

The long-wavelength limit of the Boltzmann equation: deriving relativistic dissipative fluid dynamics from a microscopic theory

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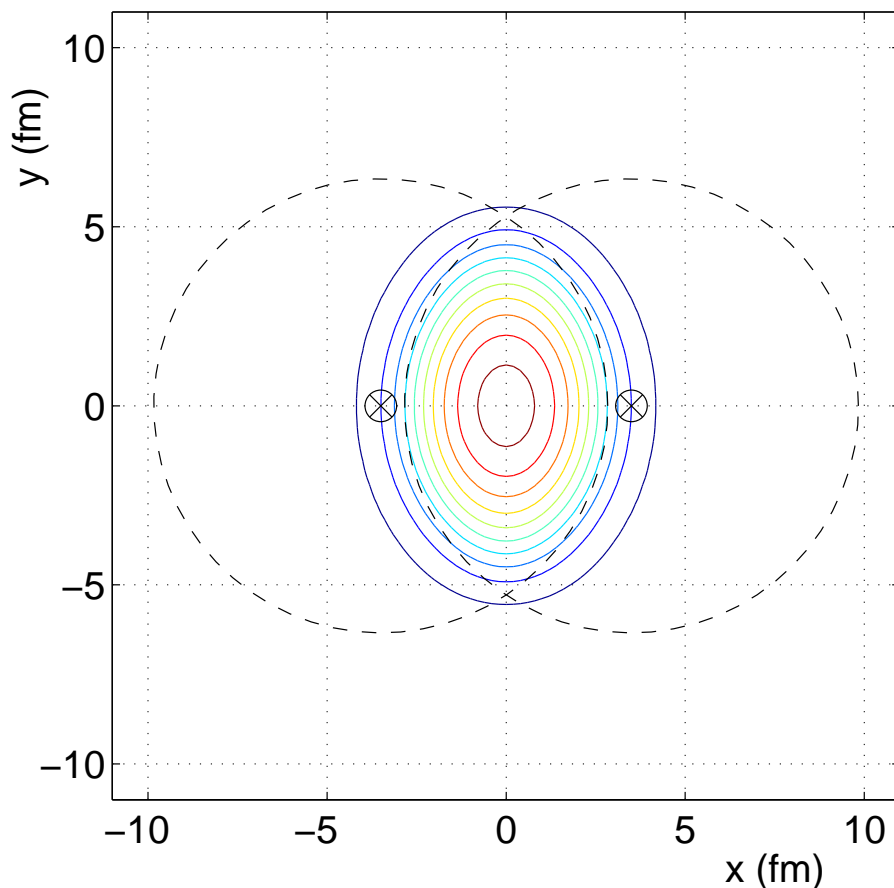


thanks to:

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Fluid dynamics implies collective flow

non-central heavy-ion collision
in transverse ($x - y$) plane ($z = 0$):



If particles do not interact with each other,
they stream freely towards the detector

⇒ single-inclusive particle spectrum:

$$E \frac{dN}{d^3\vec{p}} \equiv E \frac{dN}{dp_z d^2\vec{p}_\perp}, \quad p_\perp = \sqrt{p_x^2 + p_y^2}$$

transverse momentum

$$\equiv E \frac{dN}{dp_z p_\perp dp_\perp d\varphi} \equiv \frac{dN}{dy p_\perp dp_\perp d\varphi},$$

$$\tanh y \equiv \frac{p_z}{E}, \quad y : \text{longitudinal rapidity}$$

is independent of azimuthal angle φ

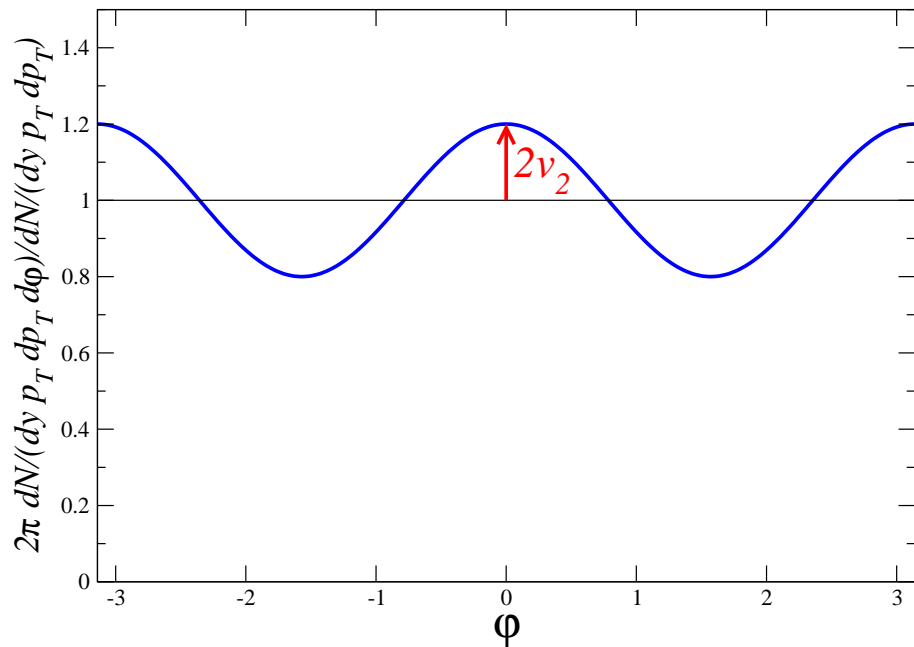
⇒ information on initial geometry is lost

But: If particles interact strongly (like in
a fluid), collective flow develops

⇒ initial spatial asymmetry is, by
difference in pressure gradients,
converted to final momentum anisotropy

Characterization of collective flow

Event-averaged single-inclusive particle spectrum at $y = 0$ as function of φ :



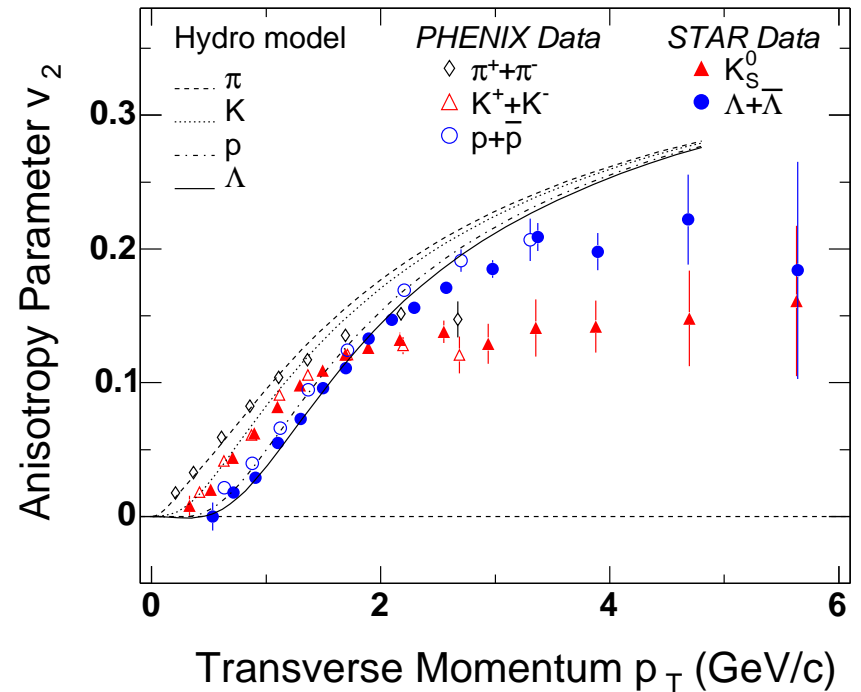
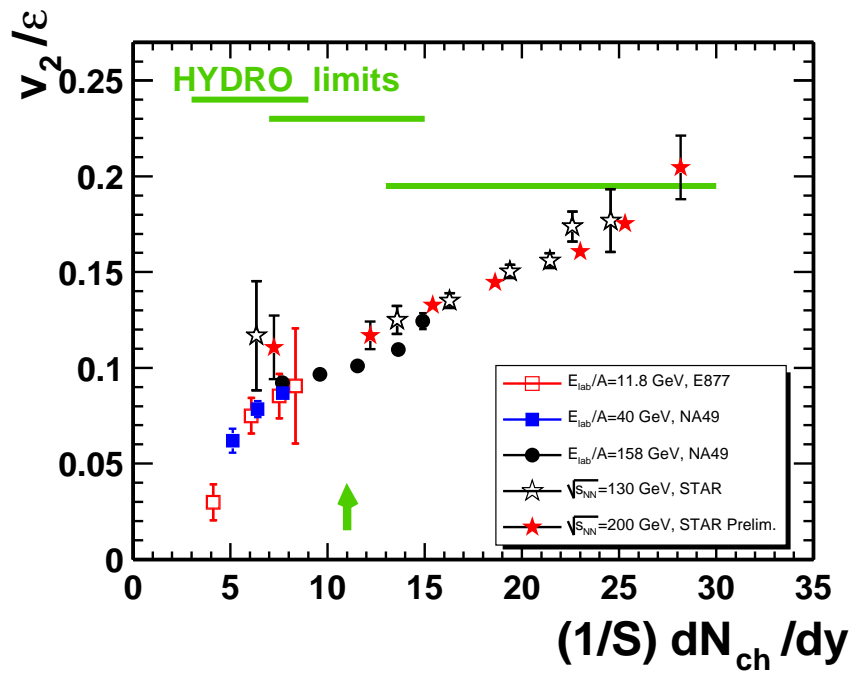
⇒ preferential emission of particles
in the **reaction** ($x - z$) **plane**

⇒ Fourier decomposition of single-inclusive particle spectrum:

$$E \frac{dN}{d^3\vec{p}} \equiv \frac{dN}{dy p_{\perp} dp_{\perp} d\varphi} \equiv \frac{1}{2\pi} \frac{dN}{dy p_{\perp} dp_{\perp}} \left[1 + 2 \sum_{n=1}^{\infty} v_n(y, p_{\perp}) \cos(n\varphi) \right]$$

v_1 : **directed flow**, v_2 : **elliptic flow** v_3 : **triangular flow** , etc.

Data confronts theory



- ⇒ approach to fluid-dynamical limit with increasing centrality and beam energy
- ⇒ quantitative description of elliptic flow at RHIC within ideal fluid dynamics
- ⇒ no dissipative effects! ⇒ “RHIC physicists serve up the perfect fluid”

Two problems (I)

1. There is no real ideal fluid!

shear viscosity $\eta \sim \frac{T}{\langle \sigma \rangle} \rightarrow 0 \iff$ average scattering cross section $\langle \sigma \rangle \rightarrow \infty$

minimal value for shear viscosity to entropy density ratio:

(i) from uncertainty principle (“quantum limit”): $\frac{\eta}{s} \gtrsim \frac{1}{12}$

P. Danielewicz, M. Gyulassy, PRD 31 (1985) 53

(ii) from AdS/CFT correspondence: conjectured lower bound $\frac{\eta}{s} = \frac{1}{4\pi}$

P. Kovtun, D.T. Son, A. Starinets, PRL 94 (2005) 111601

\implies What is $\frac{\eta}{s}$ of hot and dense hadronic matter?

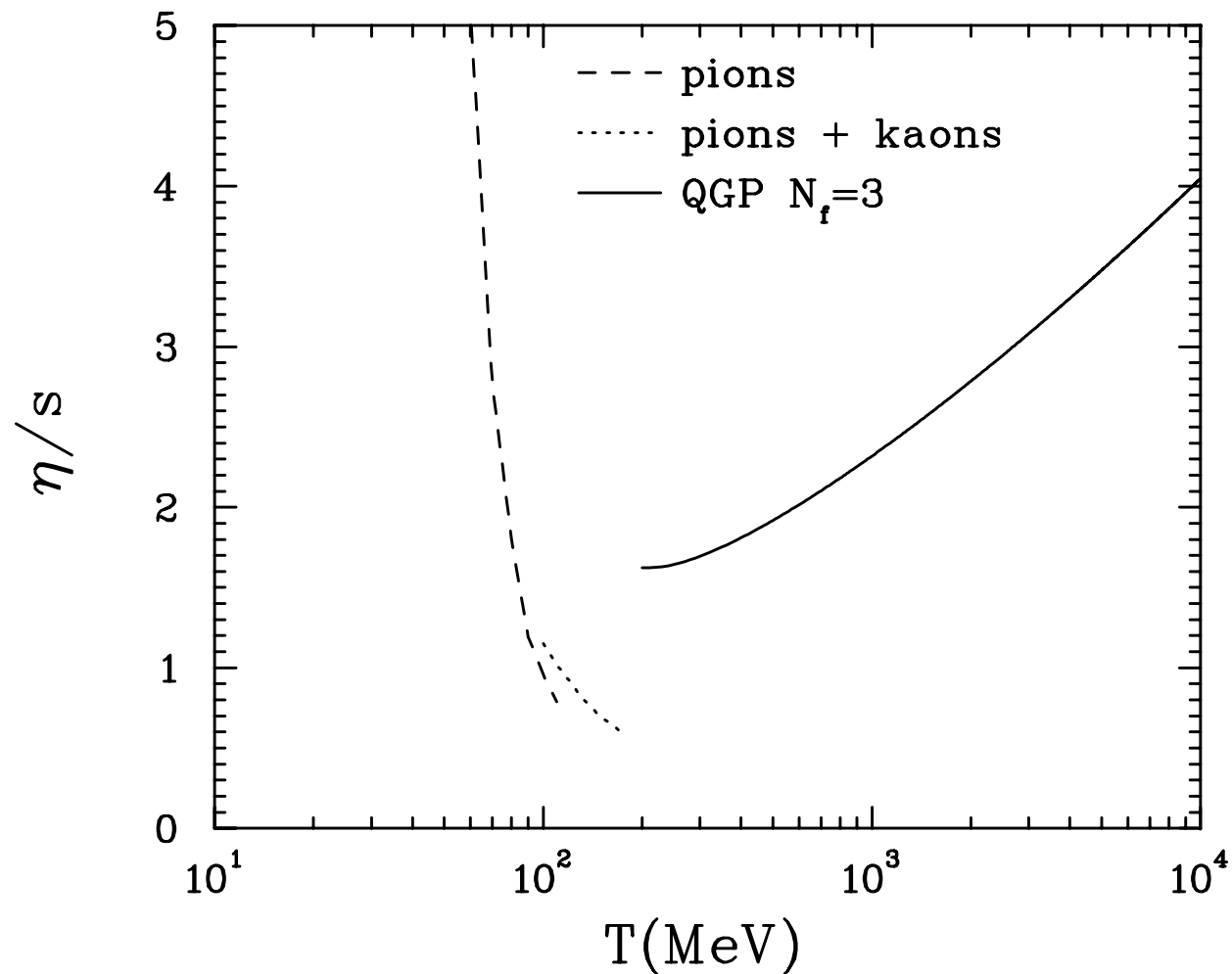
If $\frac{\eta}{s} \ll 1 \implies$ matter is strongly interacting!

\implies “strongly coupled quark-gluon plasma” (sQGP)

Two problems (I)

η/s has a **minimum** near a phase transition!

L.P. Csernai, J.I. Kapusta, L.D. McLerran, PRL 97 (2006) 152303

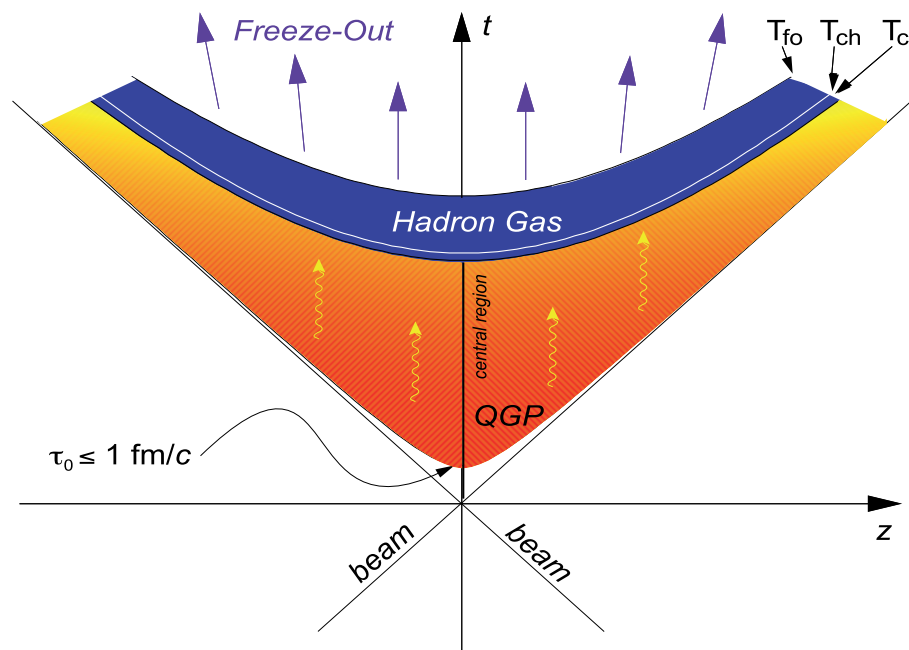


Two problems (II)

2. Fluid-dynamical equations of motion: $\partial_\mu T^{\mu\nu} = 0$

⇒ partial differential equations

⇒ require **initial conditions** on a **space-time hypersurface**



energy-momentum tensor $T^{\mu\nu}(\tau_0, \vec{x})$
on initial space-time hypersurface

$$\tau \equiv \sqrt{t^2 - z^2} \equiv \tau_0 = \text{const.}$$

⇒ continuum of parameters
to fit to experimental data

⇒ experimental data may allow
for non-zero viscosity!

⇒ need calculations within
dissipative fluid dynamics and
with **realistic initial conditions**

Fluid dynamics: degrees of freedom

1. **Net charge (e.g., baryon number, strangeness, etc.) current:** $N^\mu = n u^\mu + n^\mu$

u^μ **fluid 4-velocity**, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

$g_{\mu\nu} \equiv \text{diag}(+, -, -, -)$ (**West coast!!**) **metric tensor**

$n \equiv u^\mu N_\mu$ **net charge density in fluid rest frame**

$n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$ **diffusion current (flow of net charge relative to u^μ)**, $n^\mu u_\mu = 0$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ **projector onto 3-space orthogonal to u^μ** , $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + 2 q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ **energy density in fluid rest frame**

p **pressure in fluid rest frame**

Π **bulk viscous pressure**, $p + \Pi \equiv -\frac{1}{3} \Delta^{\mu\nu} T_{\mu\nu}$

$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ **heat flux current (flow of energy relative to u^μ)**, $q^\mu u_\mu = 0$

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ **shear stress tensor**, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0$, $\pi^\mu{}_\mu = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2} (a^{\mu\nu} + a^{\nu\mu})$ **symmetrized tensor**

$a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{(\mu} \Delta^{\nu)}_\beta - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$ **symmetrized, traceless spatial projection**

Fluid dynamics: equations of motion

1. Net charge conservation:

$$\partial_\mu N^\mu = 0 \iff \dot{n} + n\theta + \partial \cdot n = 0$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{a}_{\text{RF}} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. Energy-momentum conservation:

$$\partial_\mu T^{\mu\nu} = 0 \iff \text{energy conservation:}$$

$$u_\nu \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu} \partial_\mu u_\nu = 0$$

acceleration equation:

$$\Delta^{\mu\nu} \partial^\lambda T_{\nu\lambda} = 0 \iff$$

$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q \cdot \partial u^\mu - \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ 3-gradient,

(spatial gradient in fluid rest frame, $u_{\text{RF}}^\mu \equiv (1, 0, 0, 0)$)

Solvability

Problem:

5 equations, **but** 15 unknowns (for given u^μ): ϵ , p , n , Π , n^μ (3), q^μ (3), $\pi^{\mu\nu}$ (5)

Solution:

1. clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ
 - \implies does not help! Promotes u^μ to dynamical variable!
2. **ideal fluid limit**: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$
 - \implies 6 unknowns: ϵ , p , n , u^μ (3) (not quite there yet...)
 - \implies fluid is in local thermodynamical equilibrium
 - \implies provide **equation of state (EOS)** $p(\epsilon, n)$ to close system of equations
3. provide additional equations for dissipative quantities
 - \implies **dissipative** relativistic fluid dynamics
 - (a) **First-order** theories: e.g. generalization of **Navier-Stokes (NS)** equations to the relativistic case (Landau, Lifshitz)
 - (b) **Second-order** theories: e.g. **Israel-Stewart (IS)** equations

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order, dissipative relativistic fluid dynamics

1. bulk viscous pressure: $\Pi_{\text{NS}} = -\zeta \theta$

ζ bulk viscosity

2. diffusion current: $n_{\text{NS}}^{\mu} = \kappa_n \nabla^{\mu} \alpha$

$\beta \equiv 1/T$ inverse temperature,

$\alpha \equiv \beta \mu$, μ chemical potential,

κ_n net-charge diffusion coefficient

3. shear stress tensor: $\pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$

η shear viscosity,

$\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$ shear tensor

⇒ algebraic expressions in terms of thermodynamic and fluid variables

⇒ simple... but: unstable and acausal equations of motion!!

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

“Simplified” version:

$$\begin{aligned}\tau_{\Pi} \dot{\Pi} + \Pi &= \Pi_{\text{NS}} \\ \tau_n \dot{n}^{\langle\mu\rangle} + n^{\mu} &= n_{\text{NS}}^{\mu} \\ \tau_{\pi} \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= \pi_{\text{NS}}^{\mu\nu}\end{aligned}$$

cf. also T. Koide, G.S. Denicol, Ph. Mota, T. Kodama, Phys. Rev. C75 (2007) 034909

⇒ **dynamical** (instead of **algebraic**) equations for dissipative terms!

solution: e.g. bulk viscous pressure

$$\Pi(t) = \Pi_{\text{NS}} \left(1 - e^{-t/\tau_{\Pi}}\right) + \Pi(0) e^{-t/\tau_{\Pi}}$$

⇒ dissipative quantities Π , n^{μ} , $\pi^{\mu\nu}$ **relax** to their respective **NS** values

Π_{NS} , n_{NS}^{μ} , $\pi_{\text{NS}}^{\mu\nu}$ **on time scales** τ_{Π} , τ_n , τ_{π}

⇒ **stable and causal** fluid dynamical equations of motion!

see, e.g., S. Pu, T. Koide, DHR, PRD81 (2010) 114039

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- mean free path $\ell_{\text{mfp}} \sim (\langle \sigma \rangle n)^{-1}$
 $\langle \sigma \rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- length scale over which macroscopic fluid fields vary L_{hydro} , $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: since $\eta \sim (\langle \sigma \rangle \lambda_{\text{th}})^{-1} \implies$

$$\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{1}{\langle \sigma \rangle n} \frac{1}{\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\eta}{s}$$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\implies \frac{\eta}{s}$ solely determined by the 2 microscopic length scales!

Note: similar argument holds for $\frac{\zeta}{s}$, $\frac{\kappa_n}{\beta s}$

Power counting (II)

3 regimes:

- **dilute gas limit** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \gg 1 \iff \langle \sigma \rangle \ll \lambda_{\text{th}}^2 \implies \text{weak-coupling limit}$
- **viscous fluids** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \sim 1 \iff \langle \sigma \rangle \sim \lambda_{\text{th}}^2$
interactions happen on the scale $\lambda_{\text{th}} \implies \text{moderate coupling}$
- **ideal fluid limit** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \ll 1 \iff \langle \sigma \rangle \gg \lambda_{\text{th}}^2 \implies \text{strong-coupling limit}$

gradient (derivative) expansion:

$$\ell_{\text{mfp}} \partial_{\mu} \sim \frac{\ell_{\text{mfp}}}{L_{\text{hydro}}} \equiv K \sim \delta \ll 1$$

K **Knudsen number**

\implies equivalent to $k \ell_{\text{mfp}} \ll 1$, k typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

\implies separation of macroscopic fluid dynamics (large scale $\sim L_{\text{hydro}}$)
from microscopic particle dynamics (small scale $\sim \ell_{\text{mfp}}$)

Power counting (III)

Primary quantities: ϵ, p, n, s \iff **Dissipative quantities:** $\Pi, n^\mu, \pi^{\mu\nu}$

$$\text{If } K \sim \ell_{\text{mfp}} \partial_\mu \sim \delta \ll 1, \text{ then } \frac{\Pi}{\epsilon} \sim \frac{n^\mu}{s} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$$

Dissipative quantities are small compared to **primary quantities**
 \implies small deviations from local thermodynamical equilibrium!

Note: statement independent of value of $\frac{\zeta}{s}, \frac{\kappa_n}{\beta s}, \frac{\eta}{s}$!

Proof: Gibbs relation: $\epsilon + p = Ts + \mu n \implies \beta\epsilon \sim s$!

Estimate dissipative terms by their Navier-Stokes values:

$$\Pi \sim \Pi_{\text{NS}} = -\zeta \theta, \quad n^\mu \sim n_{\text{NS}}^\mu = \kappa_n \nabla^\mu \alpha, \quad \pi^{\mu\nu} \sim \pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$$

$$\implies \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\epsilon} \beta \theta \sim -\frac{\zeta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \theta \sim \ell_{\text{mfp}} \partial_\mu u^\mu \sim \delta,$$

$$\frac{n^\mu}{s} \sim \frac{\kappa_n}{s} \nabla^\mu \alpha \sim \frac{\kappa_n}{\beta s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \nabla^\mu \alpha \sim \ell_{\text{mfp}} \nabla^\mu \alpha \sim \delta,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2 \frac{\eta}{\beta\epsilon} \beta \sigma^{\mu\nu} \sim 2 \frac{\eta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \sigma^{\mu\nu} \sim \ell_{\text{mfp}} \nabla^{<\mu} u^{\nu>} \sim \delta, \quad \text{q.e.d.}$$

Israel-Stewart equations revisited (I)

additional **relaxation term** in **IS** equation is of second order in δ :

$$\frac{1}{\epsilon} \tau_{\Pi} \dot{\Pi} \sim \frac{1}{\epsilon} u^{\mu} \ell_{\text{mfp}} \partial_{\mu} \Pi \sim K \frac{\Pi}{\epsilon} \sim K \delta \sim O(\delta^2)$$

\implies to be consistent, have to include other second-order terms as well!

$$\begin{aligned} \tau_{\Pi} \dot{\Pi} + \Pi &= \Pi_{\text{NS}} + \mathcal{K} \\ \tau_n \dot{n}^{<\mu>} + n^{\mu} &= n_{\text{NS}}^{\mu} + \mathcal{K}^{\mu} \\ \tau_{\pi} \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi_{\text{NS}}^{\mu\nu} + \mathcal{K}^{\mu\nu} \end{aligned}$$

$$\mathcal{K} = \zeta_1 \omega_{\mu\nu} \omega^{\mu\nu} + \zeta_2 \sigma^{\mu\nu} \sigma_{\mu\nu} + \zeta_3 \theta^2 + \zeta_4 (\nabla\alpha)^2 + \zeta_5 (\nabla p)^2 + \zeta_6 \nabla\alpha \cdot \nabla p + \zeta_7 \nabla^2\alpha + \zeta_8 \nabla^2 p ,$$

$$\mathcal{K}^{\mu} = \kappa_1 \sigma^{\mu\nu} \nabla_{\nu}\alpha + \kappa_2 \sigma^{\mu\nu} \nabla_{\nu}p + \kappa_3 \theta \nabla^{\mu}\alpha + \kappa_4 \theta \nabla^{\mu}p + \kappa_5 \omega^{\mu\nu} \nabla_{\nu}\alpha + \kappa_6 \Delta^{\mu\lambda} \partial^{\nu} \sigma_{\lambda\nu} + \kappa_7 \nabla^{\mu}\theta ,$$

$$\begin{aligned} \mathcal{K}^{\mu\nu} &= \eta_1 \omega_{\lambda}^{<\mu} \omega^{\nu>\lambda} + \eta_2 \theta \sigma^{\mu\nu} + \eta_3 \sigma_{\lambda}^{<\mu} \sigma^{\nu>\lambda} + \eta_4 \sigma_{\lambda}^{<\mu} \omega^{\nu>\lambda} + \eta_5 \nabla^{<\mu}\alpha \nabla^{\nu>}\alpha \\ &+ \eta_6 \nabla^{<\mu}p \nabla^{\nu>}p + \eta_7 \nabla^{<\mu}\alpha \nabla^{\nu>}p + \eta_8 \nabla^{<\mu}\nabla^{\nu>}\alpha + \eta_9 \nabla^{<\mu}\nabla^{\nu>}p \end{aligned}$$

where $\omega^{\mu\nu} \equiv \nabla^{<\mu}u^{\nu>}$ **vorticity**

\implies **second-order gradient expansion!**

cf. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

Israel-Stewart equations revisited (II)

unfortunately, including second-order gradient terms renders eqs. of motion parabolic

⇒ acausal, unstable ⇒ in general, \mathcal{K} , \mathcal{K}^μ , $\mathcal{K}^{\mu\nu}$ have to be omitted!

... but there is more: in principle, Π , n^μ , $\pi^{\mu\nu}$ are quantities independent from θ , $\nabla^\mu\alpha$, $\nabla^\mu p$, $\sigma^{\mu\nu}$, $\omega^{\mu\nu}$

⇒ additional Lorentz-covariants:

$$\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi &= \Pi_{\text{NS}} + \mathcal{K} + \mathcal{J} + \mathcal{R} \\ \tau_n \dot{n}^{<\mu>} + n^\mu &= n_{\text{NS}}^\mu + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu \\ \tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi_{\text{NS}}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} \end{aligned}$$

$$\mathcal{J} = -\ell_{\Pi n} \nabla \cdot n - \tau_{\Pi n} n \cdot \nabla p - \delta_{\Pi\Pi} \theta \Pi - \lambda_{\Pi n} n \cdot \nabla \alpha + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\begin{aligned} \mathcal{J}^\mu &= \omega^{\mu\nu} n_\nu - \delta_{nn} \theta n^\mu - \ell_{n\Pi} \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \nabla^\lambda \pi_{\nu\lambda} + \tau_{n\Pi} \Pi \nabla^\mu p - \tau_{n\pi} \pi^{\mu\nu} \nabla_\nu p - \lambda_{nn} \sigma^{\mu\nu} n_\nu + \lambda_{n\Pi} \Pi \nabla^\mu \alpha \\ &\quad - \lambda_{n\pi} \pi^{\mu\nu} \nabla_\nu \alpha \end{aligned}$$

$$\mathcal{J}^{\mu\nu} = 2 \pi_\lambda^{<\mu} \omega^{\nu>\lambda} - \delta_{\pi\pi} \theta \pi^{\mu\nu} - \tau_{\pi\pi} \pi_\lambda^{<\mu} \sigma^{\nu>\lambda} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} - \tau_{\pi n} n^{<\mu} \nabla^{\nu>} p + \ell_{\pi n} \nabla^{<\mu} n^{\nu>} + \lambda_{\pi n} n^{<\mu} \nabla^{\nu>} \alpha$$

$$\mathcal{R} = \varphi_1 \Pi^2 + \varphi_2 n \cdot n + \varphi_3 \pi^{\mu\nu} \pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4 \pi^{\mu\nu} n_\nu + \varphi_5 \Pi n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi_\lambda^{<\mu} \pi^{\nu>\lambda} + \varphi_8 n^{<\mu} n^{\nu>}$$

Matching to kinetic theory

Fluid dynamics is an effective (**macroscopic**) theory for the **long-wavelength, small-frequency limit** of a given (**microscopic**) theory
⇒ coefficients in equations of motion can be determined by **matching** to the underlying theory, e.g. **kinetic theory**

G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD85 (2012) 114047

Details (I)

1. Boltzmann equation $k \cdot \partial f_k = C[f]$ for single-particle distribution function f_k
2. introduce irreducible tensors of rank ℓ : $k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \equiv \Delta_{\mu_1 \cdots \mu_\ell}^{\nu_1 \cdots \nu_\ell} k_{\nu_1} \cdots k_{\nu_\ell}$
 $\Delta_{\mu_1 \cdots \mu_\ell}^{\nu_1 \cdots \nu_\ell}$ are projectors onto subspaces orthogonal to u^μ , symmetric in μ_i, ν_i , and traceless
3. f_k can be expanded in terms of $k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$

$$f_k = f_{0k} + f_{0k} \tilde{f}_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{kn}^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}$$

where

- (a) $f_{0k} = [\exp(\beta u \cdot k - \alpha) + a]^{-1}$ single-particle distribution function in local equilibrium, $a = \pm 1/0$ for Fermi/Bose/Boltzmann statistics
- (b) $\tilde{f}_{0k} = 1 - a f_{0k}$
- (c) $\mathcal{H}_{kn}^{(\ell)} = \frac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{km}^{(\ell)}$, where
 $P_{kn}^{(\ell)} = \sum_{r=0}^n a_{nr}^{(\ell)} E_k^r$ are orthogonal polynomials of order n in energy $E_k \equiv u \cdot k$
 $\implies \mathcal{H}_{kn}^{(\ell)}$ are polynomials of order N_ℓ in energy E_k
- (d) irreducible moments of $\delta f_k \equiv f_k - f_{0k}$: $\rho_n^{\mu_1 \cdots \mu_\ell} = \int dK \delta f_k E_k^n k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle}$

Details (II)

4. rewrite Boltzmann equation in the form $\delta \dot{f}_k = -\dot{f}_{0k} - \frac{1}{E_k} \{k \cdot \nabla (f_{0k} + \delta f_k) - C[f]\}$
5. derive equations of motion for **irreducible moments**, e.g. up to $\ell = 2$:

$$\begin{aligned}
 \dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)} \theta - \frac{G_{2r}}{D_{20}} \theta \Pi + \frac{G_{2r}}{D_{20}} \sigma^{\mu\nu} \pi_{\mu\nu} + \frac{G_{3r}}{D_{20}} \partial \cdot n + (r-1) \sigma_{\mu\nu} \rho_{r-2}^{\mu\nu} + r \dot{u}_\mu \rho_{r-1}^\mu \\
 &\quad - \nabla_\mu \rho_{r-1}^\mu - \frac{1}{3} [(r+2) \rho_r - (r-1) m^2 \rho_{r-2}] \theta \\
 \dot{\rho}_r^{<\mu>} &= C_{r-1}^{<\mu>} + \alpha_r^{(1)} \nabla^\mu \alpha + \omega_\nu^\mu \rho_r^\nu - \frac{1}{3} [(r+3) \rho_r^\mu - (r-1) m^2 \rho_{r-2}^\mu] \theta - \Delta_\lambda^\mu \nabla_\nu \rho_{r-1}^{\lambda\nu} \\
 &\quad - \frac{1}{5} [(2r+3) \rho_r^\nu - 2(r-1) m^2 \rho_{r-2}^\nu] \sigma_\nu^\mu - \frac{1}{3} [(r+3) \rho_{r+1} - r m^2 \rho_{r-1}] \dot{u}^\mu \\
 &\quad + \frac{\beta J_{r+2,1}}{\epsilon+p} (\Pi \dot{u}^\mu - \nabla^\mu \Pi + \Delta^{\mu\nu} \partial^\lambda \pi_{\lambda\nu}) + \frac{1}{3} \nabla^\mu (\rho_{r+1} - m^2 \rho_{r-1}) \\
 &\quad + (r-1) \rho_{r-2}^{\mu\nu\lambda} \sigma_{\lambda\nu} + r \dot{u}_\nu \rho_{r-1}^{\mu\nu} \\
 \dot{\rho}_r^{<\mu\nu>} &= C_{r-1}^{<\mu\nu>} + 2 \alpha_r^{(2)} \sigma^{\mu\nu} - \frac{2}{7} [(2r+5) \rho_r^{\lambda<\mu} - 2(r-1) m^2 \rho_{r-2}^{\lambda<\mu}]] \sigma_\lambda^{\nu>} + 2 \rho_r^{\lambda<\mu} \omega_\lambda^{\nu>} \\
 &\quad + \frac{2}{15} [(r+4) \rho_{r+2} - (2r+3) m^2 \rho_r + (r-1) m^4 \rho_{r-2}] \sigma^{\mu\nu} \\
 &\quad + \frac{2}{5} \nabla^{<\mu} (\rho_{r+1}^{\nu>} - m^2 \rho_{r-2}^{\nu>}) - \frac{2}{5} [(r+5) \rho_{r+1}^{<\mu} - r m^2 \rho_{r-1}^{<\mu}]] \dot{u}^{\nu>} \\
 &\quad - \frac{1}{3} [(r+4) \rho_r^{\mu\nu} - (r-1) m^2 \rho_{r-2}^{\mu\nu}] \theta + (r-1) \rho_{r-2}^{\mu\nu\lambda\rho} \sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu} \nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} + r \rho_{r-1}^{\mu\nu\lambda} \dot{u}_\lambda
 \end{aligned}$$

$\alpha_r^{(\ell)}$, G_{nm} , D_{nq} , J_{nq} **thermodynamic functions**

$C_r^{<\mu_1 \dots \mu_\ell>} = \int dK E_k^r k^{<\mu_1 \dots \mu_\ell>} C[f]$ **irreducible moment of collision integral**

Details (III)

Remarks:

- (a) system of infinitely many coupled equations for **irreducible moments** $\rho_r^{\mu_1 \dots \mu_\ell}$
 - (b) system completely equivalent to Boltzmann equation
 - (c) by definition $\rho_0 = -\frac{3}{m^2} \Pi$, $\rho_0^\mu = n^\mu$, $\rho_0^{\mu\nu} = \pi^{\mu\nu}$
 - (d) matching conditions in Landau frame imply $\rho_1 = \rho_2 = \rho_1^\mu = 0$
6. fluid dynamics comprises tensors up to rank 2 \implies neglect $\rho_r^{\mu_1 \dots \mu_\ell}$ with $\ell > 2$

7. linearize collision integral: $C_{r-1}^{<\mu_1 \dots \mu_\ell>} = - \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1 \dots \mu_\ell} + O(\delta f_k^2)$

\implies linearized equation of motion
for irreducible moments:

$$\begin{aligned} \dot{\vec{\rho}} + \mathcal{A}^{(0)} \vec{\rho} &\simeq \vec{\alpha}^{(0)} \theta + \dots \\ \dot{\vec{\rho}}^\mu + \mathcal{A}^{(1)} \vec{\rho}^\mu &\simeq \vec{\alpha}^{(1)} \nabla^\mu \alpha + \dots \\ \dot{\vec{\rho}}^{\mu\nu} + \mathcal{A}^{(2)} \vec{\rho}^{\mu\nu} &\simeq 2 \vec{\alpha}^{(2)} \sigma^{\mu\nu} + \dots \end{aligned}$$

8. diagonalize collision matrix: $(\Omega^{-1})^{(\ell)} \mathcal{A}^{(\ell)} \Omega^{(\ell)} = \text{diag}(\chi_0^{(\ell)}, \dots, \chi_j^{(\ell)}, \dots)$
for later purposes: $\tau^{(\ell)} \equiv (\mathcal{A}^{-1})^{(\ell)} = \Omega^{(\ell)} \text{diag}(1/\chi_0^{(\ell)}, \dots, 1/\chi_j^{(\ell)}, \dots) (\Omega^{-1})^{(\ell)}$

\implies

$$\begin{aligned} \tau^{(0)} \dot{\vec{\rho}} + \vec{\rho} &\simeq \tau^{(0)} \vec{\alpha}^{(0)} \theta + \dots \\ \tau^{(1)} \dot{\vec{\rho}}^\mu + \vec{\rho}^\mu &\simeq \tau^{(1)} \vec{\alpha}^{(1)} \nabla^\mu \alpha + \dots \\ \tau^{(2)} \dot{\vec{\rho}}^{\mu\nu} + \vec{\rho}^{\mu\nu} &\simeq 2 \tau^{(2)} \vec{\alpha}^{(2)} \sigma^{\mu\nu} + \dots \end{aligned}$$

Details (IV)

9. **eigenmodes of linearized equations of motion:** $X_i^{\mu_1 \dots \mu_\ell} = \sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} \rho_j^{\mu_1 \dots \mu_\ell}$

\implies equations of motion for eigenmodes decouple:

$$\begin{aligned} \dot{X}_i + \chi_i^{(0)} X_i &= \beta_i^{(0)} \theta + \dots \\ \dot{X}_i^{<\mu>} + \chi_i^{(1)} X_i^\mu &= \beta_i^{(1)} \nabla^\mu \alpha + \dots \\ \dot{X}_i^{<\mu\nu>} + \chi_i^{(2)} X_i^{\mu\nu} &= \beta_i^{(2)} \sigma^{\mu\nu} + \dots \end{aligned}$$

where $\beta_i^{(0)} = \sum_{j=0, \neq 1, 2}^{N_0} (\Omega^{-1})_{ij}^{(0)} \alpha_j^{(0)}$, $\beta_i^{(1)} = \sum_{j=0, \neq 1}^{N_1} (\Omega^{-1})_{ij}^{(1)} \alpha_j^{(1)}$, $\beta_i^{(2)} = 2 \sum_{j=0}^{N_2} (\Omega^{-1})_{ij}^{(2)} \alpha_j^{(2)}$

10. **slowest eigenmodes (w/o r.o.g. $i = 0$) remain dynamical,**
all faster ones ($i \neq 0$) are replaced by their asymptotic (NS) values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}} \theta, \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}} \nabla^\mu \alpha, \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}} \sigma^{\mu\nu}$$

11. **Since** $\rho_i^{\mu_1 \dots \mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1 \dots \mu_\ell}$:

$$\begin{aligned} \rho_i &\simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu \alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} \end{aligned}$$

Details (V)

⇒ express X_0 , X_0^μ , $X_0^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$

$$\text{(w/o r.o.g. } \Omega_{00}^{(\ell)} \equiv 1): \quad X_0 \simeq -\frac{3}{m^2} \Pi - \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta$$

$$X_0^\mu \simeq n^\mu - \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu\alpha$$

$$X_0^{\mu\nu} \simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

⇒ express ρ_i , ρ_i^μ , $\rho_i^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$:

$$\begin{aligned} \frac{m^2}{3} \rho_i &\simeq -\Omega_{i0}^{(0)} \Pi + \left(\zeta_i - \Omega_{i0}^{(0)} \zeta_0 \right) \theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_{ni} - \Omega_{i0}^{(1)} \kappa_{n0} \right) \nabla^\mu\alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)} \pi^{\mu\nu} + 2 \left(\eta_i - \Omega_{i0}^{(2)} \eta_0 \right) \sigma^{\mu\nu} \end{aligned}$$

where $\zeta_i = \frac{m^2}{3} \sum_{r=0, \neq 1, 2}^{N_0} \tau_{ir}^{(0)} \alpha_r^{(0)}$, $\kappa_{ni} = \sum_{r=0, \neq 1}^{N_1} \tau_{ir}^{(1)} \alpha_r^{(1)}$, $\eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)} \alpha_r^{(2)}$

⇒ equations of motion for **irreducible moments** become identical with equations of motion for **dissipative quantities** Π , n^μ , $\pi^{\mu\nu}$

⇒ identify transport coefficients

Discussion (I)

1. Basis of expansion for δf_k is **orthogonal in irreducible subspaces**
 \implies truncation at **any order in ℓ and N_ℓ possible!**
2. 14-moment approximation corresponds to choice $N_0 = 2$, $N_1 = 1$, $N_2 = 0$ and leads to **IS** equations
3. approximation can be systematically improved by increasing N_ℓ
4. transport coefficients approach Chapman-Enskog values already for $N_0 = 5$, $N_1 = 4$, $N_2 = 3$ (41-moment approximation)

Example: classical massless gas with constant cross section σ , $\ell_{\text{mfp}} = (\sigma n)^{-1}$

# of moments	η	$\tau_\pi[\ell_{\text{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta)$	2	$134/77$	$0.344/\beta$	$4/3$	$-0.689/\beta$	$-0.689/n$
32	$1.268/(\sigma\beta)$	2	1.69	$0.254/\beta$	$4/3$	$-0.687/\beta$	$-0.687/n$
41	$1.267/(\sigma\beta)$	2	1.69	$0.244/\beta$	$4/3$	$-0.685/\beta$	$-0.685/n$

# of moments	κ_n	$\tau_n[\ell_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta/20$	$\beta/20$	$0.0125\beta/p$
23	$21/(128\sigma)$	2.59	1.0	0.96	0.054β	0.118β	$0.0295\beta/p$
32	$0.1605/\sigma$	2.57	1.0	0.93	0.052β	0.119β	$0.0297\beta/p$
41	$0.1596/\sigma$	2.57	1.0	0.92	0.052β	0.119β	$0.0297\beta/p$

Discussion (II)

5. approach can be further **systematically** improved:

- (a) consider also faster eigenmodes X_i , X_i^μ , $X_i^{\mu\nu}$, $i > 0$, to be dynamical
- (b) take into account irreducible moments of tensor rank $\ell > 2$
- (c) take into account second-order corrections in the collision integral
(compute coefficients $\varphi_1, \dots, \varphi_8$)

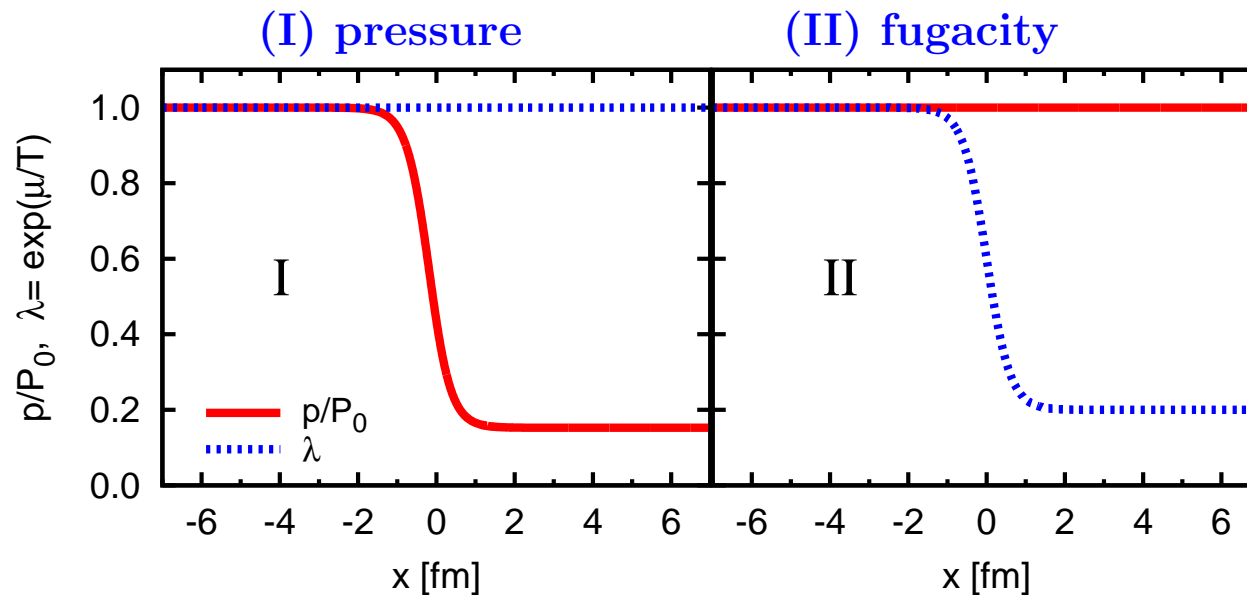
E. Molnar, H. Niemi, G.S. Denicol, DHR, PRD 89 (2014) 074010

e.g. massless Boltzmann gas: $\varphi_4 = \frac{1}{25p}$, $\varphi_7 = \frac{9}{70p}$, $\varphi_8 = \frac{8}{5\beta^2 p}$

Application: heat-flow problem (I)

G.S. Denicol, H. Niemi, I. Bouras, E. Molnár, Z. Xu, DHR, C. Greiner, arXiv:1207.6811[nucl-th]

Initial conditions: discontinuity in



⇒ first-order (NS) terms can be vanishingly small:

$$(I): \nabla^\mu \alpha \simeq 0 \qquad (II): \nabla^\mu p \simeq \dot{u}^\mu \simeq 0 \implies \sigma^{\mu\nu} \simeq 0$$

⇒ second-order terms can become larger than first-order terms!

⇒ power-counting scheme in terms of Knudsen number is invalidated!

Application: heat-flow problem (II)

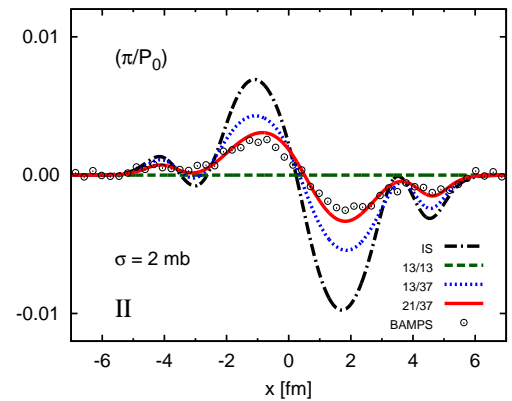
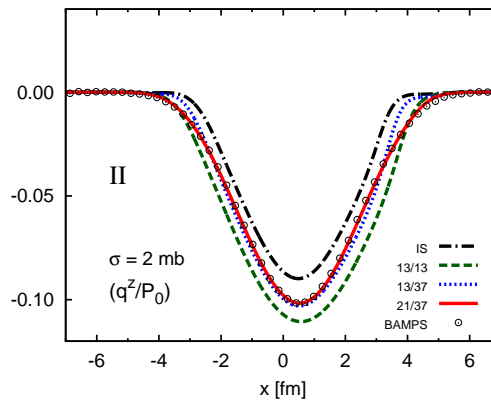
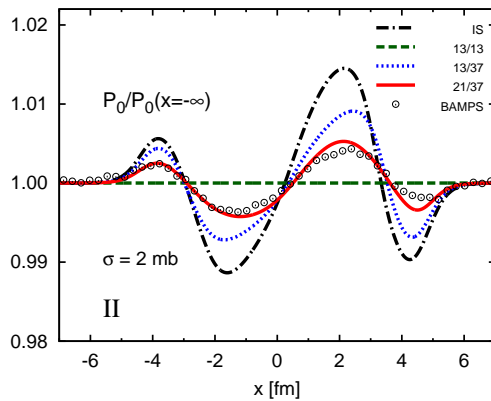
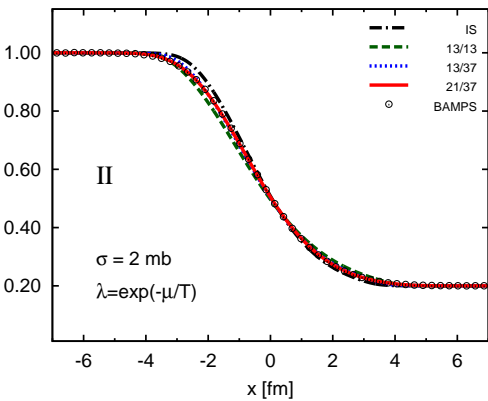
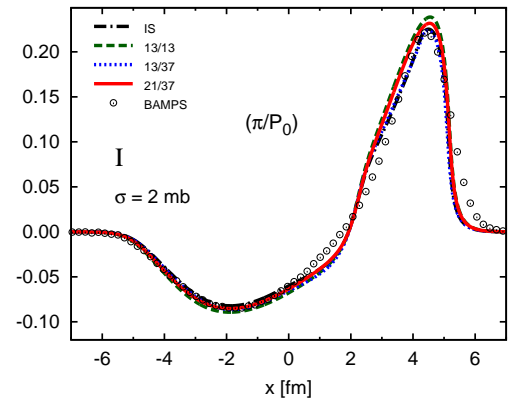
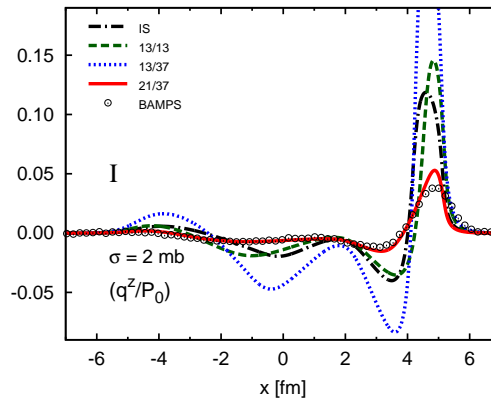
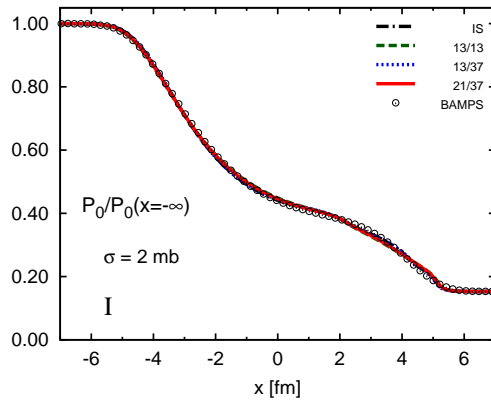
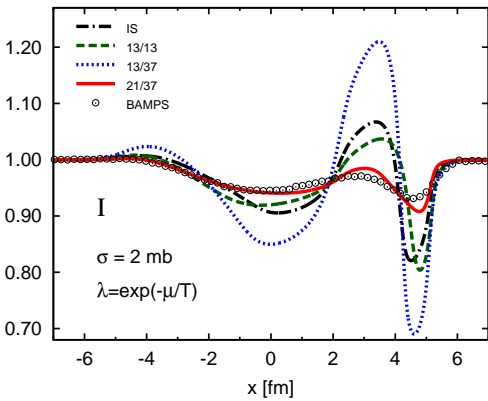
Solution: consider ρ_2^μ , $\rho_1^{\mu\nu}$ as **additional dynamical variables!**

$e^{-\alpha}$

p

q^μ

$\pi^{\mu\nu}$



Conclusions

1. Second-order fluid dynamics has been **systematically** derived as **long-wavelength, small-frequency limit** of kinetic theory
2. Transport coefficients agree with values from Chapman-Enskog expansion
3. Heat-flow problem can be solved by taking **higher** irreducible moments to be **dynamical** variables
4. Further systematic improvements are possible and should be explored