Symanzik improvement of the Yang-Mills gradient flow.

A. Ramos[a] <alberto.ramos@desy.de> and S. Sint[a,b] <sint@maths.tcd.ie>

[a] NIC, DESY. [b] Trinity College Dublin

ALPHA Collaboration

Conceptual advances in Lattice Gauge Theories. CERN, 2014.

Renormalized coupling from the Gradient Flow

Add "extra" (flow) time coordinate t ($[t] = -2$). Define gauge field $B_\mu(x,t)$

$$
\begin{aligned}
G_{\mu\nu}(x,t) &= \partial_\mu B_\nu(x,t) - \partial_\nu B_\mu(x,t) + [B_\mu(x,t), B_\nu(x,t)] \\
\frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)
\end{aligned}
$$

with initial condition $B_\mu(x, t=0) = A_\mu(x)$.

Renormalized couplings

- Define (finite quantity for $t > 0$):

$$
\langle E(t) \rangle = -\frac{1}{2} \mathrm{Tr} \langle G_{\mu\nu}(x,t) G_{\mu\nu}(x,t) \rangle
$$

- $t^2 \langle E(t) \rangle$ is dimensionless but depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: $t_0, t_1, w_0, \ldots$
- Renormalized couplings at scale $\mu = \frac{1}{\sqrt{8t}}$

$$
t^2 \langle E(t) \rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu) \left[1 + c_1 g^2_{\overline{MS}}(\mu) + \mathcal{O}(g^4_{\overline{MS}}) \right]
$$

Renormalized coupling from the Gradient Flow

Infinite volume

$$g^2_{\rm GF}(\mu) = \frac{16\pi^2}{3} t^2 \langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

- On the lattice we need a window $a \ll \sqrt{8t} \ll L$.

Running coupling: $\mu = 1/cL$

$$g^2_{\rm GF}(\mu) = \mathcal{N}^{-1} t^2 \langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

- B.C. important ($\frac{16\pi^2}{3} \to \mathcal{N}^{-1}$): Periodic, SF, Twisted (à la t'Hooft), SF-open,...
- Step scaling function

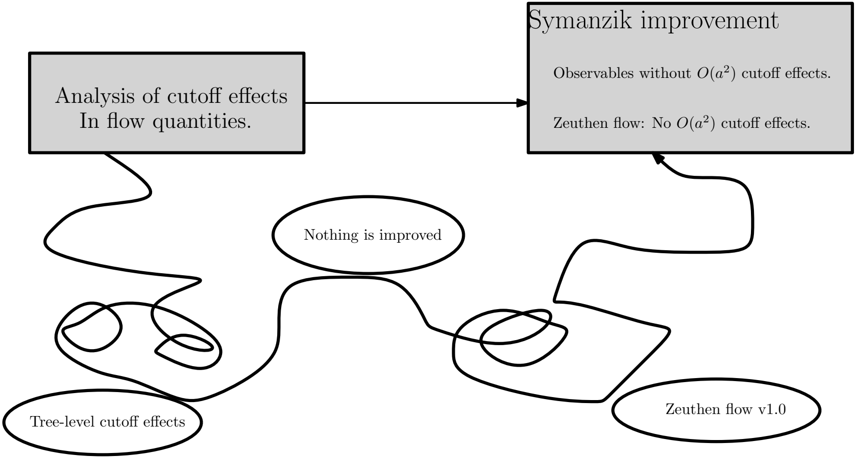
$$\sigma(u,s) = g^2_{\rm GF}(\mu/s)\Big|_{g^2_{\rm GF}(\mu)=u}$$

easily computed on the lattice ($L/a \to sL/a$ at fixed a)

$$\sigma(u,s) = \lim_{a/L\to 0} \Sigma(u,s,a/L)$$

- Continuum extrapolation only systematic.

Outline of the talk



An urban legend

The symmetric (clover) definition of $E(t)$ produce smaller cutoff effects.

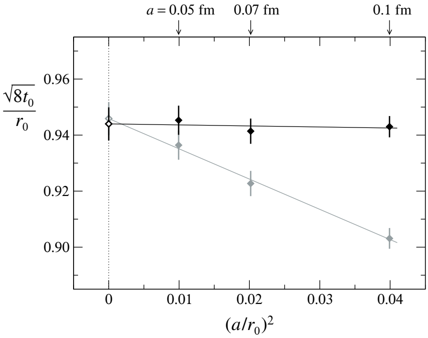


Figure : **[M. Lüscher '10]**

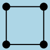
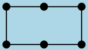
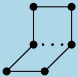
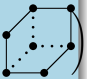
- We all jumped into the clover definition!
- This plot only shows that the Wilson action (pure gauge), with Wilson flow and clover observable produce smaller cutoff effects in $\sqrt{8t_0}/r_0$.
- But different sources of cutoff effects can be responsible of this behavior.
- In fact we think that this is an accidental cancellation.
- Not to be expected in general.

Anatomy of cutoff effects of flow observables

Tree level cutoff effects as a guide

- Compute $t^2\langle E(t)\rangle$ on the lattice to tree level.
- Compare with continuum $\Rightarrow$ cutoff effects in the coupling and $t_0, t_1, \ldots$

Contribution to $\mathcal{O}(a^2)$ cutoff effects

action : $S(c_i^{(a)}) = \frac{1}{g_0^2}\sum_x \mathrm{Tr}\Big(1 - c_0^{(a)}$  $- c_1^{(a)}$  $- c_2^{(a)}$  $- c_3^{(a)}$  $\Big)$

$$\text{flow}: \quad \frac{\mathrm{d}}{\mathrm{d}t}V_\mu(x,t) = -g_0^2\frac{\delta S(c_i^{(f)})}{\delta V_\mu(x,t)}V_\mu(x,t)$$

$$\text{obs}: \quad E(t) = -\frac{1}{2}\mathrm{Tr}\,G_{\mu\nu}(x,t)G_{\mu\nu}(x,t) = S(c^{(o)})$$

- i.e. Wilson action ($c_0^{(a)} = 1, c_1^{(a)} = c_2^{(a)} = c_3^{(a)} = 0$).
- i.e. Symanzik flow ($c_0^{(f)} = 5/3, c_1^{(f)} = -1/12, c_2^{(f)} = c_3^{(f)} = 0$).
- Clover observable. Symanzik observable (use $c_0^{(o)} = 5/3, c_1^{(o)} = -1/12$).

Anatomy of tree-level $\mathcal{O}(a^2)$ cutoff effects

To leading order each choice of action is characterized by a kernel $\hat{K}_{\mu\nu}$

$$S(c_i^{(a,f,o)}) = \frac{1}{2}\int_{-\pi/a}^{\pi/a} dp\, A_\mu(-p) K_{\mu\nu}(p; c_i^{(a,f,o)}, \lambda) A_\nu(p) + \mathcal{O}(g)$$

expanding in powers of a^2

$$K_{\mu\nu}(p; c_i^{(a,f,o)}, \lambda) = K_{\mu\nu}^{(cont)}(p;\lambda) + a^2 R_{\mu\nu}(p; c_i^{(a,f,o)}, \lambda) + \mathcal{O}(a^4)$$

Example: Wilson action

$$K_{\mu\nu}^{(\mathrm{cnt})}(p;\lambda) = p^2\left(\delta_{\mu\nu} - (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right),$$
$$R_{\mu\nu}^{(W)}(p) = -\frac{1}{12}\left[p^4\delta_{\mu\nu} - (1-\lambda)\frac{1}{2}p_\mu p_\nu(p_\mu^2 + p_\nu^2)\right].$$

Master formula

$$t^2\langle E(t)\rangle = g^2\int_{-\pi/a}^{\pi/a} dp\ \mathrm{Tr}\left\{K^{(o)}\exp(-tK^{(f)})(K^{(a)})^{-1}\exp(-tK^{(f)})\right\}$$

Anatomy of tree-level $\mathcal{O}(a^2)$ cutoff effects

Master formula

$$t^2\langle E(t)\rangle = g^2 \int_{-\pi/a}^{\pi/a} dp\ \mathrm{Tr}\left\{K^{(o)} \exp(-tK^{(f)})(K^{(a)})^{-1} \exp(-tK^{(f)})\right\}$$

Example action arbitrary dependent on c_0, c_1

$$t^2\langle E(t)\rangle = \frac{3g^2}{16\pi^2}\Big\{1 \quad + \frac{a^2}{t}\Big[(\frac{1}{4} + 2c_1^{(o)})J_{4,-2} + c_1^{(o)} J_{2,0} - -(\frac{1}{4} + 2c_1^{(a)})J_{4,-2} - c_1^{(a)} J_{2,0} - -(\frac{1}{4} + 2c_1^{(f)})J_{4,0} - c_1^{(f)} J_{2,2}\Big]\Big\}$$

With

$$J_{i,j} = \frac{\int_p e^{-2tp^2} p^i p^j}{\int_p e^{-2tp^2}} \qquad (J_{2,0} = 2, J_{4,-2} = 1, J_{2,2} = 6, J_{4,0} = 3)$$

Conclusions and the clover observable

Important conclusions:

- Observable competes in cutoff effects with the action and the flow.
- Flow produces ~ 3 times more cutoff effects than either the action or the observable.

Observable	Action	Flow	Total
Clover	Wilson	Wilson	
15	-3	-9	3
Clover	Lüscher-Weisz	Symanzik	
15	1	3	19

One can tune one parameter and cancel tree-level cutoff effects (but **no improvement**):

- Wilson action, Wilson flow: Use as observable

$$\frac{1}{4}E^{\mathrm{plaq}}(t) + \frac{3}{4}E^{\mathrm{cl}}(t)$$

- LW action, Wilson flow: Use as observable

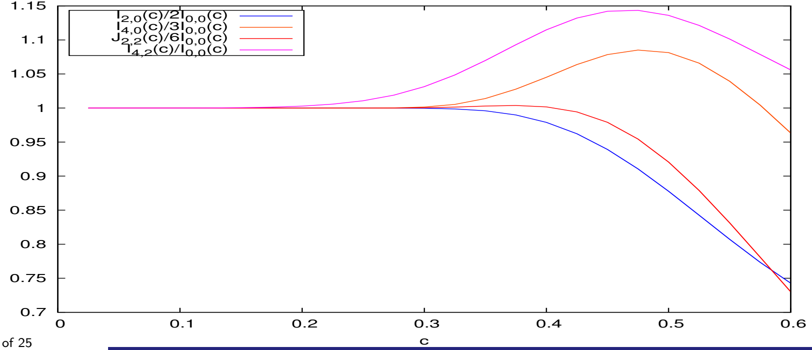
$$\frac{7}{12}E^{\mathrm{plaq}}(t) + \frac{5}{12}E^{\mathrm{cl}}(t)$$

Finite volume analysis

Choose Twisted boundary conditions scheme

- Invariance under translations. Very similar to infinite volume.
- New parameter into the game $c = \sqrt{8t}/L$

$$J_{2,0} = \frac{\int_p e^{-2tp^2} p^2}{\int_p e^{-2tp^2}} \longrightarrow J_{2,0}(c) = \frac{\sum_P e^{-\frac{c^2L^2}{4}P^2} P^2}{\sum_P e^{-\frac{c^2L^2}{4}P^2}}$$



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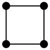
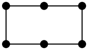
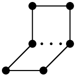
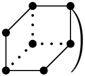
$$t^2\langle E(t)\rangle = \#\Big\{1 \quad + \\ \frac{a^2}{t}\Big[(\frac{1}{4} + 2c_1^{(o)})J_{4,-2}(c) + c_1^{(o)}J_{2,0}(c) - \\ -(\frac{1}{4} + 2c_1^{(a)})J_{4,-2}(c) - c_1^{(a)}J_{2,0}(c) - \\ -(\frac{1}{4} + 2c_1^{(f)})J_{4,0}(c) - c_1^{(f)}J_{2,2}(c)\Big]\Big\}$$

Improvement requires each coefficient to vanish independently!

Finite volume analysis: A more general flow

Try a general action:

$S(c_i) = \frac{1}{g_0^2} \sum_x \mathrm{Tr} \Big(1 - c_0$  $- c_1$  $- c_2$  $- c_3$  $\Big)$

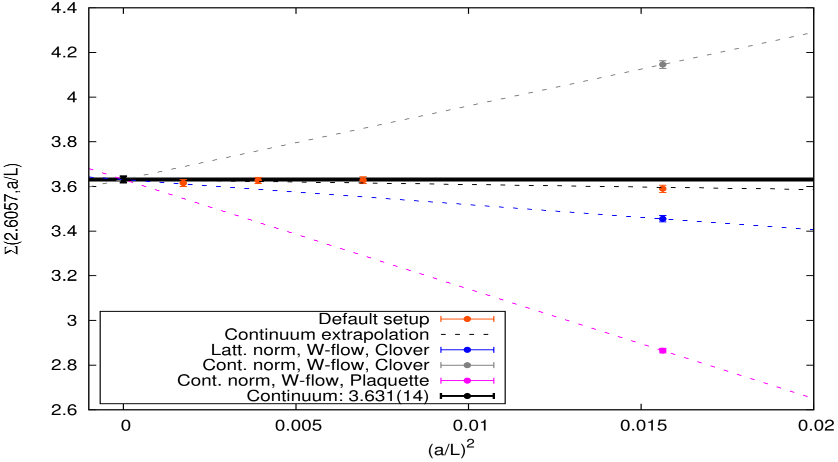
In this generalized case

$$t^2 \langle E(t) \rangle = \# \Big\{ 1 \quad + \frac{a^2}{t} \Big[(\frac{1}{4} + 2c_1^{(o)}) J_{4,-2}(c) + c_1^{(o)} J_{2,0}(c) - -(\frac{1}{4} + 2c_1^{(a)}) J_{4,-2}(c) - c_1^{(a)} J_{2,0}(c) - -(\frac{1}{4} + 2c_1^{(f)} - 2c_2^{(f)} - 2c_3^{(f)}) J_{4,0}(c) - (c_1^{(f)} + 2c_2^{(f)} + 2c_3) J_{2,2}(c) \Big] \Big\}$$

- Zeuthen flow v1.0: $c_1^{(f)} = -1/12, c_2^{(f)} = 1/24, c_3^{(f)} = 0$
- Improved action (i.e. LW), cancels an improved observable

Zeuthen flow v1.0

Test: $SU(3)$ pure gauge step scaling function, with SF boundary conditions. LW action, improved observable.



Zeuthen flow v1.0

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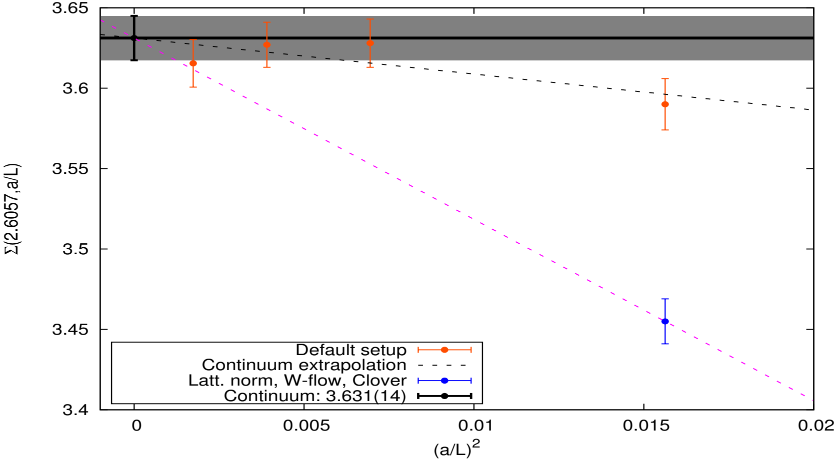
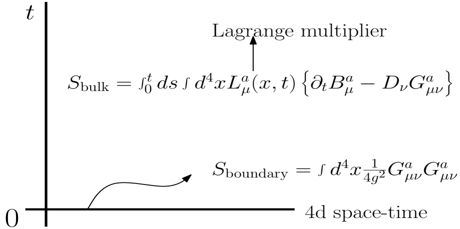


Symanzik improvement program

Look at the flow as a 5d field theory **[M. Lüscher, P. Weisz '11, M. Lüscher '13]**



$$S = S_{\text{bulk}} + S_{\text{boundary}}$$

- Bulk action: No loops → classical improvement is **non-perturbative improvement**
- Boundary action: List all (modulo flow equation) possible counterterms ($J_{\mu\nu\rho} = D_\mu F_{\nu\rho}$)

$$O_1 = \mathrm{Tr}\left\{J_{\mu\nu\rho} J_{\mu\nu\rho}\right\} \qquad O_4 = \mathrm{Tr}\left\{L_\mu(0,x) J_{\nu\mu\nu}\right\}$$
$$O_2 = \mathrm{Tr}\left\{J_{\mu\mu\rho} J_{\nu\nu\rho}\right\} \qquad O_5 = \mathrm{Tr}\left\{L_\mu(0,x) L_\mu(0,x)\right\}$$
$$O_3 = \mathrm{Tr}\left\{J_{\mu\mu\rho} J_{\mu\mu\rho}\right\}$$

Symanzik improvement program

$(J_{\mu\nu\rho} = D_\mu F_{\nu\rho})$

$$O_1 = \mathrm{Tr}\{J_{\mu\nu\rho}J_{\mu\nu\rho}\} \qquad O_4 = \mathrm{Tr}\{L_\mu(0,x)J_{\nu\mu\nu}\}$$
$$O_2 = \mathrm{Tr}\{J_{\mu\mu\rho}J_{\nu\nu\rho}\} \qquad O_5 = \mathrm{Tr}\{L_\mu(0,x)L_\mu(0,x)\}$$
$$O_3 = \mathrm{Tr}\{J_{\mu\mu\rho}J_{\mu\mu\rho}\}$$

Write a lattice action general enough to expand all boundary counterterms

$$S_{\mathrm{boundary}} = S(c_i^{(a)})$$

Other boundary couterterms implemented via a change in the initial condition

$$V_\mu(t,x)\Big|_{t=0} = U_\mu(x)\exp(a^2 c_4(\partial S))$$

Improvement conditions:

- Classical improvement of the flow equation.
- Classical improvement of observables for $t > 0$
- $c_i^{(a)}$ Determined by requiring an observables at $t = 0$ to be improved (i.e. LW action).
- $c_4 = 0$ to tree level.

Classical a-expansion of gradient flow observables

Gradient flow: ultraviolet modes are exponentially damped

$\Rightarrow$ usual quantum field theory complications are eliminated!

$\Rightarrow$ no power divergences, no mixing.

$\Rightarrow$ non-perturbative $O(a^n)$ improvement is simply achieved by classical improvement!

Observables

Study

$$E(t,x) = -\frac{1}{2}\mathrm{tr}\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\},$$

or, separately, colour magnetic and electric components:

$$E_{mag}(t,x) = -\frac{1}{2}\mathrm{tr}\{G_{kl}(x,t)G_{kl}(x,t)\}, \qquad E_{el}(t,x) = -\frac{1}{2}\mathrm{tr}\{G_{0k}(x,t)G_{0k}(x,t)\}.$$

On the lattice these are obtained from small Wilson loops (plaquettes, rectangles,...)

Classical a-expansion of gradient flow observables

Expansion of link variables:

Smooth underlying continuum gauge field $B_\mu(x)$ (t-dependence suppressed), link variables are induced:

$$\begin{aligned} V_\mu(x) &= \mathcal{P}\exp\left\{a\int_0^1 d\lambda B_\mu\left(x+(1-\lambda)a\hat{\mu}\right)\right\} \\ &= \mathbb{1} + a\int_0^1 d\lambda\; B_\mu\left(x+(1-\lambda)a\hat{\mu}\right) \\ &\quad + a^2\int_0^1 d\lambda_1\int_0^{\lambda_1} d\lambda_2\; B_\mu\left(x+(1-\lambda_1)a\hat{\mu}\right) B_\mu\left(x+(1-\lambda_2)a\hat{\mu}\right)+\ldots \\ &= \mathbb{1} + aB_\mu(x) + a^2\frac{1}{2}\left(\partial_\mu B_\mu(x)+B_\mu^2(x)\right) \\ &\quad +\frac{1}{6}a^3\left(\partial_\mu^2 B_\mu(x)+2B_\mu(x)\partial_\mu B_\mu(x)+\left(\partial_\mu B_\mu(x)\right)B_\mu(x)+B_\mu^3(x)\right)+\ldots \end{aligned}$$

Note: gauge transformations in the continuum and on the lattice are compatible:

$$B_\mu(x)\to g(x)B_\mu(x)g(x)^{-1}+g(x)\partial_\mu g(x)^{-1} \quad\Leftrightarrow\quad V_\mu(x)\to g(x)V_\mu(x)g(x+a\hat{\mu})^{-1}$$

Expansion of small Wilson loops (plaquettes, rectangles etc.)

Plaquette fields

Plaquette fields P, Q, R, S appearing in the clover leaf around x:

$$\begin{aligned}
P_{\mu\nu}(x) &= V_\mu(x)V_\nu(x+a\hat{\mu})V_\mu(x+a\hat{\nu})^\dagger V_\nu(x)^\dagger \\
Q_{\mu\nu}(x) &= V_\nu(x-a\hat{\nu})^\dagger V_\mu(x-a\hat{\nu})V_\nu(x+a\hat{\mu}-a\hat{\nu})V_\mu(x)^\dagger \\
R_{\mu\nu}(x) &= V_\mu(x-a\hat{\mu})^\dagger V_\nu(x-a\hat{\mu}-a\hat{\nu})^\dagger V_\mu(x-a\hat{\mu}-a\hat{\nu})V_\nu(x-a\hat{\nu}) \\
S_{\mu\nu}(x) &= V_\nu(x)V_\mu(x+a\hat{\mu}+a\hat{\nu})^\dagger V_\nu(x-a\hat{\mu})^\dagger V_\mu(x-a\hat{\mu})
\end{aligned}$$

Expansion of the plaquette field

Using the gauge fixing tricks by Lüscher & Weisz '85 one obtains relatively quickly:

$$\begin{aligned}
P_{\mu\nu}(x) &= \mathbb{1} + a^2 G_{\mu\nu}(x) + a^3 \frac{1}{2}(D_\mu + D_\nu)G_{\mu\nu}(x) \\
&\quad + a^4 \frac{1}{6}\left\{\left(D_\mu^2 + \frac{3}{2}D_\nu D_\mu + D_\nu^2\right) G_{\mu\nu}(x) + 3G_{\mu\nu}^2(x)\right\} + \mathrm{O}(a^5) \\
D_\mu &= \partial_\mu + [B_\mu, \cdot], \quad [D_\mu, D_\nu] = [G_{\mu\nu}, \cdot]
\end{aligned}$$

similar expansions are obtained for $Q_{\mu\nu}, R_{\mu\nu}, S_{\mu\nu}$; easily generalisable to rectangles.

Improved observables $E(x, t)$

Action densities

- start from the tree-level improved action Lüscher-Weisz action $S_{\mathrm{LW}}(V)$ and define

$$g_0^2 S_{\mathrm{LW}}(V) = a^4 \sum_x E(x, t)$$

- Note: the density is not unique (can be used to eliminate total derivative terms)
- Similar for electric or magnetic components only: restrict to the corresponding parts of the action

alternative: linear combinations

Besides plaquette definition $E(x, t)|_{pl}$ (from Wilson action density), consider the clover definition

$$E(x, t)|_{cl} \quad = \quad -\frac{1}{2}\mathrm{tr}\{G^{cl}_{\mu\nu}(x, t) G^{cl}_{\mu\nu}(x, t)\}$$

From the classical a-expansion find

$$\frac{4}{3}E(x, t)|_{pl} - \frac{1}{3}E(x, t)_{cl}$$

is $O(a^2)$ improved! (also separately for magnetic/electric components)

Translation invariance & total derivative terms

Potential problems:

So far we assumed:

- Observable $E(t,x)$ is only used in 1-point function
- infinite volume or finite volume with (twisted) periodic boundary conditions

$\Rightarrow$ total derivative terms can be neglected!

Open or SF boundary conditions break translation invariance in the Euclidean time direction; spatial directions remain periodic:

- total derivatives contribute extra terms at all orders of a:

$$a(\partial_\mu + \partial_\nu)\mathrm{tr}\{G_{\mu\nu}^2\}, \qquad a^2(\partial_\mu^2, \partial_\nu^2, \partial_\mu\partial_\nu)\mathrm{tr}\{G_{\mu\nu}^2\}, \tag{1}$$

- odd powers of a are total derivative terms only;
- The clover definition eliminates the $\mathrm{O}(a)$, but not the $\mathrm{O}(a^2)$ total derivatives; in particular

$$\frac{4}{3}E_{el}(x,t)|_{pl} - \frac{1}{3}E_{el}(x,t)|_{cl} = -\frac{1}{2}\mathrm{tr}\{G_{0k}^2\} + \frac{1}{4}a^2\partial_0^2\mathrm{tr}\{G_{0k}^2\} + \mathrm{O}(a^3)$$

- We have found $\mathrm{O}(a^2)$ improved combinations involving rectangles, however, cancelling $\mathrm{O}(a^3)$ still requires further efforts.

Proposed solution for running couplings

Use only magnetic components:

For the running coupling in finite volume with open or SF boundary conditions use

$$\langle E_{mag}(x,t)\rangle|_{x_0=T/2,c=\sqrt{8t}/L} = \mathcal{N}(c, a/L, T/L)\bar{g}^2_{GF}(L) \qquad (2)$$

- total derivatives in spatial directions can still be omitted; in particular

$$\frac{4}{3}E_{mag}(x,t)|_{pl} - \frac{1}{3}E_{mag}(x,t)|_{cl} = -\frac{1}{2}\mathrm{tr}\, G^2_{kl}(x,t) + \mathrm{O}(a^4)$$

- Statistical errors do not seem to increase compared to definition including all components; the quenched scaling test was done with magnetic components only!
- The influence of the SF boundary counterterm is reduced significantly!

Expansion of the gradient flow equation

Gradient flow equation:

$$\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} = -\partial_{x,\mu}\left[g_0^2 S_{\mathrm{lat}}(V)\right]$$

Under gauge transformations both sides transform in the adjoint representation, e.g.

$$\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} \to g(x)\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} g(x)^{-1}$$

Straightforward expansion yields

$$\left(a^2\partial_t V_\mu\right) V_\mu^{-1} = a^3\partial_t B_\mu + \frac{1}{2}a^4 D_\mu\partial_t B_\mu + \frac{1}{6}D_\mu^2\partial_t B_\mu + \mathrm{O}(a^6)$$

For a lattice action parameterized by c_1, c_2, the RHS gives:

$$\begin{aligned}
-\partial_{x,\mu}\left[g_0^2 S_{\mathrm{lat}}(V)\right] \quad = \quad & \sum_{\nu=0}^{3}\Bigg\{ a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu} \\
& + \frac{1}{12}a^5\Bigg[\left(1+12(c_1-c_2)\right)\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu \\
& + 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Bigg] G_{\nu\mu}\Bigg\} + \mathrm{O}(a^6)
\end{aligned}$$

Expansion of the gradient flow equation

Observations

- When iteratively solving this equation the O(a) term always cancels

$$\partial_t B_\mu(x,t) = \sum_{\nu=0}^{3} D_\nu G_{\nu\mu}(x,t) + \mathrm{O}(a^2)$$

  However, odd powers of a do not cancel in general!

- No choice of c_1, c_2 cancels the O(a^2) terms!
- However, the Symanzik/LW flow ($c_1 = -1/12$, $c_2 = 0$), is "almost" O(a^2) improved

$$\partial_t B_\mu = \sum_{\nu=0}^{3} \left\{ D_\nu G_{\nu\mu}(x,t) - \frac{1}{12} a^2 D_\mu^2 D_\nu G_{\nu\mu} + \mathrm{O}(a^3) \right\}$$

- This suggests to define an O(a^2) improved flow equation by setting

$$\begin{aligned} \left(a^2 \partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} &= -\left(1 + \frac{1}{12} a^2 \nabla_\mu^* \nabla_\mu\right) \partial_{x,\mu} \left[g_0^2 S_{\mathrm{LW}}(V)\right] \\ a\nabla_\mu F(x) &= V_\mu(x,t) F(x+a\hat{\mu}) V_\mu(x,t)^\dagger - F(x), \ldots \end{aligned}$$

Expansion of the gradient flow equation

Odd powers of a

The expansion of the LHS of the flow equation can be obtained to all orders:

$$\frac{1}{a}\left(\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} = \partial_t B_\mu(x,t) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}(aD_\mu)^n \partial_t B_\mu(x,t)$$

- There is an asymmetry in the treatment of the end points of the link, $x + a\hat{\mu}$ and x!

$\Rightarrow$ expand covariantly about the midpoint $\tilde{x} = x + \frac{1}{2}a\hat{\mu}$ of the link:

$$\frac{1}{a}\Omega_\mu\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1}\Omega_\mu^\dagger = \partial_t B_\mu(\tilde{x},t) + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left(\frac{a}{2}D_\mu\right)^{2n} \partial_t B_\mu(\tilde{x},t)$$

Here, $\Omega_\mu(x,t)$ is the "half link variable" from $\tilde{x}$ to x

$\Rightarrow$ all odd powers of a cancel!

- Expect the same to happen on the RHS, so odd powers are completely eliminated!

$\Rightarrow$ However, the midpoint $\tilde{x}$ is not a point on the lattice, have to live with the asymmetry!(?)

Expansion of the gradient flow equation

Absence of O(a^3) in the improved flow equation

Pushing the expansion of the improved flow equation (around x) to the next order it turns out that the O(a^3) correction is given by:

$$\frac{1}{24} a^3 D_\mu^3 \left(\sum_{\nu=0}^{3} D_\nu G_{\nu\mu} - \partial_t B_\mu \right)$$

so that, after recursive use of the flow equation one gets:

$$\partial_t B_\mu(x,t) = \sum_{\nu=0}^{3} D_\nu G_{\nu\mu}(x,t) + \mathrm{O}(a^4)$$

Some remarks

- Note: expect the explict derivative to introduce odd powers of a, starting at O(a^5).
- In infinite volume or with (twisted) periodic b.c.'s this solution is ready to be used!
- With open or SF b.c.'s need to discuss how to treat the derivative at the physical boundaries; from Martin's orbifold construction (Lüscher '14) expect e.g. that the force term for the SF satisfies Neumann conditions.

Conclusions and Outlook

We have applied the Symanzik procedure to the pure gauge theory with the gradient flow.

- The classical nature of the gradient flow equation allows to completely eliminate $\mathrm{O}(a^2)$ effects originating from the observables at $t > 0$ and from the gradient flow equation
- Remaining cutoff effects originate from the 4D action and a couple of 4D boundary counterterms
- Some technical details at open or SF boundaries still need to be worked out.
- Need to repeat the quenched scaling test using the improved flow equation.
- The generalization to include the fermion flow should be straightforward.

A personal note to Martin Lüscher by Stefan Sint

As your former Ph.D. student, I would like to express my gratitude

- for accepting me as a student, despite the fact that this created the highly anomalous situation of having simultaneously two Ph.D. students. It is hard to imagine better guidance for a student!
- for your generous advice and many discussions over the years, and for leading by example.
- for taking the long term view. In an increasingly breathless world I often find it helpful to ask myself "What would Martin say?"
- for your scientific contributions which set a very high standard. I am sure that many will pass the test of time and be standard references for decades to come!

Thank you, Martin!

Wishes for the future:

- Many more healthy and productive years!
- Enjoy your swiss-made new house and the alps!
- Visit us occasionally!