A new and unifying formalism for the study of particle-spin dynamics using tools distilled from the theory of principal bundles

K. Heinemann\textsuperscript{a}, D. P. Barber\textsuperscript{b}, J. A. Ellison\textsuperscript{a}, and M. Vogt\textsuperscript{b}

\textsuperscript{a} Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.

\textsuperscript{b} Deutsches Elektronen–Synchrotron, DESY, 22607 Hamburg, Germany

22 October 2014

\textsuperscript{a} Work supported by DOE under DE-FG02-99ER41104 and DESY
The aims

- Define and classify invariant “spin” fields in storage rings
- Try to discover new invariant fields of physical significance
- Move towards criteria for existence of such fields (off orbital resonance!)
- In particular, can the expectation that invariant spin fields (ISF) always exist for integrable orbital motion in storage rings (the “ISF Conjecture”) be vindicated?
- Draw inspiration from underlying theory of fibre bundles

For today, an impression of an ongoing programme of exploration.

Expose you to deeper aspects of the EoS_M, to another way of thinking, to new points of view, ..... Not much explicit mathematics today but there’s lots available and a big paper in the works!

Useful pre-requisite knowledge:

- The **T-BMT equation** for spin motion driven by integrable orbital motion:
  \[ \frac{d\vec{S}}{d\theta} = \vec{\Omega}(\theta, \phi(\theta), J)) \times \vec{S}; \quad \theta = 2\pi s/C, \quad J \equiv J_1, J_2, J_3, \quad \phi \equiv \phi_1, \phi_2, \phi_3 \]

- The **invariant spin field** (ISF) \( \hat{n}(\theta, \phi, J) \) — a function 2\pi-periodic in \((\theta, \phi)\) – no history. For an equilibrium beam, the max. attainable equilibrium polarisation on a torus \( J \) is \( \langle \hat{n} \rangle_\phi \) Does \( \hat{n}(\theta, \phi, J) \) always exist? Recall the ISF Conjecture.

- The **invariant frame field** (IFF): \( \hat{\nu}_1(\theta, \phi, J), \hat{\nu}_2(\theta, \phi, J), \hat{n}(\theta, \phi, J) \) — local coordinate frames in phase space.

- The **equivalence class of amplitude dependent spin tunes** (ADST) — a countable infinity of members \( \nu(J) \).
  Spin-orbit resonance: \( \nu(J) = m_0 + \sum_i m_i Q_i \) (not! with \( \nu_0 \)) (see Barber: Spin2010)
  First ideas on ISF, IFF, ADST from Derbenev and Kondratenko, 1973 (41 years ago! but different terminology)

- The ADST does **not** exist on orbital resonance! – but the beam would be unstable anyway.
  On orbital resonance an “ISF-like” object always exists but it need not be continuous in \( \phi \) or unique (up to a sign) — it’s messy (see Barber + Vogt: Spin2006, Barber: Spin2002)
Basic equations

Simpler notation: \( \vec{S} \mapsto S \)

The 1-turn spin maps from the T-BMT eqn. starting at \((\theta_0, \phi(\theta_0), J)\):

\[
S(\theta_0 + 2\pi) = A(\theta_0, \phi(\theta_0), J)S(\theta_0)
\]

where the 1-turn spin map \( A(\theta, \phi, J) \) is an orthogonal \( 3 \times 3 \) matrix, an element of \( SO(3) \), and it is a function of \((\theta, \phi, J)\).

Over 1 turn, the position \( \phi \) on the \( d \)-torus \( \mathbb{T}^d \) is transformed to \( j_J(\phi) \) — allows for generalisations

Usually \( d = 1, 2 \) or \( 3 \).

Ignore (ambiguous and miniscule) Stern-Gerlach forces
The invariant spin field – discrete “time”

The invariant spin field (ISF) $f_\nu (\equiv \hat{n}(\theta, \phi, J))$ – a T-BMT solution which is a $2\pi$-periodic function of $\theta, \phi$

$$f_\nu(\theta + 2\pi, j_J(\phi), J) = A(\theta, \phi, J)f_\nu(\theta, \phi, J)$$

Switch to discrete “time” $\Rightarrow$ choose a fixed $\theta$

We are only interested in being off orbital resonance with $f_\nu$ continuous in $\phi$.

$$f_\nu(j_J(\phi), J) = A(\theta, \phi, J)f_\nu(\phi, J)$$

The invariant spin-1/2 density matrix on $\mathbb{T}^d$

$$\rho_{1/2}^1(\phi, J) = \frac{1}{2}\{I + P_{eq}(J)\hat{n}(\phi, J) \cdot \vec{\sigma}\}$$

Now ignore $J$ as it’s just a parameter.

$$\rho_{1/2}^1(\phi) = \frac{1}{2}\{I + P_{eq}(J)\hat{n}(\phi) \cdot \vec{\sigma}\}$$
More invariant fields

The invariant spin-1 density matrix in terms of spin-1 ang. mtm. matrices $\hat{J}$ (Barber + Vogt: Spin2008):

$$\rho_1^I(\phi) = \frac{1}{3} \left\{ I + \frac{3}{2} P_{eq}(J) \hat{n} \cdot \vec{J} + \sqrt{\frac{3}{2}} \xi_{eq}(J) \sum_{i,j} T^I_{ij} (\hat{J}_i \hat{J}_j + \hat{J}_j \hat{J}_i) \right\}$$

$T$ is the rank–2, $3 \times 3$, real, symmetric, traceless, Cartesian, polarization tensor

The invariant tensor field (ITF) $T^I$:

$$T^I(j_J(\phi)) = A(\phi) T^I(\phi) A(\phi)^T$$

See (see Barber + Vogt: Spin2008)!:

$$T^I = \pm \sqrt{\frac{3}{2}} \left\{ \hat{n} \hat{n}^T - \frac{1}{3} I \right\}$$
Unify—

the various kinds of transform involving $A(\phi)$ by using the operator $l$ – an “$SO(3)$–action”

Then over $1$ turn the field $f$ becomes the field $f'$ where $f'(\phi) := l(A(j_j^{-1}(\phi)); f(j_j^{-1}(\phi)))$ or:

\[ f \mapsto f' = l(A \circ j_j^{-1}; f \circ j_j^{-1}) \]

By definition, an invariant field maps into itself (the whole field): $f' = f$.

Then

\[ f(j_j(\phi)) = l(A(\phi); f(\phi)) \]

The field dynamics is induced by the particle-spin dynamics: $(j_j, A)$ defines $f$.

We don’t need to know that $A$ comes from the T-BMT equation! — we are exploring structures.

Encompass the “spin” dynamics in $SO(3)$-spaces $(E, l)$ where $E$ is a topological space and $l$ is a continuous $SO(3)$–action on $E$ i.e., $l: SO(3) \times E \to E$ is continuous and $l(I; x) = x$ and $l(r_1 r_2; x) = l(r_1; l(r_2; x))$ with $x \in E$.

With the flexibility in the choice of $(E, l)$, we have a unified way to study the dynamics of spin-$1/2$ and spin-$1$ particles, the density matrices of the bunches and ....

Are there other invariant fields?
Subgroups of $SO(3) \iff$ invariant fields

It will turn out that (see the Normal Form Theorem later):

The ISF is associated with the subgroup $SO(2)$ of $SO(3)$

The ITF is associated with the subgroups $SO(2) \rtimes Z(2)$ (see Zappa-Szép products) and $SO(3)_{\text{diag}}$ of $SO(3)$

The invariant spin-1/2 density matrix is associated with the subgroup $SO(2)$ of $SO(3)$

The invariant spin-1 density matrix is associated with the subgroups $SO(2), SO(2) \rtimes Z(2)$ and $SO(3)_{\text{diag}}$ of $SO(3)$ — and perhaps others.

In fact these subgroups are so-called isotropy groups.

Also (see later) with this apparatus we can find a

$$T^I = \pm \sqrt{\frac{3}{2}} \left\{ \hat{n} \hat{n}^T - \frac{1}{3} I \right\}$$

Is there a way of connecting and classifying invariant fields?
Draw inspiration from underlying bundle theory

The tools:

- Geometrical tools distilled from the theory of fibre bundles
  — recall the use of bundles in gauge theories in particle physics too
- A fixed principal $SO(3)$-bundle induces the dynamics of fields via its associated bundles
- The theory was developed in the 1980’s by Zimmer, Feres et al.

- Simple introductions to the basic idea of bundles:
- Thorough treatments:
- For this work:
  R. Feres, “Dynamical systems and semisimple groups: an introduction”
  R. Feres, A. Katok in “Handbook of dynamical systems Vol. 1A” .
Associated bundles

Particle Physics

Base-space of associated bundle = space-time

The temporal motion of $\sigma$ is driven by the principal bundle and depends on the choice of $(E, \ell)$ (e.g. Klein-Gordon particle, Dirac particle, ... with wave function $\psi$).

Spin-Orbit System

Base-space of associated bundle = $T^d$

The temporal motion of $\sigma$ is driven by the principal bundle and depends on the choice of $(E, \ell)$ (e.g. $(\mathbb{R}^3, \ell_V)$, $(E_i, \ell_i)$, ... with field $f$).

Here, $z \equiv (\phi, J)$
The principal bundle underlying the spin motion

Base-space of principal $SO(3)$-bundle $= T^d$

fibre $\sim SO(3)$
Brief remarks w.r.t. bundles

Recall that for an invariant field: \[ f' = f \]

Since the dynamics of \( f \) comes from the cross sections it has its peculiar form \[ f' = l(A \circ j^{-1}_j; f \circ j^{-1}_j) \]

This is very convenient since it works in discrete time i.e., there is no need for derivation from a continuous time system.

Thus the dynamics of \( f \) need not be Yang-Mills-like

However, since our fixed principal bundle is smooth, one can study the path lifting motions. The first results are encouraging (planar spin motion).
Four central theorems

- The Normal Form Theorem
- The Decomposition Theorem
- The Invariant Reduction Theorem
- The Cross Section Theorem — not today.
The Normal Form Theorem

shows that:

The ISF is associated with the subgroup $SO(2)$ of $SO(3)$

The ITF is associated with the subgroups $SO(2) \rtimes Z(2)$ and $SO(3)_{\text{diag}}$ of $SO(3)$

The invariant spin-$1/2$ density matrix is associated with the subgroup $SO(2)$ of $SO(3)$

The invariant spin-$1$ density matrix is associated with the subgroups $SO(2), SO(2) \rtimes Z(2)$ and $SO(3)_{\text{diag}}$ of $SO(3)$

The 3rd column ($\equiv$ unit vector) of an IFF is the ISF.

The NFT generalises this to invariant fields from $SO(2)$ to arbitrary subgroups $H$ of $SO(3)$

$\implies$ we have a new view of the IFF.
The Decomposition Theorem

Since invariant fields are tied to subgroups of $SO(3)$, these subgroups can be used to classify and relate invariant fields

The Decomposition Theorem:
"If two invariant fields $f$ and $g$ are tied to conjugate subgroups $H$ and $H'$ then $f$ and $g$ are related by a homeomorphism"

This provides an avenue to construct a:

$$ T^1 = \pm \sqrt{\frac{3}{2}} \left\{ \hat{\mathbf{n}}^T - \frac{1}{3} I \right\} $$

SO!

We can generate (say) the ITF from the ISF without recourse to physics! — we have a machine to generate non-arbitrary invariant fields.

Are they always physically relevant?
The Invariant Reduction Theorem

- Definition: for subgroups $H$ of $SO(3)$, $H$–reductions are principal bundles which are subbundles of the underlying $SO(3)$–bundle
- Reduction theorem: every $H$–reduction relies on a field $f$ tied to $H$.
- Since the dynamics of $f$ comes from a cross-section it induces a dynamics on the $H$–reduction leading to:

The Invariant Reduction Theorem:

“Invariance of $f$ $\iff$ the $H$–reduction is invariant”

$\Rightarrow$ geometrization of invariant fields $\Rightarrow$ a new view!

NOTE: the IRT and the NFT are closely related

- Thus the cross-section dynamics gives new insights into ISF’s and ITF’s e.g., into the ISF Conjecture.
Summary and plans

- We have a new formalism for defining, generalising and classifying invariant fields in storage rings using tools inspired by bundle theory (– gauge theories also exploit bundles).

- We will pursue the analogy with bundles to find parallels useful for studying invariant fields.

- We hope to vindicate the ISF Conjecture or find the conditions for its validity.