Finite-volume methods for hadrons and their interactions in lattice QCD

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21 July - 1 August, 2014, CERN

@Martin’s Fest, 31 July, 2014

Martin Lüscher

A leader/guardian, who keeps high standards and morals of our lattice community.

Chris Michael:

" The lattice and QCD community in Europe* has benefitted tremendously from Martin Luescher's contributions. He has combined rigor with practicality in a way that underpinned the work of many groups (myself included).

The list of his innovations is long and varied -- for me: the understanding and exploitation of finite size effects and the importance of many -scale studies are my top two."

Europe* -> the World (SA)

I regard Martin as my unofficial supervisor (without his permission): whenever I write a paper, I ask myself "how does Martin judge the quality of this work ?" .

I regret very much a fact that I have never collaborated with him. So I know Martin mainly from his papers/talks, and discussions/conversations during conferences.

When I read his papers, I always feel that what I think as a goal of the research is what Martin regards as merely a start.

I have never thought him as a competitor in research. He is always running too far ahead of me.

Finite-Volume methods by Martin (selected papers)

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States
M. Luscher (DESY)
Commun.Math.Phys. 104 (1986) 177
Cited by 503 records

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States
M. Luscher (DESY)
Commun.Math.Phys. 105 (1986) 153-188
Cited by 588 records

How to Calculate the Elastic Scattering Matrix in Two-dimensional Quantum Field Theories by Numerical Simulation
Martin Luscher (DESY) , Ulli Wolff (Kiel U. & Bielefeld U.)
Nucl.Phys. B339 (1990) 222-252
Cited by 519 records

Two particle states on a torus and their relation to the scattering matrix
Martin Luscher (DESY)
Nucl.Phys. B354 (1991) 531-578
Cited by 531 records

Signatures of unstable particles in finite volume
Martin Luscher (DESY)
Nucl.Phys. B364 (1991) 237-254
Cited by 234 records

Weak transition matrix elements from finite volume correlation functions
Laurent Lellouch (Annecy, LAPTH) , Martin Luscher (CERN)
Commun.Math.Phys. 219 (2001) 31-44
Cited by 210 records

1. Finite-volume effects to masses of stable particles
2. Finite-volume method to scattering lengths
3. Finite-volume method to scattering phase shifts
4. Finite-volume method to weak matrix elements
5. From Finite-volume method to “Potential”
6. Conclusion

1. Finite-volume effects to masses of stable particles

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States

M. Luscher (DESY)

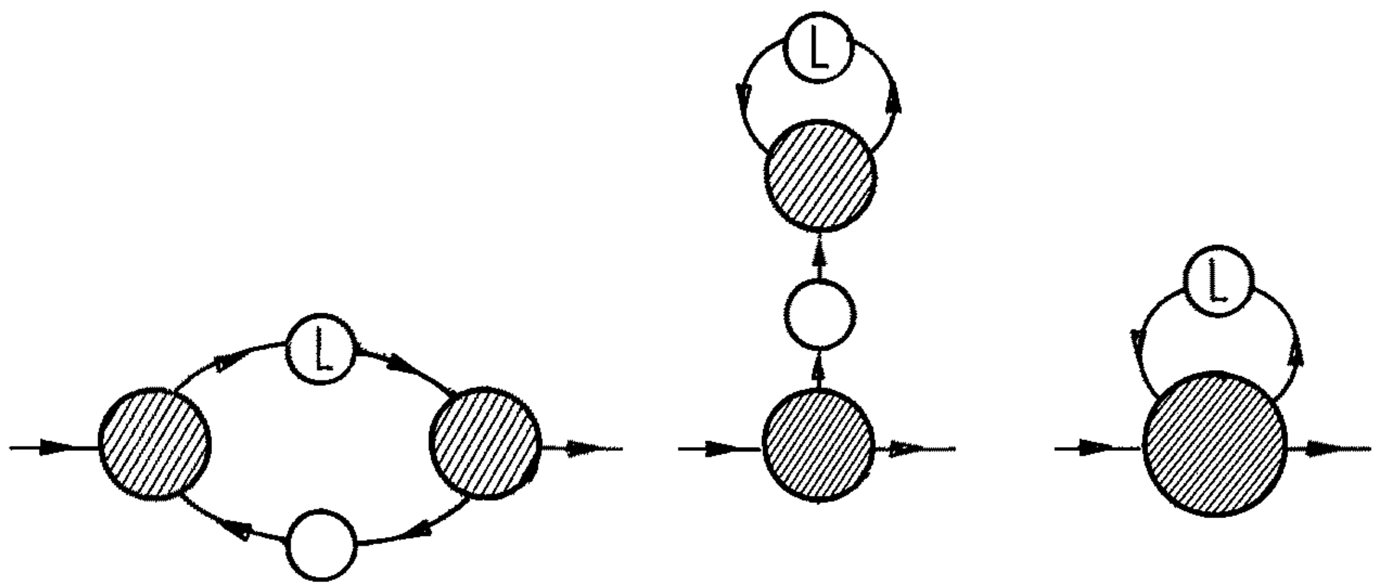
Commun.Math.Phys. 104 (1986) 177

Stable particle

$$M(L) - m = \Delta m = -\frac{3}{16\pi m^2 L}\left\{\lambda^2 e^{-\frac{\sqrt{3}}{2}mL} + \frac{m}{\pi}\int_{-\infty}^{\infty} dy\, e^{-\sqrt{m^2+y^2}L} F(iy) + O(e^{-\bar{m}L})\right\}$$

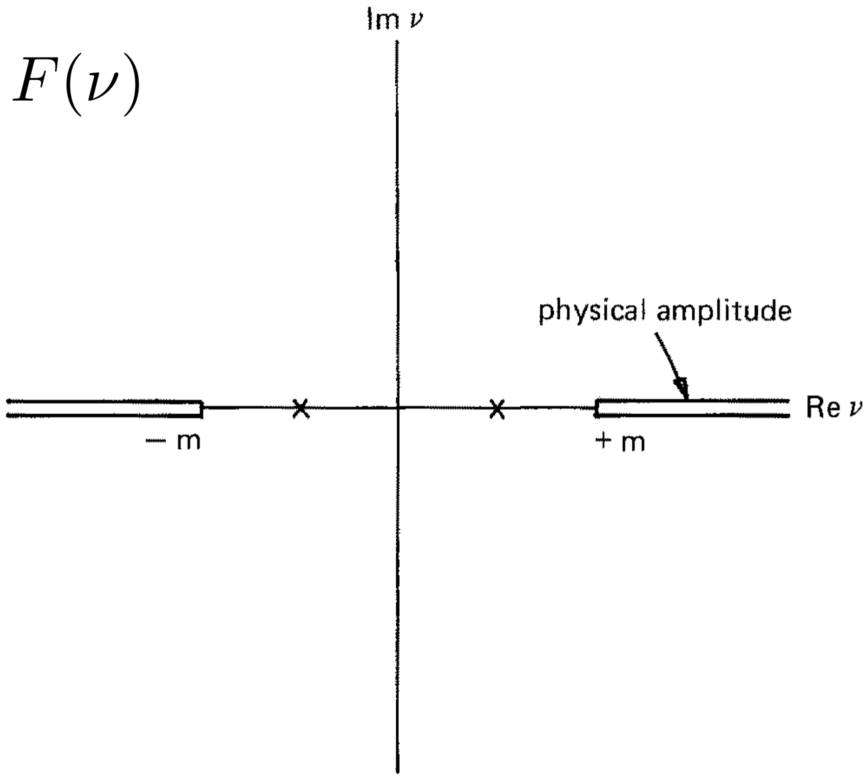
3-pt coupling   forward scattering amplitude

$$F(\nu) = T(\boldsymbol{p},\boldsymbol{q}|\boldsymbol{p},\boldsymbol{q}), \quad \nu = (p_0 q_0 - \boldsymbol{p}\cdot\boldsymbol{q})/m$$



(a) (b) (c)

$F(\nu)$

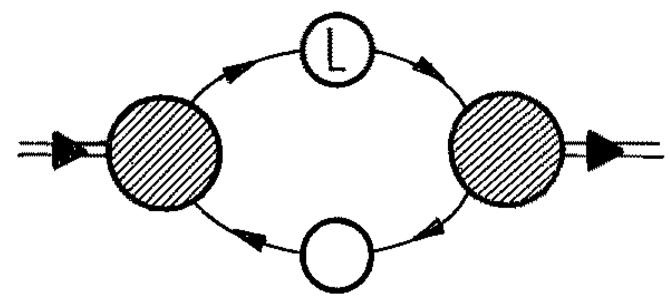


bound state

$$M_B(L) - m_B = \Delta m_B = -\frac{3g^2}{16\pi m_B^2 L} e^{-\mu L} + O(e^{-\bar{m}_B L})$$

$$\bar{m}_B \geqq \sqrt{2}\mu\,,$$
$$\mu = \sqrt{m^2 - \tfrac{1}{4}m_B^2}\,.$$

mixed 3-pt coupling



(c)

interactions $\Rightarrow$ estimate of finite size effect to masses

I were satisfied with this, but Martin not ...

interactions $\Leftarrow$ finite size effect

2. Finite-volume method to scattering lengths

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States

M. Luscher (DESY)

Commun.Math.Phys. 105 (1986) 153-188

lowest energy of two particles in a finte box → **S-wave scattering length**

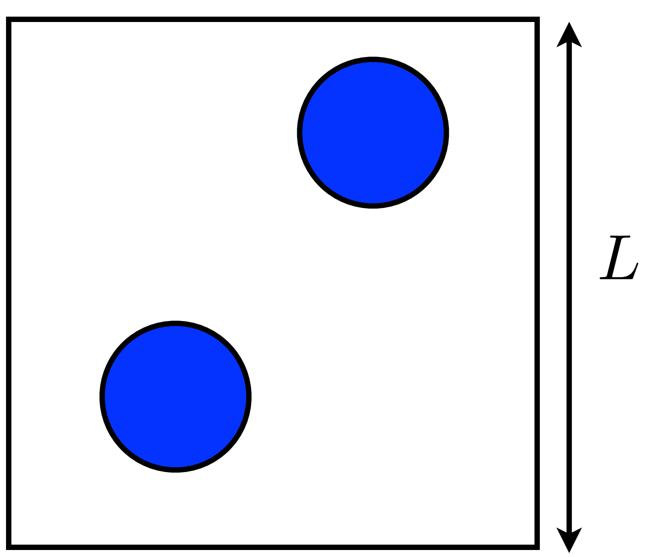
$$W = 2m - \frac{4\pi a_0}{mL^3}\left\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right\} + O(L^{-6})$$

$$c_1 = -2.837297\,,$$

$$c_2 = 6.375183\,,$$

$$a_0 = \lim_{p\to 0}\frac{1}{2ip}\left(e^{2i\delta_0} - 1\right)$$

special case: Huang-Yang, PR 105(1957) 767.



This is a great result, but Martin did not stop thinking.

3. Finite-volume method to scattering phase shifts

Two particle states on a torus and their relation to the scattering matrix
Martin Luscher (DESY)
Nucl.Phys. B354 (1991) 531-578

two particle energy in a finite box (center of mass system below inelastic threshold)

$$W = 2\sqrt{m^2 + \boldsymbol{k}^2} < 4m \qquad \boldsymbol{k} \neq \frac{2\pi}{L}\boldsymbol{n} \;\; (\boldsymbol{n} \in Z^3)$$

due to the interaction between two particles

scattering phase shifts

Ex: S-wave phase shift $\delta_0(k)$

$$k \cot \delta_0(k) = \frac{2}{\sqrt{\pi} L} Z_{00}(1; q^2) \qquad k = |\boldsymbol{k}| \qquad q = \frac{kL}{2\pi}$$

generalized zeta-function

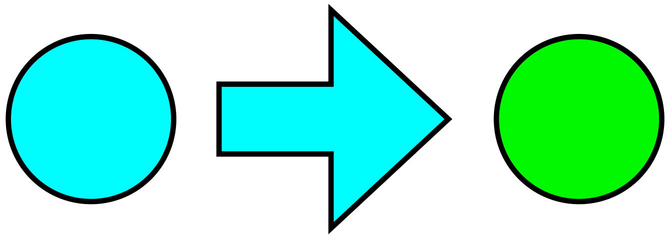
$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in Z^3} (\mathbf{n}^2 - q^2)^{-s}$$

Martin has finally obtained the satisfactory answer.

Some generalizations

1. extension to the moving frame

Rummukainen-Gotlieb, NPB450(1995)397.



2. extension to multiple two-body channel

Liu-Feng-He, IJMP A21(2006)847.

Lage-Meissner-Rusetsky, PLB681(2009)439.

Bernard-Lage-Meissner-Rusetsky, JHEP1101(2011)019.

Doring-Meissner-Oset-Rusetsky, EPJ A47(2011)139.

Hansen-Sharpe, PRD86(2012)016007.

Briceno-Davoudi, PRD88(2013) 094507.

Briceno, PRD89(2014) 074507.

$$A + B \to A + B$$

$$A + B \to C + D$$

3. extension to three-body channel

Kreuzer-Hammer, PLB694(2011)424.

Polejaeva-Rusetsky, EPJ A48(2012)67

Kreuzer-Griesshammer, EPJ A48(2012)93.

Briceno-Davoudi, PRD87(2013)094507.

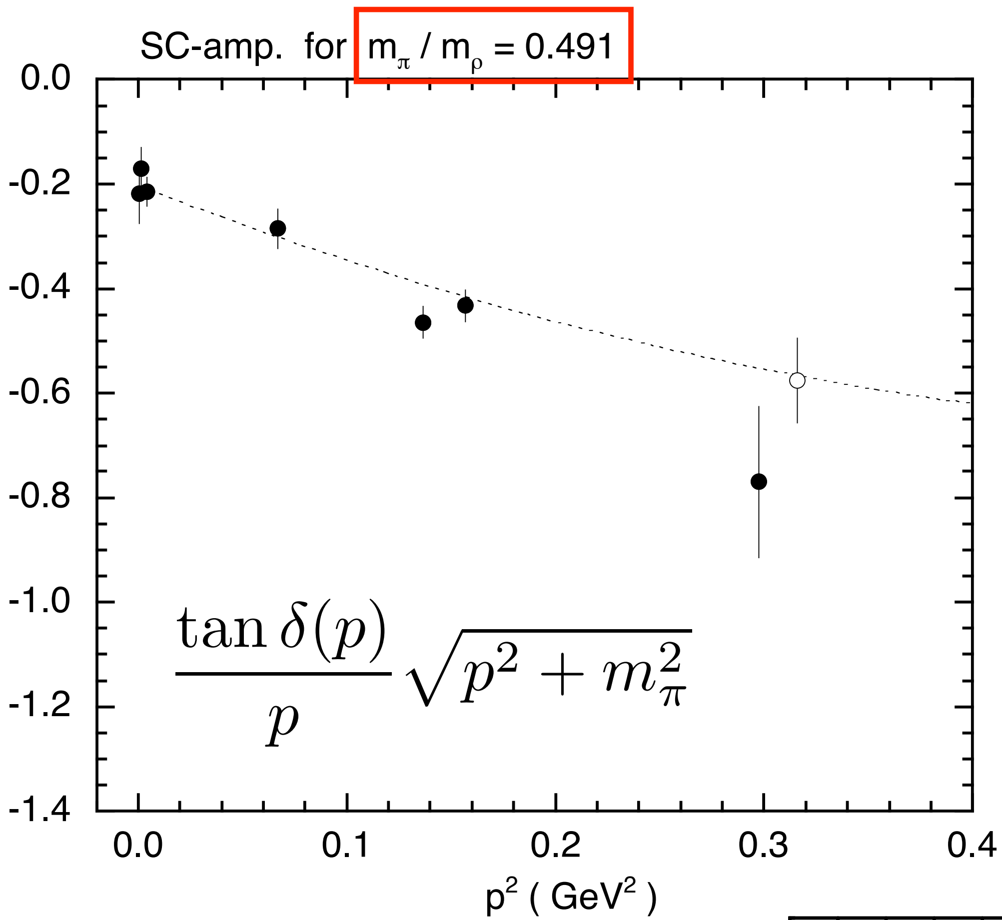
Hansen-Sharpe, arXive:1311.4848[hep-lat].

$$NNN \to NNN$$

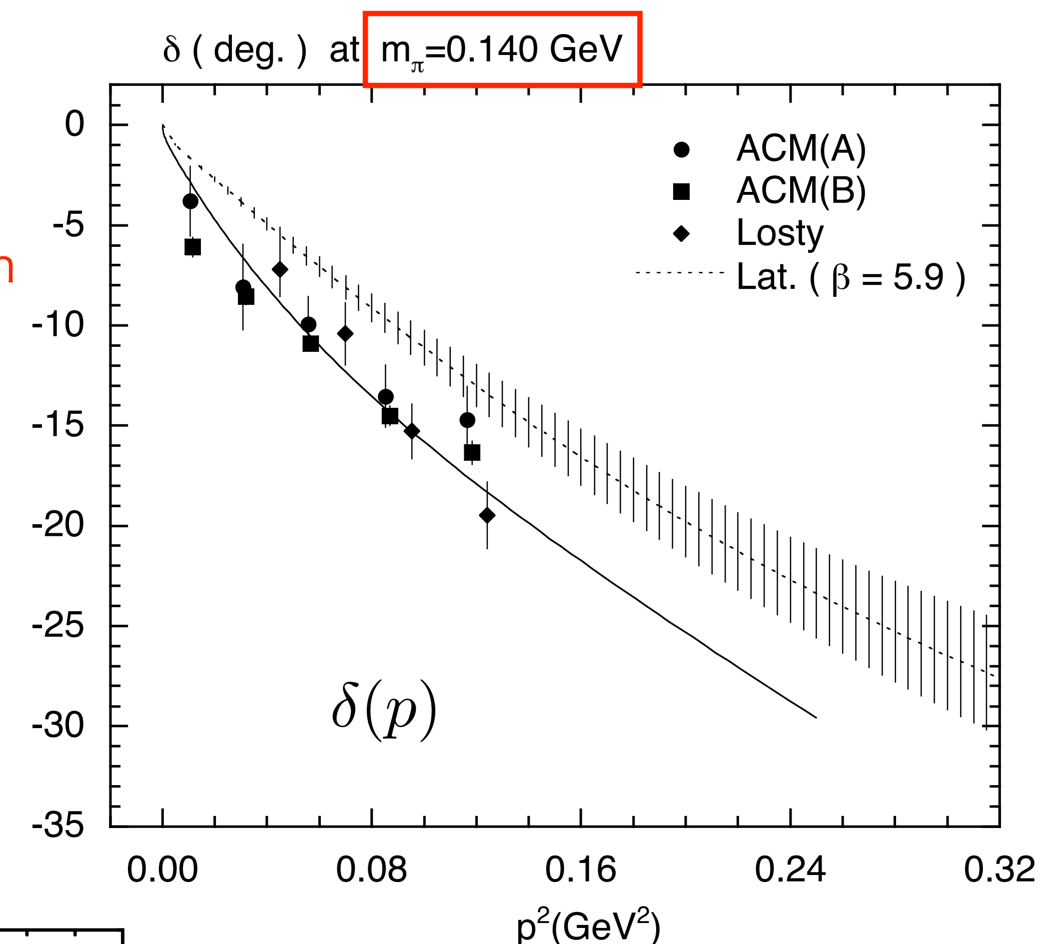
Some numerical results

$I = 2\ \pi\pi$ scattering phase shift

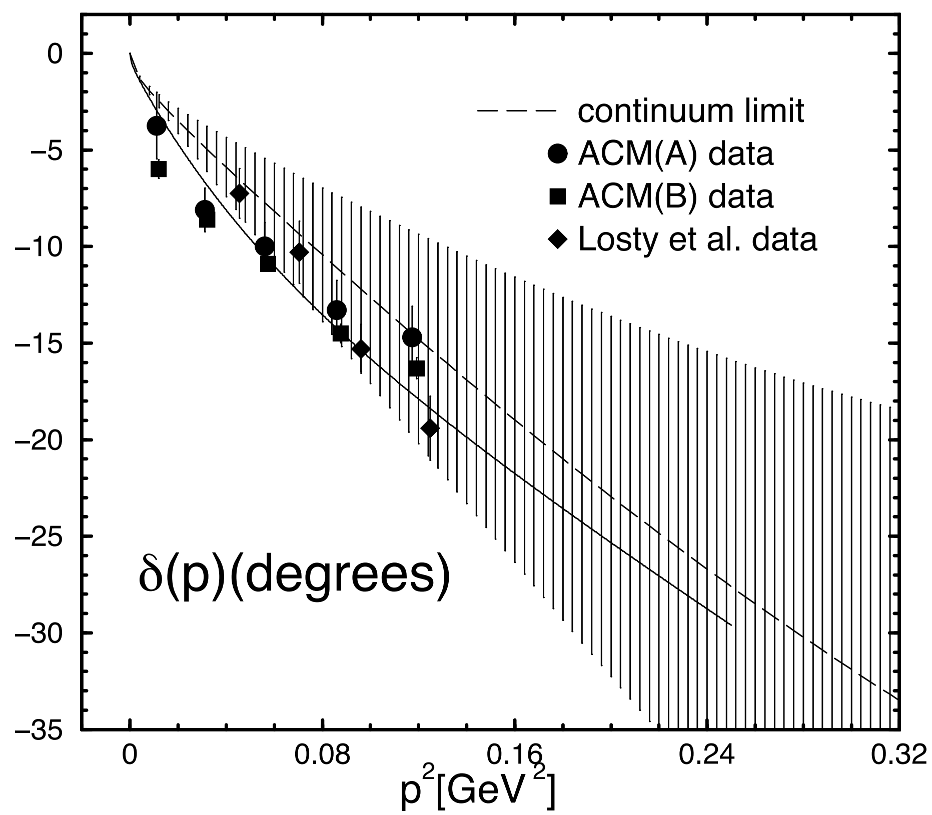
CP-PACS Col. , PRD67(2003)014502



chiral extrapolation



continuum extrapolation

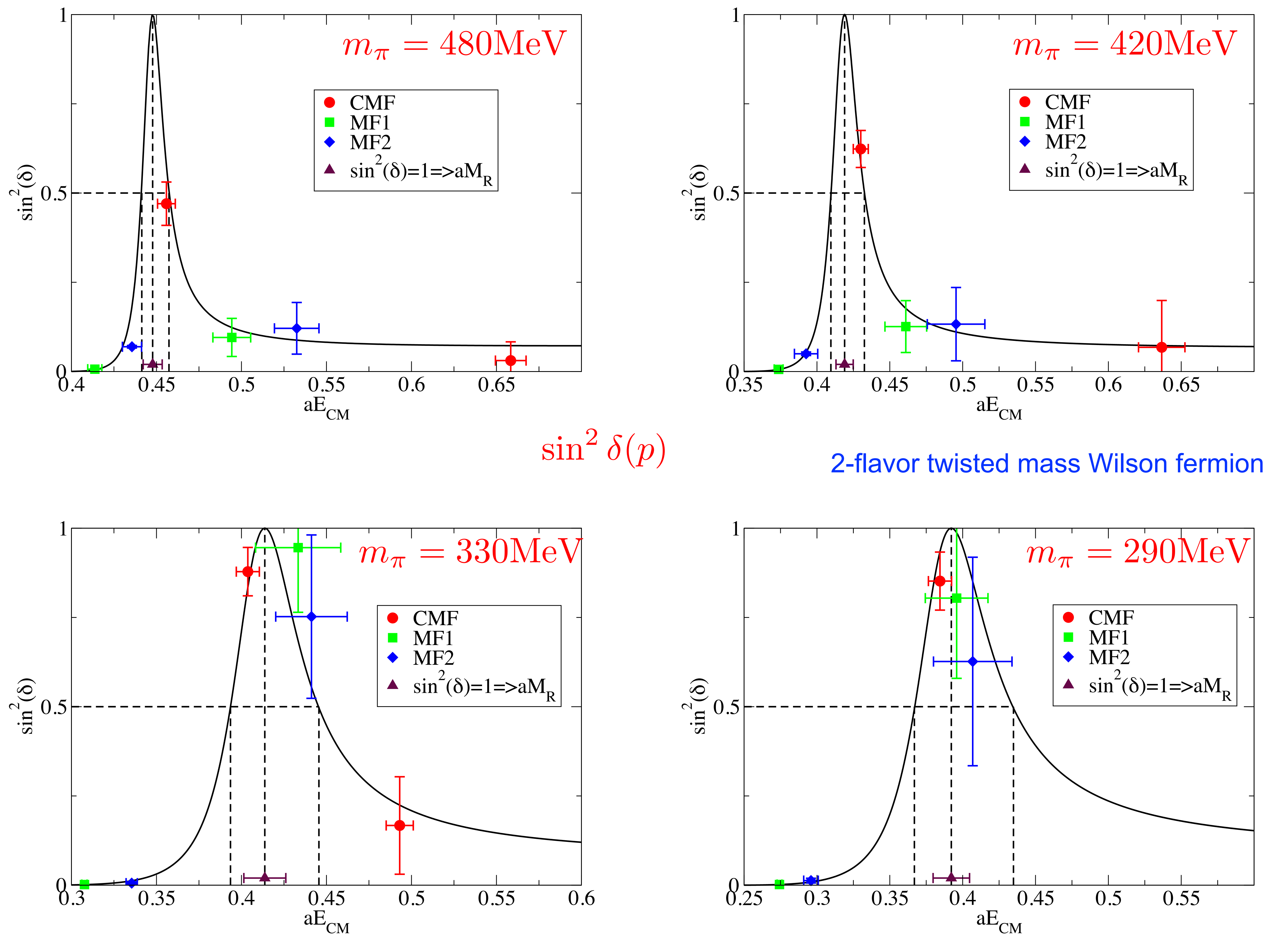


2-flavor Wilson fermions

Yamazaki *et al.* (CP-PACS Col.), PRD70(2004)074513

$I = 1\ \pi\pi$ scattering (ρ resonance) P-wave

Feng-Jansen-Renner, PRD83(2011)094505

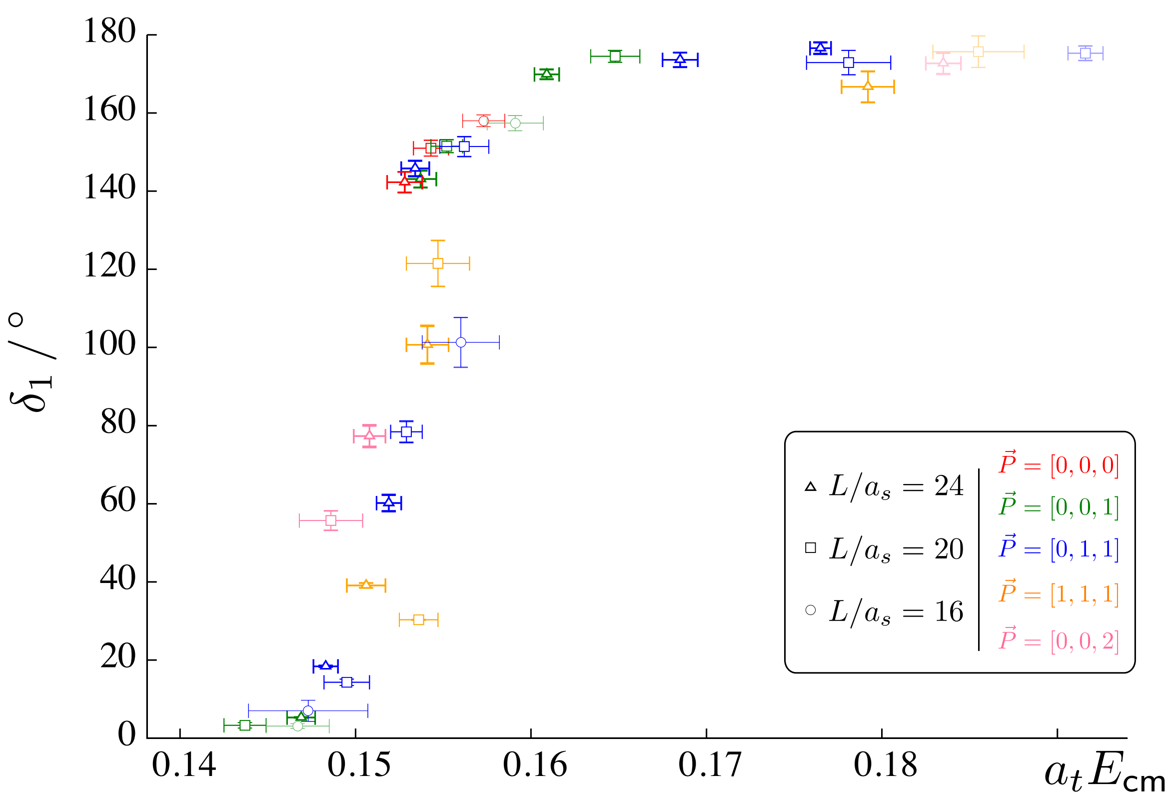


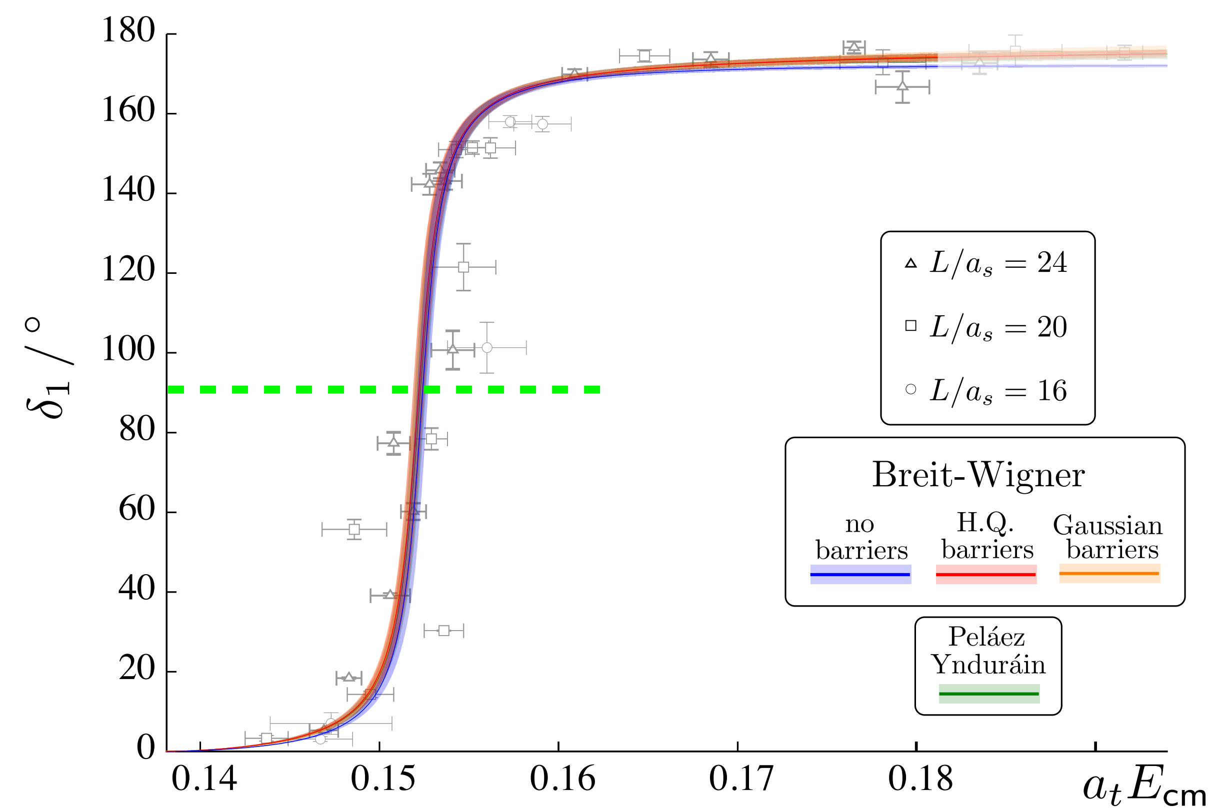
$$\tan \delta_1 = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{p^3}{E_{CM}(m_\rho^2 - E_{CM}^2)}\,, \quad p = \sqrt{E_{CM}^2/4 - m_\pi^2}\,,$$

$I = 1\ \pi\pi$ scattering (ρ resonance)

Dudek-Edwards-Thomas, PRD87(2013)034505

$\delta_1(E_{\rm cm})$





2-flavor anisotropic clover fermion

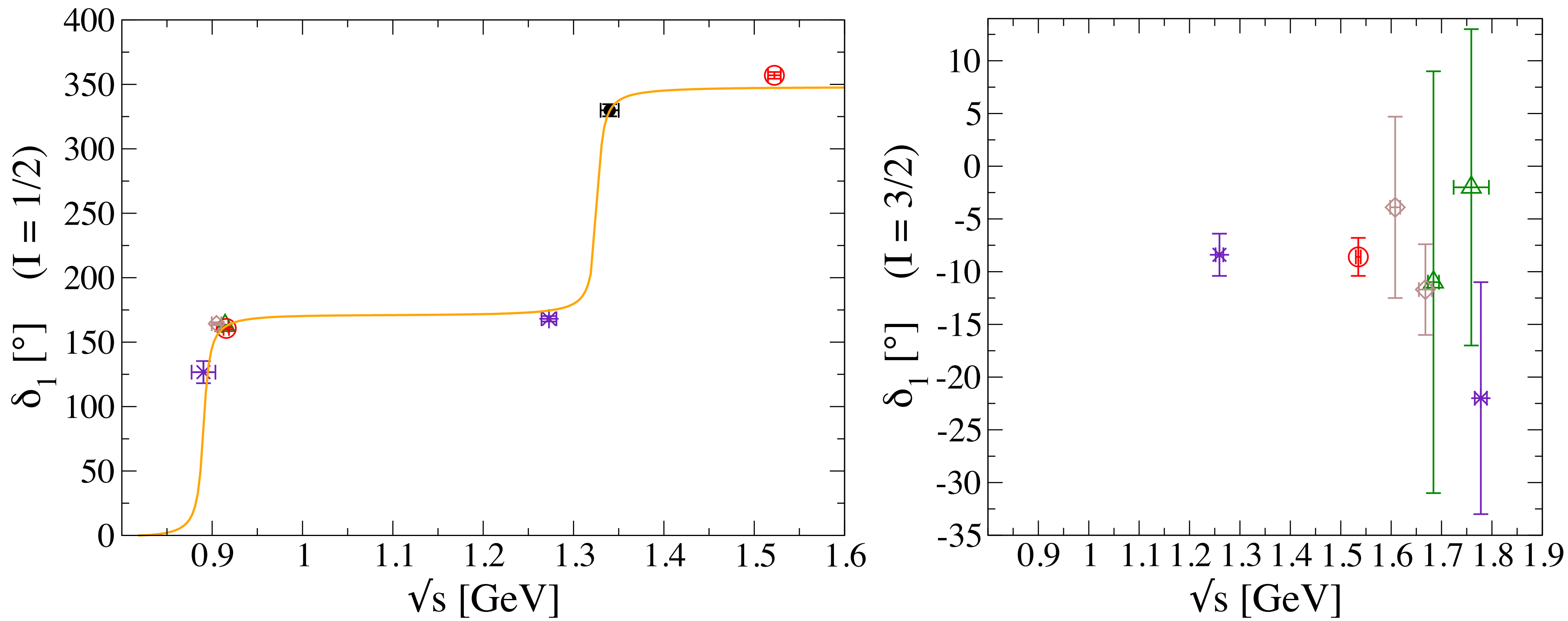
$a_s \sim 0.12\,\mathrm{fm}$

$m_\pi \sim 400\,\mathrm{MeV}$

Other resonances

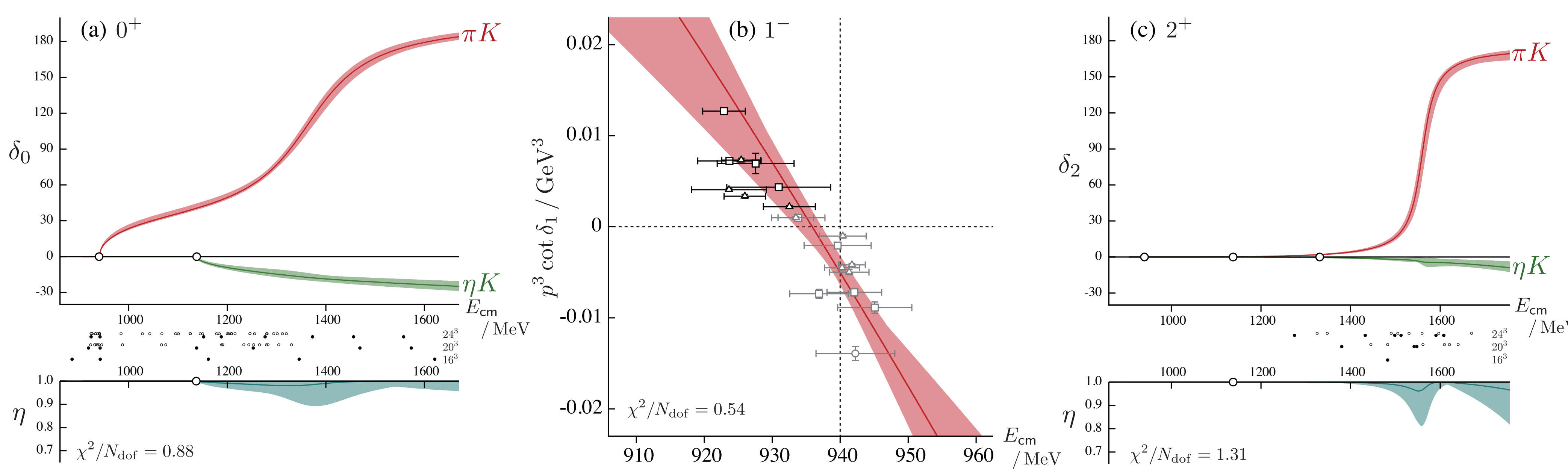
$K^*(\pi K, \text{P-wave})$

Prelovsek-Leskovec-Lang-Mohler, PRD88(2013)054508



$\pi K, \eta K$

Dudek-Edwards-Thomas-Wilson, arXiv:1406.4158



4. Finite-volume method to weak matrix elements

Weak transition matrix elements from finite volume correlation functions
Laurent Lellouch (Annecy, LAPTH) , Martin Luscher (CERN)
Commun.Math.Phys. 219 (2001) 31-44

$K \to \pi\pi$ decay amplitude

$$T(K \to \pi\pi) = \langle \pi\ p_1, \pi\ p_2\ \mathrm{out}|\mathcal{L}_{\mathrm{w}}(0)|K\ p\rangle = A e^{i\delta_0}$$

decay rate in infinite volume

$$\Gamma = \frac{k_\pi}{16\pi m_K^2}|A|^2, \qquad k_\pi \equiv \frac{1}{2}\sqrt{m_K^2 - 4m_\pi^2}$$

infinite volume

$$|A|^2 = 8\pi \left\{ q\frac{\partial\phi}{\partial q} + k\frac{\partial\delta_0}{\partial k} \right\}_{k=k_\pi} \left(\frac{m_K}{k_\pi}\right)^3 |M|^2$$

finite volume

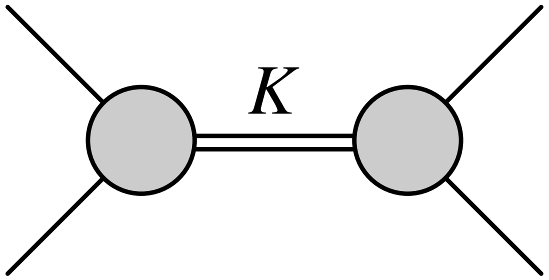
finite volume matrix element

$$M = \langle \pi\pi|H_{\mathrm{w}}|K\rangle,$$

$$W = 2\sqrt{m_\pi^2 + k^2},$$

$$n\pi - \delta_0(k) = \phi(q), \qquad q \equiv \frac{kL}{2\pi},$$

$\pi\pi$ scattering phase shift in finite vplume



Some results

$K \to (\pi\pi)_{I=2}$ decay amplitude

T. Blum *et al.*, PRL108(2012)141061
T. B.um *et al.*, PRD86(2012)074513

Lattice

$$\boxed{\begin{aligned} \mathrm{Re}A_2 &= 1.381(46)_{\mathrm{stat}}(258)_{\mathrm{syst}}10^{-8}\ \mathrm{GeV}, \\ \mathrm{Im}A_2 &= -6.54(46)_{\mathrm{stat}}(120)_{\mathrm{syst}}10^{-13}\ \mathrm{GeV}. \end{aligned}}$$

$a^{-1} = 1.364$ GeV, $m_\pi = 142$ MeV, $m_K = 506$ MeV

$W_{2\pi} = 486$ MeV

LL formula

$$\mathcal{A}_i = \left[\frac{\sqrt{2^{n_{\mathrm{tw}}}}}{2\pi q_\pi}\sqrt{\frac{\partial\phi}{\partial q_\pi} + \frac{\partial\delta}{\partial q_\pi}}\right]\frac{2}{\sqrt{2^{n_{\mathrm{tw}}}}}L^{3/2}\sqrt{m_K}E_{\pi\pi}\mathcal{M}_i,$$

Experiment

$$\mathrm{Re}A_2 = 1.479(4) \times 10^{-8}\mathrm{GeV}$$

K^+ decays

$\Delta I = 1/2$ rule

P. Boyle *et al.*, PRL110(2013)152001

Lattice

$$\boxed{\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = \begin{cases} 9.1(2.1) & \text{for } m_K = 878\ \mathrm{MeV}, \quad m_\pi = 422\ \mathrm{MeV} \\ 12.0(1.7) & \text{for } m_K = 662\ \mathrm{MeV}, \quad m_\pi = 329\ \mathrm{MeV}. \end{cases}}$$

$a^{-1} = 1.73$ GeV

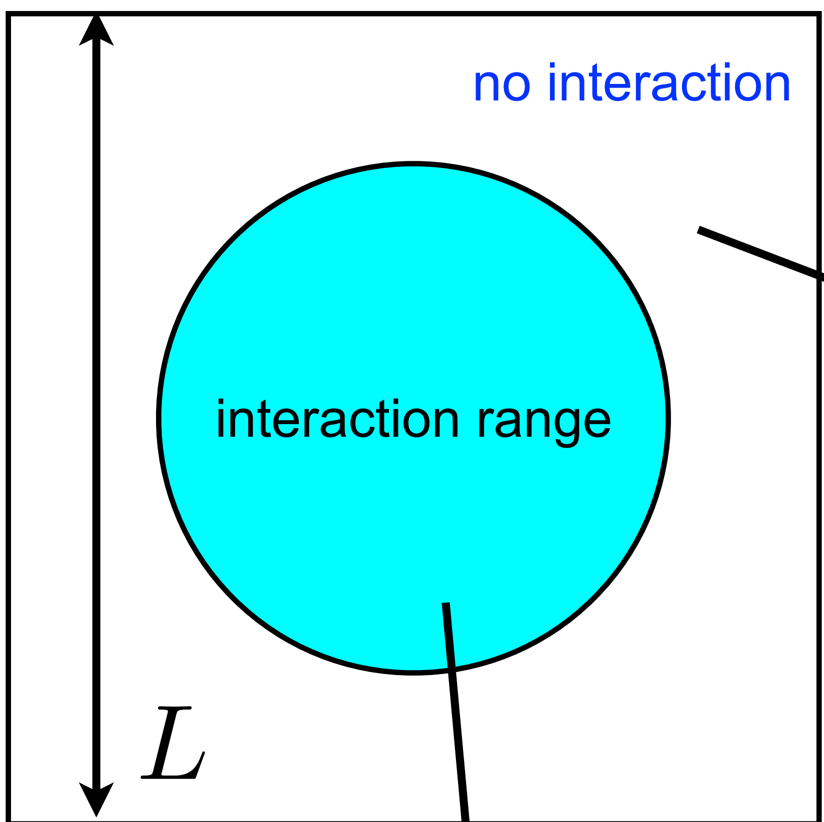
Experiment

$$\frac{\mathrm{Re}\ A_0}{\mathrm{Re}\ A_2} = 22.45(6)$$

5. From Finite-volume method to “Potential”

CP-PACS Collaboration (Aoki *et al.*) PRD71(2005)094504 Ishizuka

2-particles in finite box



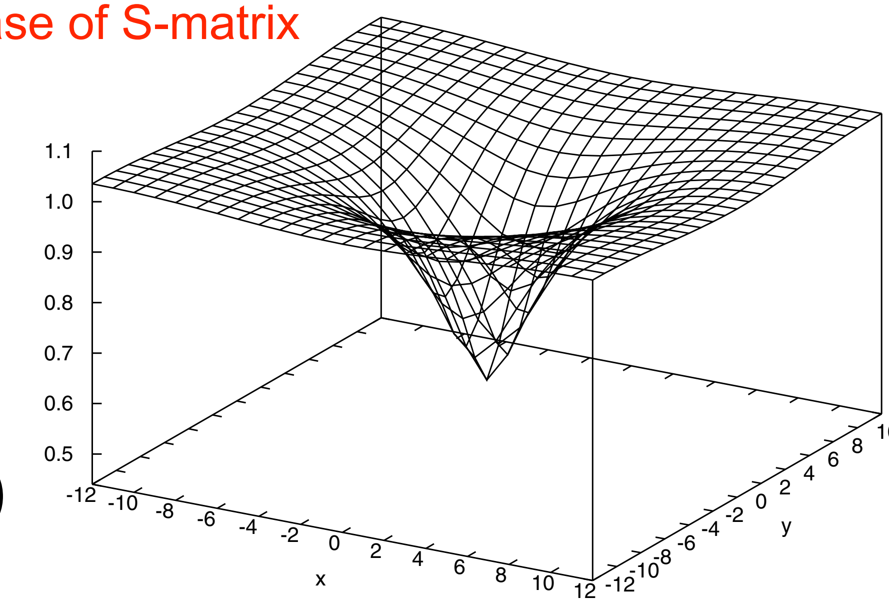
$I = 2$ $\pi\pi$ system

$$\varphi(\boldsymbol{r};k) = \sum_{\boldsymbol{x}} \langle 0|\pi(\boldsymbol{r}+\boldsymbol{x},0)\pi(\boldsymbol{x},0)|\pi\pi,k\rangle$$

wave function

$$\simeq \frac{e^{i\delta_l(k)}}{kr}\sin(kr - l\pi/2 + \delta_l(k))$$

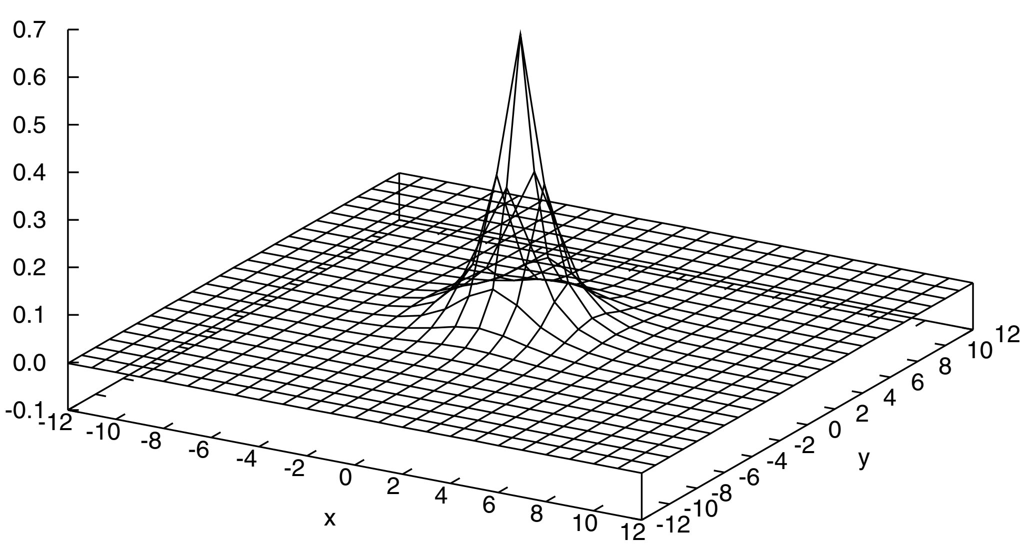
phase of S-matrix



$$\boldsymbol{k} \neq \frac{2\pi}{L}\boldsymbol{n}$$

Martin’s formula

$$k\cot\delta_0(k) = \frac{2}{\sqrt{\pi}L} Z_{00}(1;q^2)$$



interaction kernel

$$V(\boldsymbol{r};k) = \frac{\boldsymbol{\nabla}^2\varphi(\boldsymbol{r};k)}{\varphi(\boldsymbol{r};k)}$$

“Potential” ?

Without Martin's rigor and stoicism,

Ishii-Aoki-Hatsuda, PRL90(2007)0022001

we rashly defined a NN "potential" as

Nambu-Bethe-Salpeter (NBS) Wave function

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0 | N(\mathbf{x}+\mathbf{r},0) N(\mathbf{x},0) | NN, W_k \rangle$$

$$\boxed{[\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, \underline{U(\mathbf{x},\mathbf{y})} \varphi_{\mathbf{k}}(\mathbf{y})}$$

non-local "potential"

$$\epsilon_k = \frac{\mathbf{k}^2}{2\mu}$$ kinetic energy

$$H_0 = \frac{-\nabla^2}{2\mu}$$ free Hamiltonian

$$\mu = m_N/2$$ reduced mass

$$U(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\nabla)\delta^3(\mathbf{x}-\mathbf{y})$$ derivative expansion

$$V(\mathbf{x},\nabla) = V_0(r) + V_\sigma(r)(\sigma_{\mathbf{1}}\cdot\sigma_{\mathbf{2}}) + V_T(r)S_{12} + V_{\rm LS}(r)\mathbf{L}\cdot\mathbf{S} + O(\nabla^2)$$

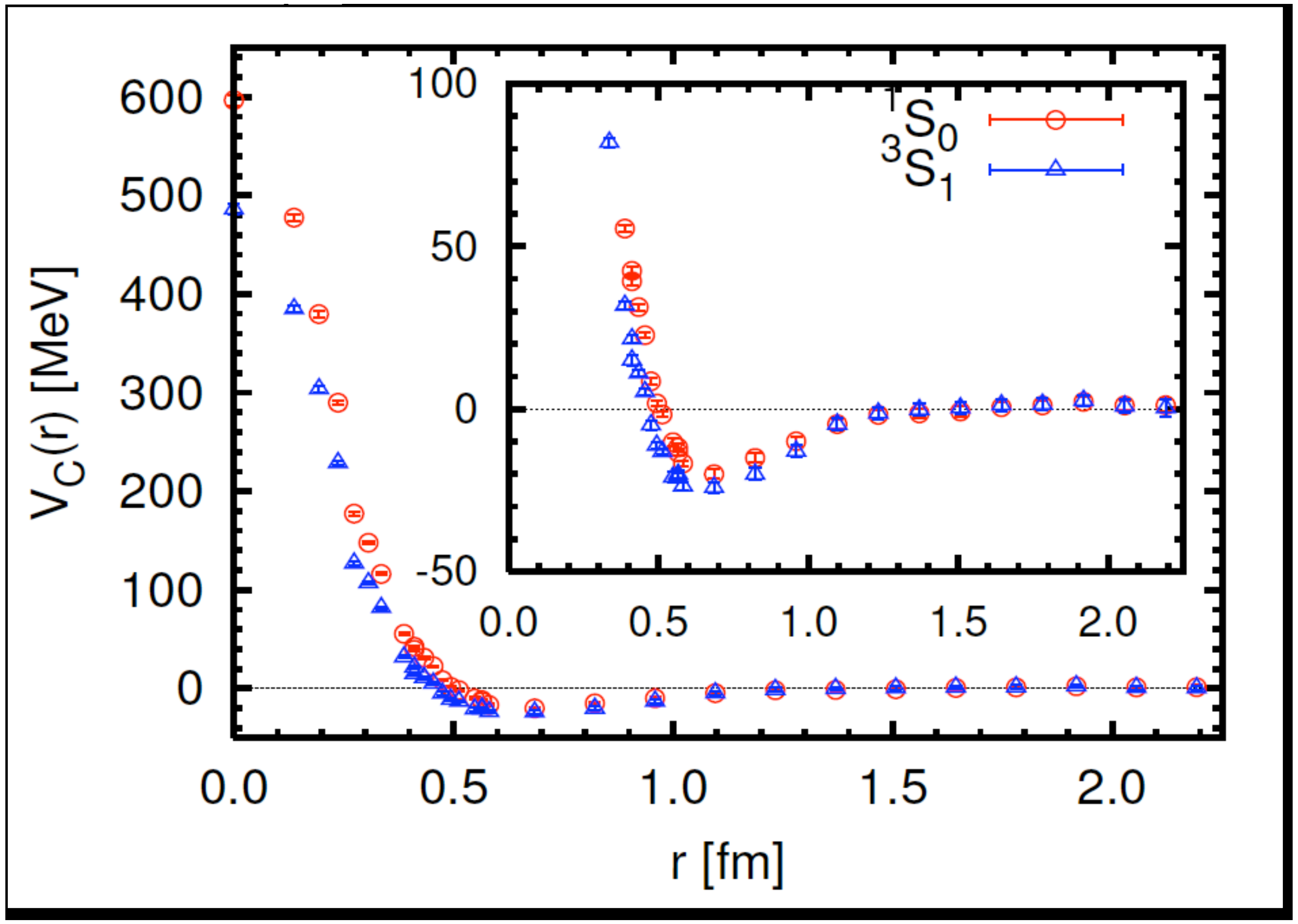
LO LO LO NLO NNLO

$$S_{12} = \frac{3}{r^2}(\sigma_{\mathbf{1}}\cdot\mathbf{x})(\sigma_{\mathbf{2}}\cdot\mathbf{x}) - (\sigma_{\mathbf{1}}\cdot\sigma_{\mathbf{2}})$$

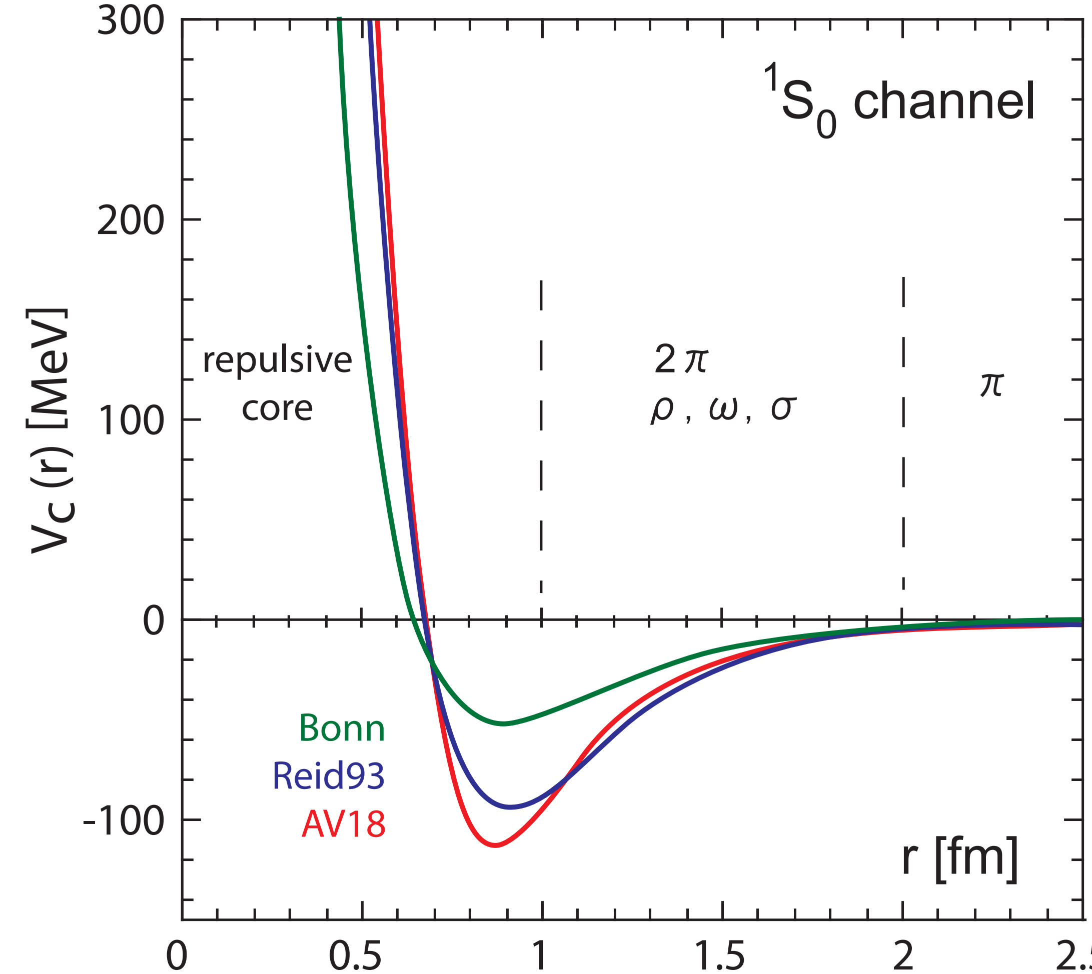
and calculated it in lattice QCD.

$$\boxed{V_{\rm LO}(\mathbf{x}) = \frac{[\epsilon_k - H_0]\varphi_{\mathbf{k}}(\mathbf{x})}{\varphi_{\mathbf{k}}(\mathbf{x})}}$$

our result (quenched QCD)



potential from experiments



a=0.137 fm L=4.4fm $m_\pi \simeq 0.53$ GeV

Ishii-Aoki-Hatsuda, PRL90(2007)0022001

Surprisingly, lattice NN "potential" reproduces qualitative features of NN potentials extracted from experimental data, including the short distance repulsion (repulsive core).

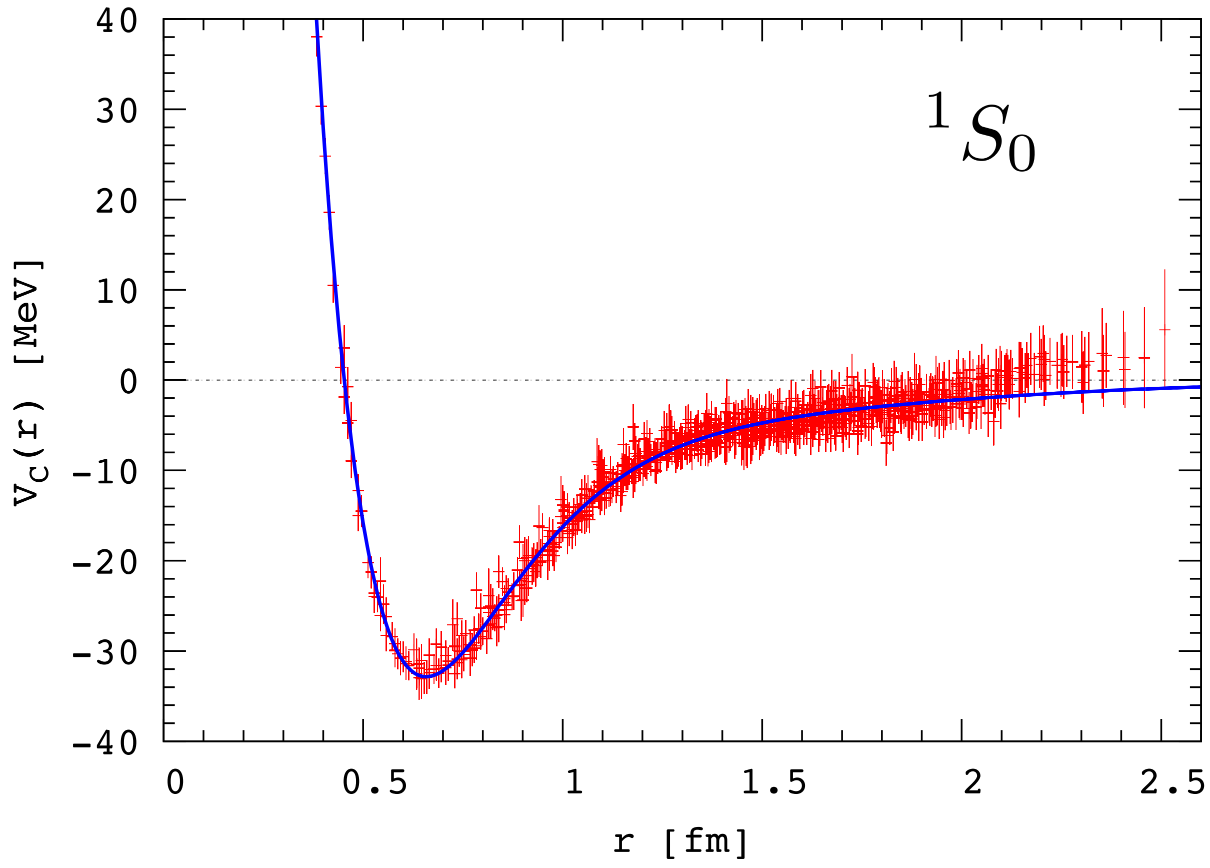
2+1 flavor QCD

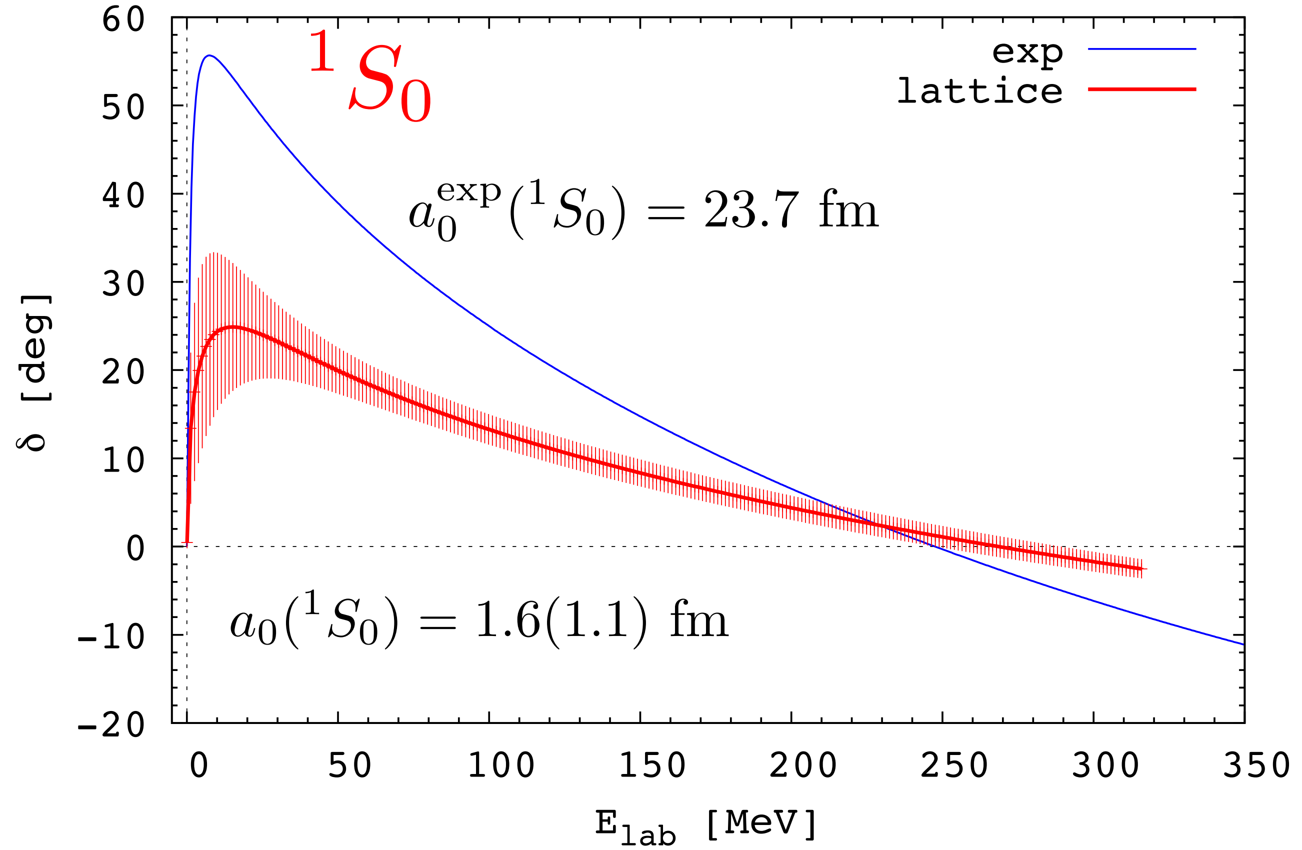
a=0.09fm, L=2.9fm

$m_\pi \simeq 700$ MeV

NN potential

phase shift



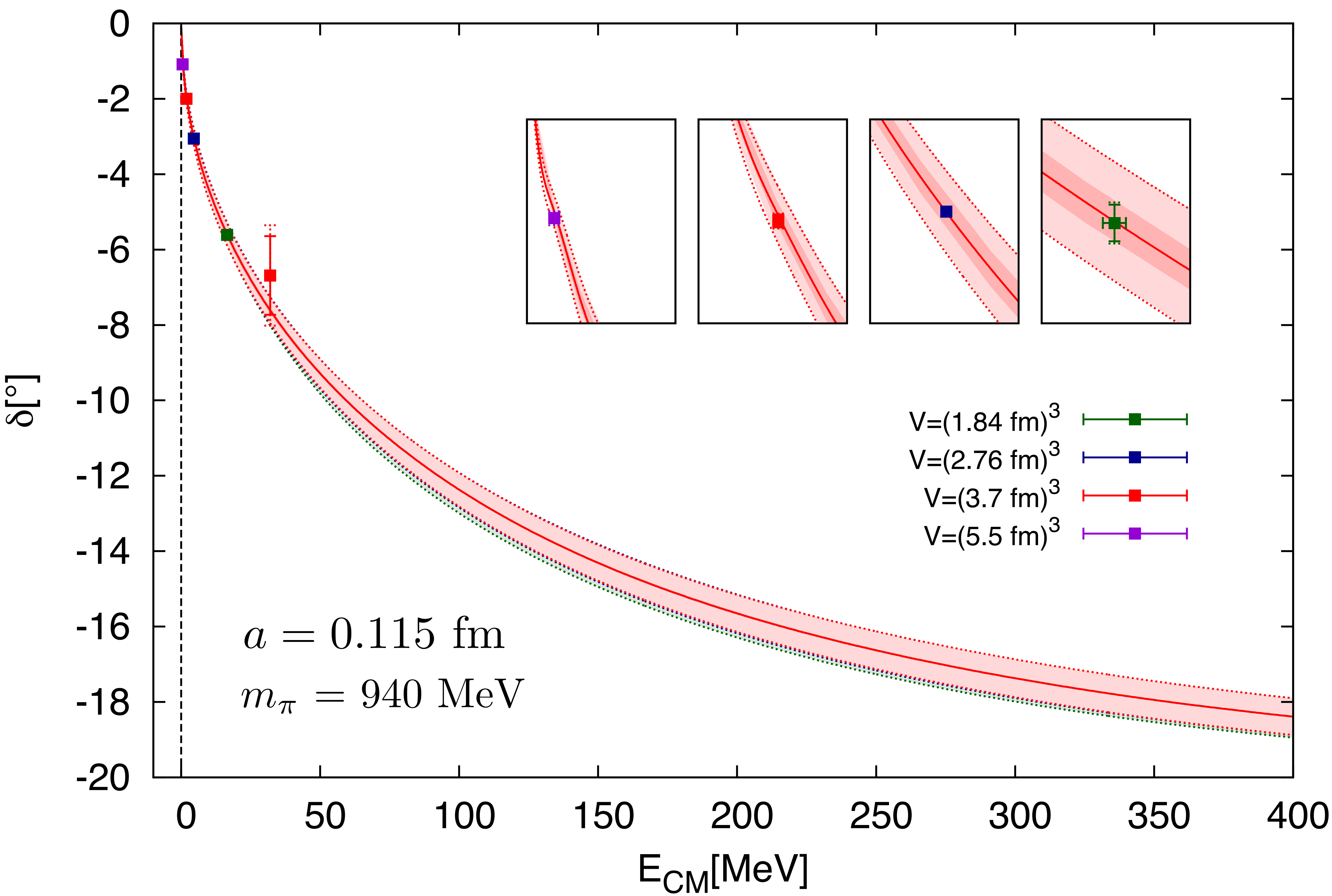


Ishii *et al.* ,PLB712(2012)437

Phase shift has a reasonable shape.
The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass on K-computer.

"Potential" vs Finite-volume method (I=2 pi-pi scattering. Quenched QCD)



Kurth-Ishii-Doi,-Aoki-Hatsuda, JHEP 1312(2013)015

Both methods agree very well.

6. Conclusion

Martin’s Finite-Volume methods have brought tremendous impacts to our lattice community.

cf. Stone Age -> Bronze Age

Since hadron spectroscopy in lattice QCD is about to be completed, these methods become more and more important in the next generation, to investigate hadron interactions in lattice QCD.

Without the non-scientific obligations he had before, I am sure that Martin will bring more exciting and important contributions to our lattice community.

cf. Bronze Age -> Iron Age

I wish Martin has fruitful and enjoyable life in Bern, and sometimes visit us in Japan !