Conformal Field Theories

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work with Martin Lüscher

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1. M. Lüscher, G. Mack, *Global Conformal Invariance in Quantum Field Theory*, Commun. Math. Phys. **41** 203-234 (1975)
2. M. Lüscher, G. Mack, *The energy momentum tensor of critical quantum field theories in* $1+1$ *dimensions*, unpublished (1976)
3. M. Lüscher, *Analytic Representations of Simple Lie Groups and their Continuation to Contractive Representations of Holomorphic Semigroups* DESY 75/51 (1975) PhD thesis

Global Conformal Invariance in Quantum Field Theory

Commun. math. Phys. 41, 203—234 (1975)

Global Conformal Invariance in Quantum Field Theory

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Abstract. Suppose that there is given a Wightman quantum field theory (QFT) whose Euclidean Green functions are invariant under the Euclidean conformal group $\mathfrak{G} \simeq SO_e(5, 1)$. We show that its Hilbert space of physical states carries then a unitary representation of the universal (∞-sheeted) covering group $\mathfrak{G}^*$ of the Minkowskian conformal group $SO_e(4, 2)/\mathbb{Z}_2$. The Wightman functions can be analytically continued to a domain of holomorphy which has as a real boundary an ∞-sheeted covering $\tilde{M}$ of Minkowski-space M^4. It is known that $\mathfrak{G}^*$ can act on this space $\tilde{M}$ and that $\tilde{M}$ admits a globally $\mathfrak{G}^*$-invariant causal ordering; $\tilde{M}$ is thus the natural space on which a globally $\mathfrak{G}^*$- invariant local QFT could live. We discuss some of the properties of such a theory, in particular the spectrum of the conformal Hamiltonian $H = \frac{1}{2}(P^0 + K^0)$.

As a tool we use a generalized Hille-Yosida theorem for Lie semigroups. Such a theorem is stated and proven in Appendix C. It enables us to analytically continue contractive representations of a certain maximal subsemigroup $\mathfrak{S}$ of $\mathfrak{G}$ to unitary representations of $\mathfrak{G}^*$.

1. Introduction

Conformal invariant quantum field theory (QFT) is of interest from the point 2
of view of constructive quantum field theory because such theories can be analyzed to a remarkable extent by nonperturbative methods, i.e. without recourse to

The problem:

Finite conformal transformations can take points of Minkowski space to infinity.

The appropriately compactified Minkowski space does not admit a conformal invariant causal structure:

(Finite conformal transformations can take relatively spacelike pairs of points to relatively timelike points)

$\Rightarrow$ bad reputation of conformal symmetry

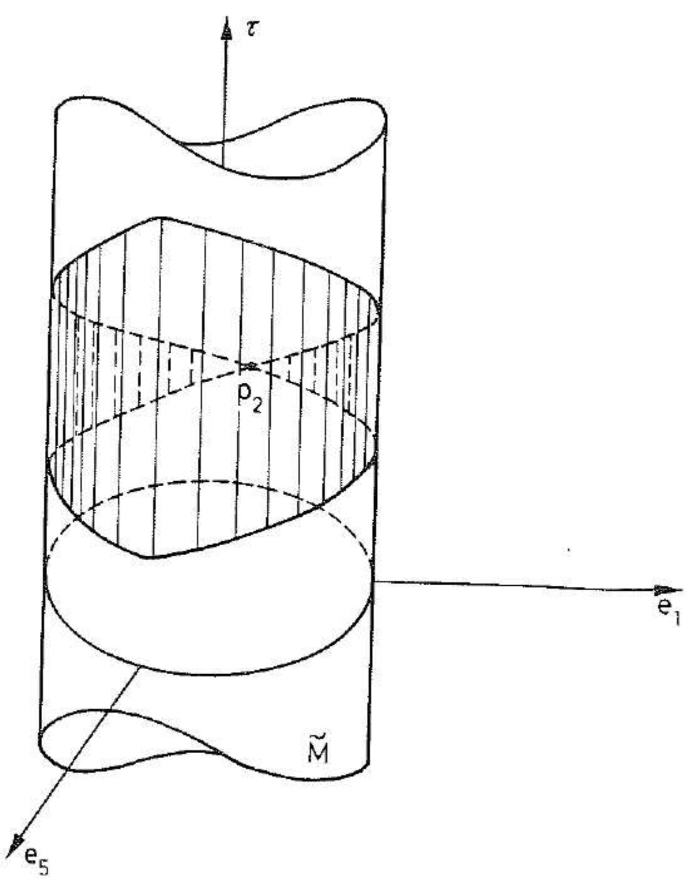
but infinitesimal conformal transf. are ok solution: QM admits projective (rather than unitary) representations of symmetry groups, and therefore unitary representations of the universal covering $\mathfrak{G}^*$ of $SO(4,2)$,

$$SO(4,2) = \mathfrak{G}^*/\mathbf{Z}_2 \times \mathbf{Z}$$

$\mathfrak{G}^*$ can act on an ∞-sheeted covering $\tilde{M}$ of conformally compactified Minkowski space.

$\tilde{M}$ admits an invariant causal ordering

subtlety: The eigenvalues of $\mathbf{Z}$ do not distinguish superselection sectors (Schroer and Swieca showed 2-dim models) Contrast rotations


τ
p_2
e_1
$\tilde{M}$
e_5

Proposition: 1. Assuming conformal invariance of Euklidean Green functions, the Hilbert space of physical states admits a unitary representation of the universal covering $\mathfrak{G}^*$ of the conformal group $SO(4,2)$.

2. The Euklidean N-point Green functions admit analytic continuation to a complex domain with N-tuples of points on the $\infty-$sheeted universal covering $\tilde{M}$ of compactified Minkowski space as its boundary.

Mathematical basis: prove and apply a generalized Hille Yosida Theorem.

1. Reflection positivity (Osterwalder Schrader positivity) of Euklidean Green functions permits to reconstruct the Hilbert space $\mathcal{H}$ of physical states and its scalar product from the Euklidean Green functions

2. Elements of a maximal subsemigroup $\mathfrak{S}$ of the Euklidean conformal group $\mathfrak{G} \simeq SO(5,1)$ can act on $\mathcal{H}$ as contraction operators.

3. This contractive representation of $\mathfrak{S}$ can be analytically continued to a unitary representation of $\mathfrak{G}^*$ acting in $\mathcal{H}$

Let $SO(4,1) = \mathfrak{U} \subset \mathfrak{G},\ H = -iJ_{64}$
$\mathfrak{S}^0 = \{\Lambda = u_1 e^{-H\tau} u_2,\ \tau > 0,\ u_i \in \mathfrak{U}\} \subset \mathfrak{G}$
maps the half space $x^4 > 0$ into itself.

Hille- Yosida theorem

Given a 1-parameter semigroup of operators T_t on $\mathcal{H}$, $t \geq 0$ with $||T_t\Psi - \Psi|| \mapsto 0$ for $t \mapsto 0$ for all $\Psi \in \mathcal{H}$

Then $T_t = \exp(-Ht)$ with positive selfadjoint generator $H \Rightarrow$

The semigroup can be analytically continued to a 1-parameter group of unitary operators $\exp(-iHs)$.

with a view towards generalization

$$H \in \mathfrak{g}_-, \qquad iH \in i\mathfrak{g}_-$$

Lie group $\mathfrak{G}$ with Lie algebra $\mathfrak{g}$ (Euklidean), automorphism θ of $\mathfrak{g}$, $\theta^2 = 1$.

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$$

$\theta(X) = \pm X$ for $X \in \mathfrak{g}_\pm$. Consider

$$\mathfrak{g}^* = \mathfrak{g}_+ + i\mathfrak{g}_-$$

Lie algebra of $\mathfrak{G}^*$ (Minkowskian)
Both contain $\mathfrak{U}$ generated by $\mathfrak{g}_+$.
Suppose open convex cone $V \subset \mathfrak{g}_-$
  V is invariant under $\mathfrak{U}$, (i.e. $uXu^{-1} \in V$)
  V and $\mathfrak{g}_+$ span $\mathfrak{g}$.$\Rightarrow$ Semigroups $\subset \mathfrak{G}$
$\mathfrak{S}^0 = \{\Lambda = e^{X_1} ... e^{X_k} u \in \mathfrak{G}, k \geq 1, X_i \in V\}$
$\mathfrak{S} = \mathfrak{S}^0 \cup \mathfrak{U}$

generalized Hille Yosida theorem
for Lie semigroups
Let T be a representation of $\mathfrak{S}$ by contraction operators on $\mathcal{H}$ (viz.$||T(\Lambda)|| \leq 1$).
Suppose that

$$||T(\Lambda)\Psi - \Psi|| \mapsto 0 \text{ if } \Lambda \mapsto 1$$

$$< \Psi|T(\Lambda)\Phi > = < T(\bar{\Lambda})\Psi|\Phi >, \; \bar{\Lambda} = \theta(\Lambda^{-1})$$

Then T can be analytically continued to a unitary representation of the simply connected Lie group $\mathfrak{G}^*$ with Lie algebra $\mathfrak{g}^* = \mathfrak{g}_+ + i\mathfrak{g}_-$. The selfadjoint generators $T(X)$ are positive for $-X \in V$.
conformal Hamiltonian $H = J_{60} = \frac{1}{2}(P^0 + K^0)$ and its conjugates

The energy momentum tensor
in 1+1 dimensions is a Lie field
assuming dilatation symmetry

The energy momentum tensor of critical quantum field theories in 1+1 dimensions.

by

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Abstract: We show that the energy momentum tensor of a scaling invariant quantum field theory in two dimensional space time is always a Lie field. Its commutation relations are unique up to two positive real numbers which depend on the dynamics of the model considered

$x_\pm = x^0 \pm x^1$, dilatation symmetry $\Rightarrow$
$\theta^\mu{}_\mu(x) = 0, \quad \theta_{00} \pm \theta_{01} = \theta_\pm(x_\pm)$

$$
\begin{aligned}
[\theta_+(x_+), \theta_+(y_+)] &= N_+ \frac{i^3}{12\pi} \delta'''(x_+ - y_+) \mathbf{1} \\
&\quad + 4i\delta'(x_+ - y_+)\theta_+(y_+) \\
&\quad - 2i\delta(x_+ - y_+)\theta'_+(y_+)
\end{aligned}
$$

and similarly for θ_-

$$\Big[\theta_+(x_+), \theta_-(y_-)\Big] = 0$$

We showed

$$N_+ \geq 1 \qquad \text{if } N_+ \neq 0$$

In case of parity invariance $N_+ = N_- \equiv N$.
Later notation $N_+ = N_- = 2c$ central charge

Change of variables: $x_+ = tg\frac{1}{2}\tau$, $\tau \in (-\pi, +\pi)$

$$T(\tau) \;=\; (\cos\frac{1}{2}\tau)^{-4}\theta_+(x_+(\tau))$$
$$X_k \;=\; -\frac{1}{8}\int_{-\pi}^{\pi} d\tau e^{-ik\tau}T(\tau)$$

satisfy the Virasoro algebra, known from dual resonance models (k integer).

$$[X_l, X_k] = \frac{N}{24}k(k^2-1)\delta_{k+l,0} + (k-l)X_{k+l}$$

$X_k^+ = X_{-k}$, $X_k|0>= 0$ for $k \leq 1$

The allowed values of c were found a decade later

All vacuum expectation values (Wightman)

$$W(x_1, ..., x_n) = < 0|\theta(x_1)...\theta(x_n)|0 >$$

can be constructed by splitting θ into positive and negative frequencies and commuting θ^- to the right until it annihilates the vacuum $|0>$.

Special case $N_+ = 1$ was constructed:

$$\theta_+(x_+) = i : \varphi(x_+) \frac{\partial}{\partial x_+} \varphi(x_+) :$$

$\varphi(x_+)$ anticommuting field obeying CAR

$$\{\varphi(x_+), \varphi(y_+)\} = \delta(x - y)$$

. Derivation: step 1: Show that

$$\theta^{\mu}{}_{\mu}(x) = 0$$

This follows from $\theta^{\mu}{}_{\mu}(x)|0>= 0$ by the Reeh Schlieder Theorem. The latter follows from dilatation symmetry by studying the two point function of $\theta_{\mu\nu}$

It follows that $\theta_{\mu\nu}$ has two independent components $\theta_{\pm}$ which depend on single variables $x_{\pm} = x^0 \pm x^1$

step 2: Let $x = x_+$, $y = y_+$,$\theta = \theta_+$ Study the two-point function of the bilocal operator

$$F(z, y) = [\theta(x), \theta(y)]\,, \qquad z = y - x$$

which is supported at $z = 0$, and its moments

$$O_k(y) = \frac{(-)^k}{k!} \int dz \;\, z^k f(z) F(z, y)$$

where $f(z) \equiv 1$ in a neighborhood of $z = 0$.

Key step: Positivity of

$$< 0|O_k^+(x), O_k(y)]0 >= B_k(x - y - i\epsilon)^{2k-6}$$

implies (according to Gelfand and Shilov vol.1.) that $O_k(y)|0 >= 0$ and therefore $O_k(y) = 0$ for $k > 3$, hence.

$$[\theta(x), \theta(y)] = \sum_{l=0}^{3} \delta^{(l)}(x-y) O_l(y)$$

Exploit antisymmetry $[\theta(x), \theta(y)] = -[\theta(y), \theta(x)]$ and translational invariance to get CR

step 3: Derive inequality $N \geq 1$ on $N \equiv N_+$, assuming $N \neq 0$. : Compute the norm

$$< \Psi|\Psi >= 56N(N-1)(5N+44)$$

of a special vector $|\Psi>$ The vector $|\Psi>$ obeys $X_0|\Psi>. = 9|\Psi>$ and $X_{-1}|\Psi>= 0.$

$$
\begin{aligned}
|\Psi> \;=\; & \{8X_9 + 6X_7X_2 + 12X_6X_3 \\
& -8X_2X_5X_2 + 12X_3X_4X_2 \\
& -5X_3X_3X_3\}|0>
\end{aligned}
$$