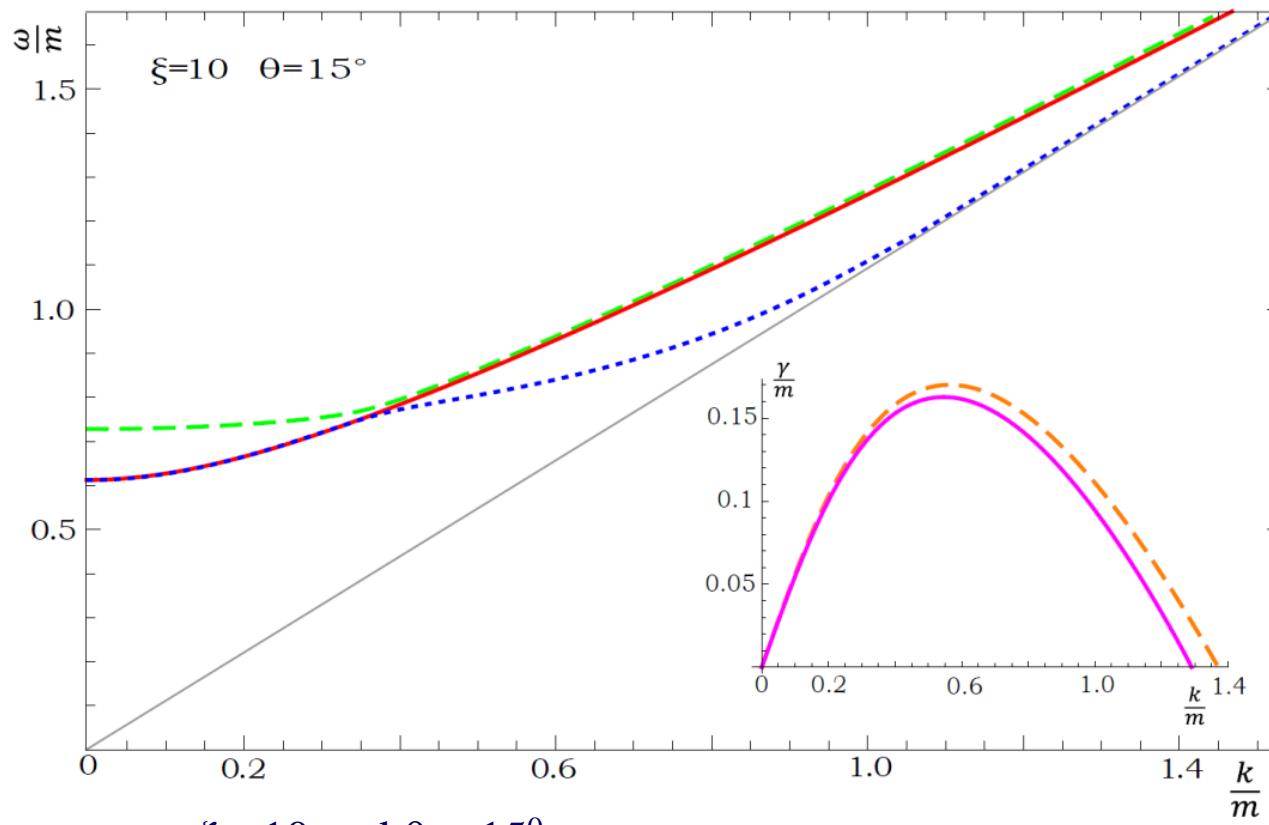
Finite prolateness

8 (numerical) solutions (6 real and 2 imaginary modes)



Finite oblateness

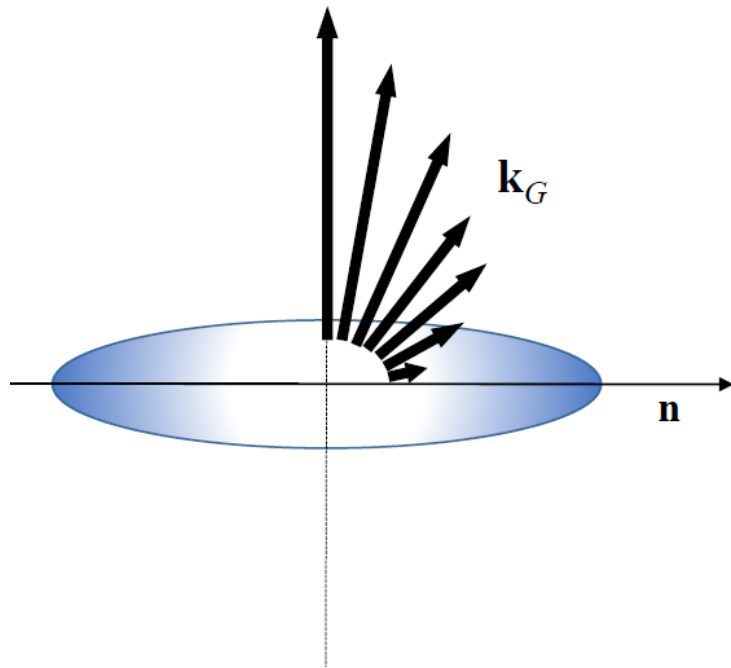
10 (numerical) solutions (6 real and 4 imaginary modes)



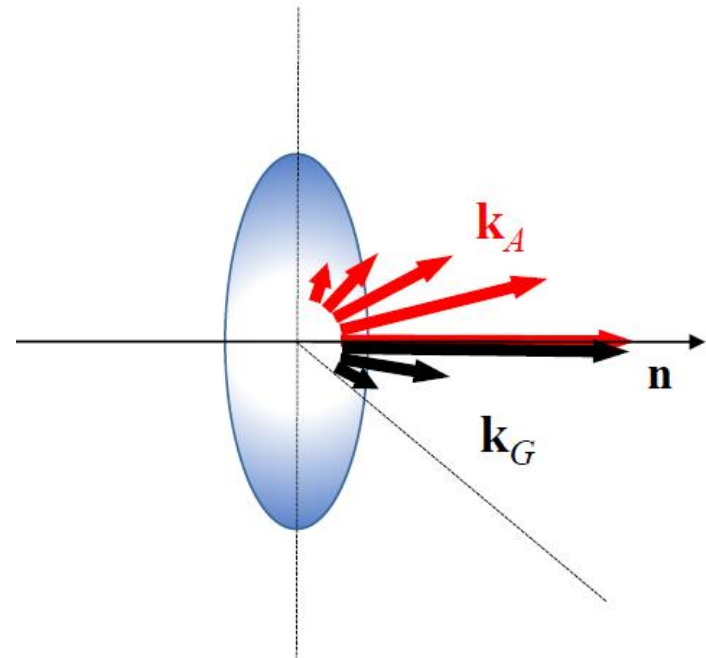
$\xi = 10$ and $\theta = 15^\circ$

Prolate versus oblate

Maximal wave vectors of unstable modes



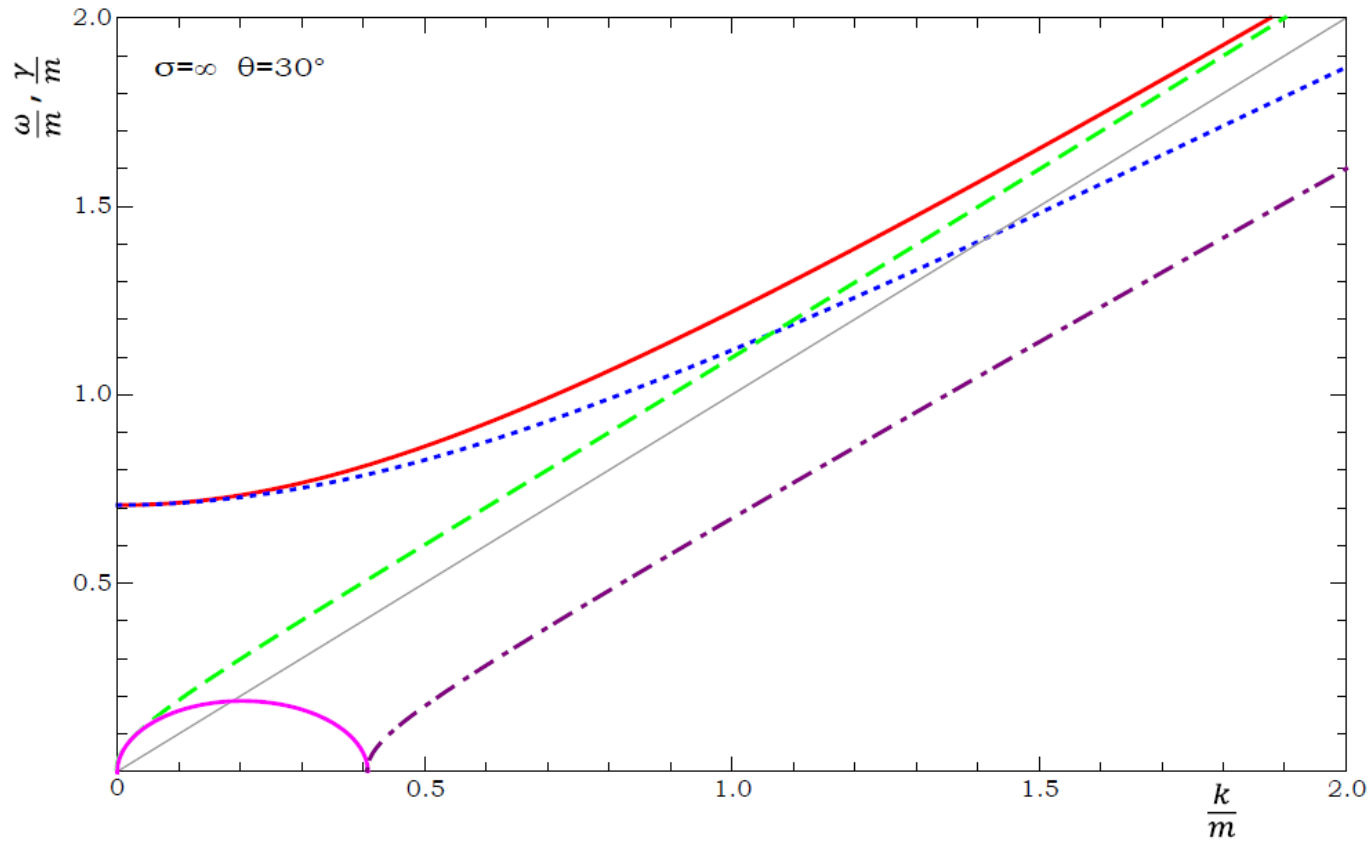
prolate



oblate

Extremely prolate QGP

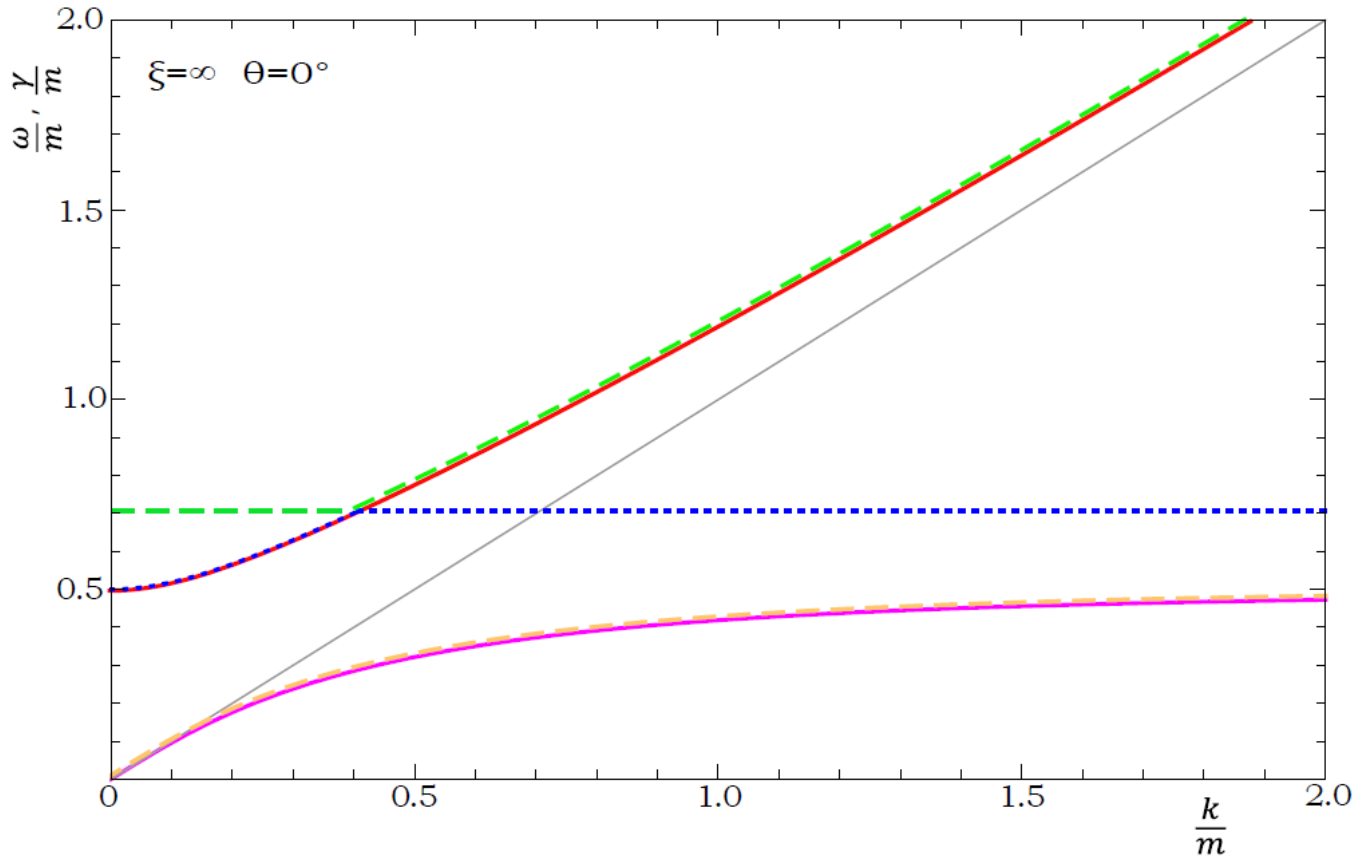
$f(\mathbf{p}) \sim \delta(p_T)$ 8 (analytic) solutions



$\theta = 30^\circ$

Extremely oblate QGP

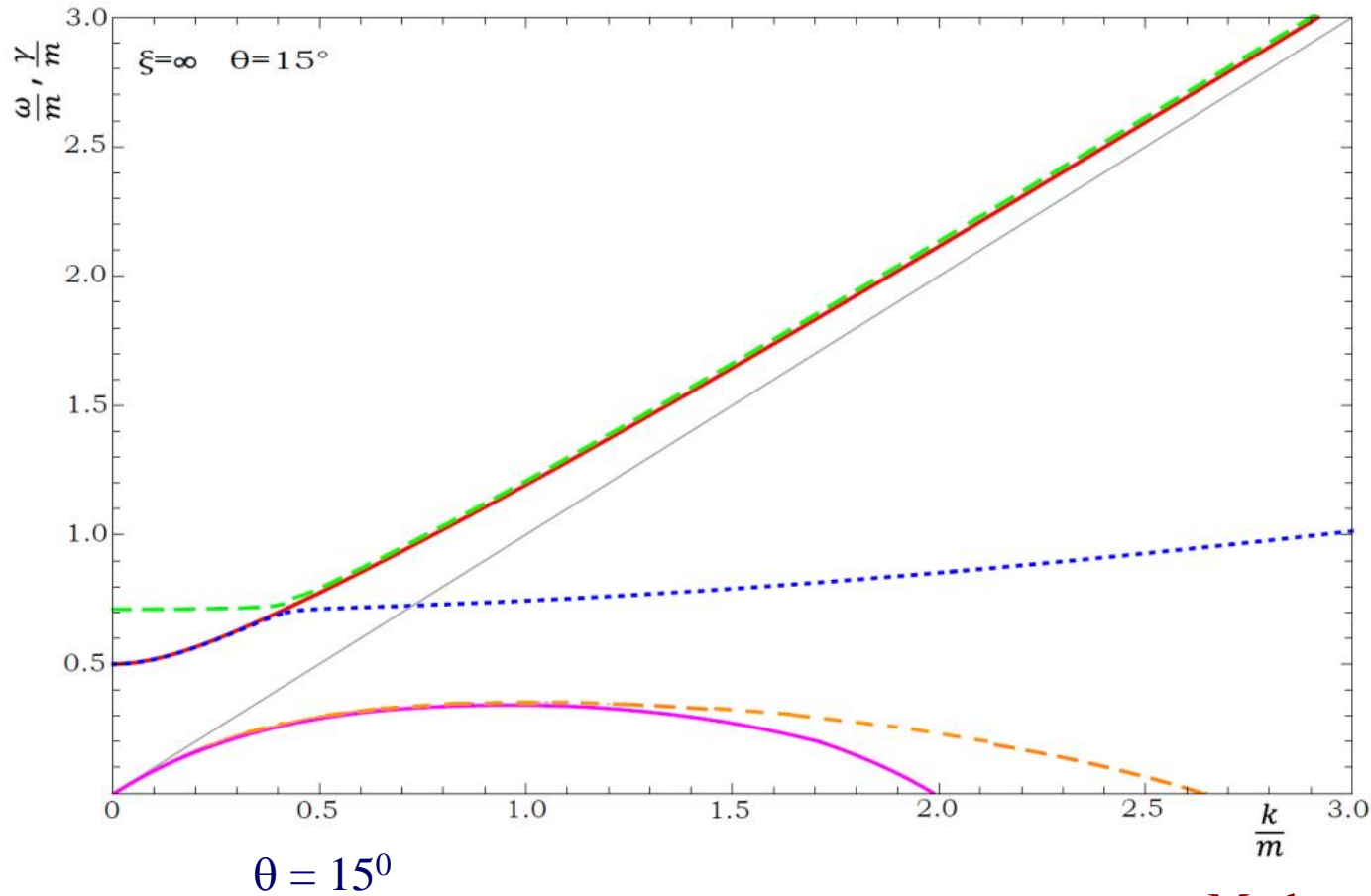
$$f(\mathbf{p}) \sim \delta(p_L) \quad \text{8 or 10 solutions}$$



How to connect the lines?

Extremely oblate QGP

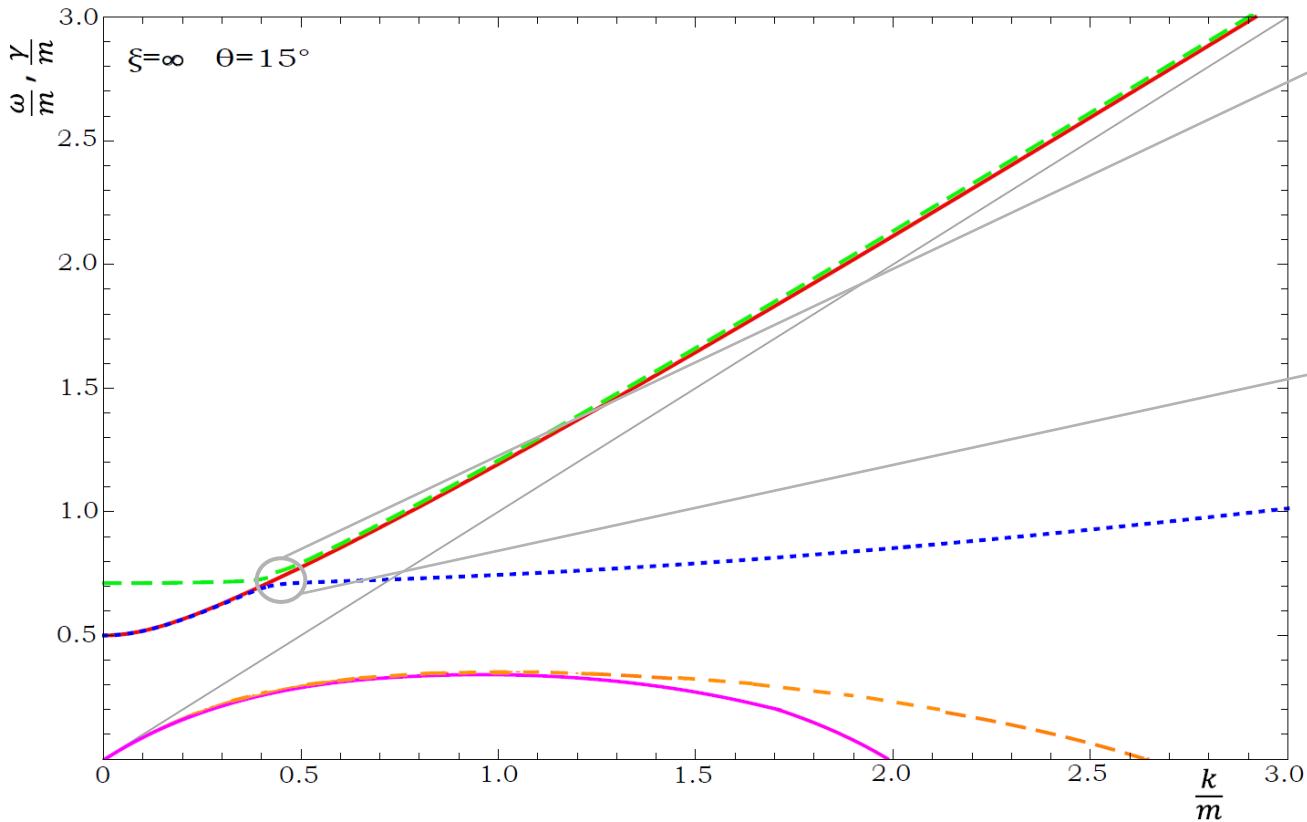
$f(\mathbf{p}) \sim \delta(p_L)$ 8 or 10 solutions



Mode crossing ?

Extremely oblate QGP

$$f(\mathbf{p}) \sim \delta(p_L) \quad \text{8 or 10 solutions}$$



$$\theta = 15^\circ$$

No mode crossing
but
mode coupling!

Number of solutions

The number of modes for each system

| Momentum distribution | Number of real modes | Number of imaginary modes | Total number of modes | Maximal number of modes |
|-----------------------|------------------------|---|---|-------------------------|
| extremely prolate | $6 + 2\Theta(k - k_p)$ | $2\Theta(k_p - k)$ | 8 | 8 |
| weakly prolate | 6 | $2\Theta(k_C - k)$ | $6 + 2\Theta(k_C - k)$ | 8 |
| isotropic | 6 | 0 | 6 | 6 |
| weakly oblate | 6 | $2\Theta(k_A - k) + 2\Theta(k_C - k)$ | $6 + 2\Theta(k_A - k) + 2\Theta(k_C - k)$ | 10 |
| extremely oblate | 6 | $2\Theta(k_{oA} - k) + 2\Theta(k_{oG} - k)$ | $6 + 2\Theta(k_{oA} - k) + 2\Theta(k_{oG} - k)$ | 10 |

Conclusions

- ▶ Systematic analysis of the complete mode spectrum is performed for any anisotropy.
- ▶ The number of modes is found in every case.
- ▶ Analytical and numerical solutions are given.
- ▶ There is no anisotropy threshold for existence of instabilities.
- ▶ Complete spectrum of modes is needed to compute various plasma characteristics e.g. the energy loss in anisotropic QGP.

for more see:

M. Carrington, K. Deja and St. Mrówczyński, arXiv:1407.2764, to appear in Phys. Rev. C