A new and unifying formalism for the study of particle-spin dynamics using tools distilled from the theory of principal bundles [a]

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The aims

- Define and classify invariant "spin" fields in storage rings
- Try to discover new invariant fields of physical significance
- Move towards criteria for existence of such fields (off orbital resonance!)
- In particular, can the expectation that invariant spin fields (ISF) always exist for integrable orbital motion in storage rings (the "*ISF Conjecture*") be vindicated?
- Draw inspiration from underlying theory of fibre bundles

For today, an impression of an ongoing programme of exploration.

Expose you to deeper aspects of the EoSM, to another way of thinking, to new points of view, ...

Not much explicit mathematics today but there's lots available and a big paper in the works!

See http://arxiv.org/abs/1409.4373 for an impression
and Klaus Heinemann's thesis: University of New Mexico, May 2010.

Useful pre-requisite knowledge:

- The **T-BMT equation** for spin motion driven by integrable orbital motion:
  $d\vec{S}/d\theta = \vec{\Omega}(\theta, \phi(\theta), J)) \times \vec{S}; \quad ; \qquad \theta = 2\pi s/C, \quad J \equiv J_1, J_2, J_3, \quad \phi \equiv \phi_1, \phi_2, \phi_3$

- The **invariant spin field** (ISF) $\hat{n}(\theta, \phi, J)$ — a function 2π-periodic in (θ, ϕ) – no history.
  For an equilibrium beam, the max. attainable equilibrium polarisation on a torus J is $< \hat{n} >_\phi$
  Does $\hat{n}(\theta, \phi, J)$ always exist? Recall the ISF Conjecture.
- The **invariant frame field** (IFF): $\hat{u}_1(\theta, \phi, J), \quad \hat{u}_2(\theta, \phi, J), \quad \hat{n}(\theta, \phi, J)$ — local coordinate frames in phase space.

- The **equivalence class of amplitude dependent spin tunes** (ADST) — a countable infinity of members $\nu(J)$.
  Spin-orbit resonance: $\nu(J) = m_0 + \sum_i m_i Q_i$ (not! with ν_0) (see Barber: Spin2010)
  First ideas on ISF, IFF, ADST from Derbenev and Kondratenko, 1973 (41 years ago! but different terminology)

- The ADST does **not** exist on orbital resonance! – but the beam would be unstable anyway.
  On orbital resonance an "ISF-like" object always exists but it need not be continuous in ϕ or unique (up to a sign) —– it's messy (see Barber + Vogt: Spin2006, Barber: Spin2002)

Basic equations

Simpler notation : $\vec{S} \Longrightarrow S$

The 1-turn spin maps from the T-BMT eqn. starting at $(\theta_0, \phi(\theta_0), J)$:

$$S(\theta_0 + 2\pi) = A(\theta_0, \phi(\theta_0), J) S(\theta_0)$$

where the 1-turn spin map $A(\theta, \phi, J)$ is an orthogonal 3×3 matrix, an element of $SO(3)$, and it is a *function* of (θ, ϕ, J).

Over 1 turn, the position ϕ on the d–torus $\mathbb{T}^d$ is transformed to $j_J(\phi)$ — allows for generalisations
Usually $d = 1, 2$ or 3.

Ignore (ambiguous and miniscule) Stern-Gerlach forces

The invariant spin field – discrete "time"

The invariant spin *field* (ISF) f_v ($\equiv \hat{n}(\theta, \phi, J)$) – a T-BMT solution which is a 2π-periodic *function* of θ, ϕ

$$f_v(\theta + 2\pi, j_J(\phi), J) = A(\theta, \phi, J) f_v(\theta, \phi, J)$$

Switch to discrete "time" $\Rightarrow$ choose a fixed θ

We are only interested in being off orbital resonance with f_v continuous in ϕ.

$$f_v(j_J(\phi), J)) = A(\theta, \phi, J) f_v(\phi, J)$$

The invariant spin-1/2 density matrix on $\mathbb{T}^d$

$$\rho^{\mathrm{I}}_{1/2}(\phi, J) = \frac{1}{2}\{I + P_{\mathrm{eq}}(J)\, \hat{n}(\phi, J) \cdot \vec{\sigma}\}$$

Now ignore J as it's just a parameter.

$$\rho^{\mathrm{I}}_{1/2}(\phi) = \frac{1}{2}\{I + P_{\mathrm{eq}}(J)\, \hat{n}(\phi) \cdot \vec{\sigma}\}$$

More invariant fields

The invariant spin-1 density matrix in terms of spin-1 ang. mtm. matrices $\mathfrak{J}$ (Barber + Vogt: Spin2008):

$$\rho_1^{\mathrm{I}}(\phi) = \frac{1}{3}\left\{ I + \frac{3}{2} P_{\mathrm{eq}}(J)\,\hat{n}\cdot\vec{\mathfrak{J}} + \sqrt{\frac{3}{2}}\mathfrak{T}_{\mathrm{eq}}(J)\sum_{i,j} T_{ij}^{\mathrm{I}}\;(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)\right\}$$

T is the rank–2, 3×3, real, symmetric, traceless, Cartesian, polarization tensor

The invariant tensor field (ITF) T^I:

$$T^I(j_J(\phi)) = A(\phi)T^I(\phi)A(\phi)^T$$

See (see Barber + Vogt: Spin2008)!:

$$T^{\mathrm{I}} = \pm\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\mathrm{T}} - \frac{1}{3}I\right\}$$

Unify–

the various kinds of transform involving $A(\phi)$ by using the operator l – an "$SO(3)$–action"

Then over 1 turn the field f becomes the field f' where $f'(\phi) := l(A(j_J^{-1}(\phi)); f(j_J^{-1}(\phi)))$ or:

$$\boxed{\mathrm{f} \mapsto f' = l(A \circ j_J^{-1}; f \circ j_J^{-1})}$$

By definition, an invariant field maps into itself (the whole field): $f' = f$.

Then

$$\boxed{\mathrm{f}(\ \mathrm{j}_J(\phi)) = l(A(\phi); f(\phi))}$$

The field dynamics is induced by the particle-spin dynamics: (j_J, A) defines f.

We don't need to know that A comes from the T-BMT equation! — we are exploring structures.

Encompass the "spin" dynamics in $SO(3)$-spaces (E, l) where E is a topological space and l is a continuous $SO(3)$–action on E i.e., $l : SO(3) \times E \to E$ is continuous and $l(I; x) = x$ and $l(r_1 r_2; x) = l(r_1; l(r_2; x))$ with $x \in E$.

With the flexibility in the choice of (E, l), we have a unified way to study the dynamics of spin-1/2 and spin-1 particles, the density matrices of the bunches and .....

Are there other invariant fields?

Subgroups of $SO(3)$ $\Longleftrightarrow$ invariant fields

It will turn out that (see the Normal Form Theorem later):

The ISF is associated with the subgroup $SO(2)$ of $SO(3)$

The ITF is associated with the subgroups $SO(2) \bowtie Z(2)$ (see Zappa-Szép products) and $SO(3)_{\mathrm{diag}}$ of $SO(3)$

The invariant spin-1/2 density matrix is associated with the subgroup $SO(2)$ of $SO(3)$

The invariant spin-1 density matrix is associated with the subgroups $SO(2)$, $SO(2) \bowtie Z(2)$ and $SO(3)_{\mathrm{diag}}$ of $SO(3)$) —- and perhaps others.

In fact these subgroups are so-called ***isotropy groups*** .

ALSO (see later) with this apparatus we can find a

$$T^{\mathrm{I}} = \pm\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\mathrm{T}} - \frac{1}{3}I\right\}$$

Is there a way of connecting and classifying invariant fields?

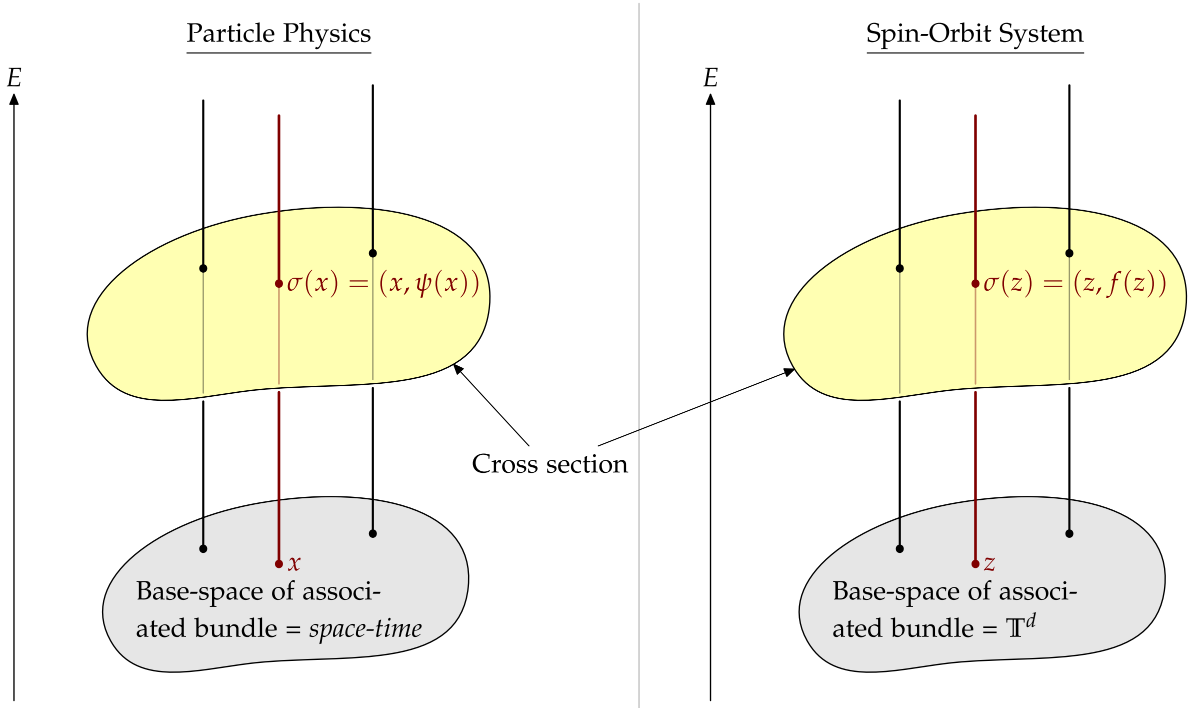
Draw inspiration from underlying bundle theory

The tools:

- Geometrical tools distilled from the theory of fibre bundles
  — recall the use of bundles in gauge theories in particle physics too
- A fixed principal $SO(3)$-bundle induces the dynamics of fields via its associated bundles
- The theory was developed in the 1980's by Zimmer, Feres et al.

- Simple introductions to the basic idea of bundles:
  H.J. Bernstein and A.V Phillips, "Fiber Bundles and Quantum Theory?" Scientific American 1981, for lifts, connections etc for the A-B effect with magnetic fields.
- Thorough treatments:
  G.L.Naber, "Topology and Geometry of Gauge Fields", D. Husemoller, "Fiber Bundles".
- For this work:
  R. Feres, "Dynamical systems and semisimple groups: an introduction"
  R. Feres, A. Katok in "Handbook of dynamical systems Vol. 1A" .

Associated bundles

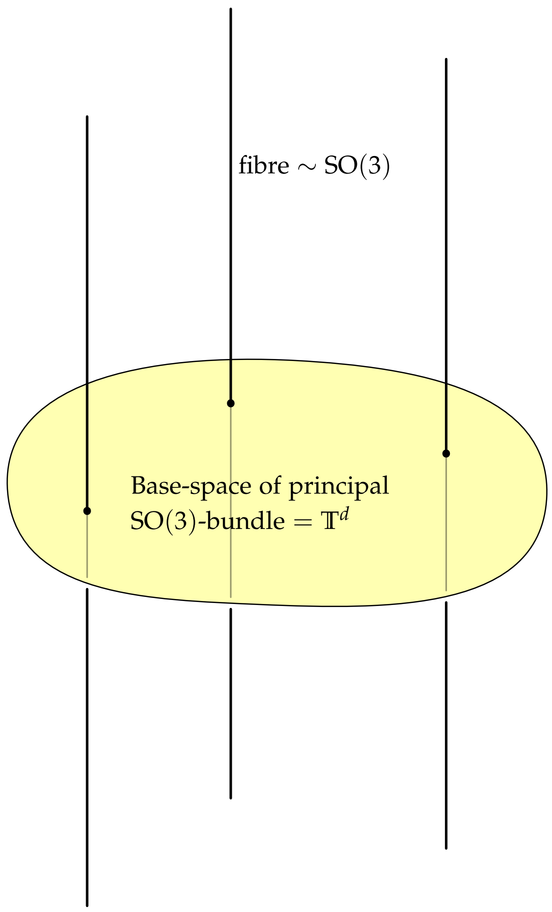


The temporal motion of σ is driven by the principal bundle and depends on the choice of (E, ℓ) (*e.g.* Klein-Gordon particle, Dirac particle, . . . with wave function ψ).

The temporal motion of σ is driven by the principal bundle and depends on the choice of (E, ℓ) (*e.g.* $(\mathbb{R}^3, \ell_V)$, (E_t, ℓ_t), . . . with field f).

Here, $z \equiv (\phi, J)$

The principal bundle underlying the spin motion



Brief remarks w.r.t. bundles

Recall that for an invariant field: $f' = f$

Since the dynamics of f comes from the cross sections it has its peculiar form
$f' = l(A \circ j_J^{-1}; f \circ j_J^{-1})$

This is very convenient since it works in discrete time i.e., there is no need for derivation from a continuous time system.

Thus the dynamics of f need not be Yang-Mills-like

However, since our fixed principal bundle is smooth, one can study the path lifting motions. The first results are encouraging (planar spin motion).

Four central theorems

- The Normal Form Theorem
- The Decomposition Theorem
- The Invariant Reduction Theorem
- The Cross Section Theorem — not today.

The Normal Form Theorem

shows that:

The ISF is associated with the subgroup $SO(2)$ of $SO(3)$

The ITF is associated with the subgroups $SO(2) \bowtie Z(2)$ and $SO(3)_{\mathrm{diag}}$ of $SO(3)$

The invariant spin-1/2 density matrix is associated with the subgroup $SO(2)$ of $SO(3)$

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The 3rd column ($\equiv$ unit vector) of an IFF is the ISF.

The NFT generalises this to invariant fields from $SO(2)$ to arbitrary subgroups H of $SO(3)$

$\Longrightarrow$ we have a new view of the IFF.

The Decomposition Theorem

Since invariant fields are tied to subgroups of $SO(3)$, these subgroups can be used to classify and relate invariant fields

The Decomposition Theorem:

"If two invariant fields f and g are tied to conjugate subgroups H and H'
then f and g are related by a homeomorphism"

This provides an avenue to construct a:

$$T^{\mathrm{I}} = \pm\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\mathrm{T}} - \frac{1}{3}I\right\}$$

SO!

We can generate (say) the ITF from the ISF without recourse to physics! — we have a machine to generate non-arbitrary invariant fields.

Are they always physically relevant?

The Invariant Reduction Theorem

- Definition: for subgroups H of $SO(3)$, H–reductions are principal bundles which are subbundles of the underlying $SO(3)$–bundle
- Reduction theorem: every H–reduction relies on a field f tied to H.
- Since the dynamics of f comes from a cross-section it induces a dynamics on the H–reduction leading to:

  The Invariant Reduction Theorem:

  $$\text{“Invariance of } f \iff \text{the } H\text{–reduction is invariant”}$$

  $$\Longrightarrow \text{geometrization of invariant fields —-a new view!}$$

  NOTE: the IRT and the NFT are closely related
- Thus the cross-section dynamics gives new insights into ISF's and ITF's e.g., into the ISF Conjecture.

Summary and plans

- We have a new formalism for defining, generalising and classifying invariant fields in storage rings using tools inspired by bundle theory (– gauge theories also exploit bundles).

- We will persue the analogy with bundles to find parallels useful for studying invariant fields.

- We hope to vindicate the ISF Conjecture or find the conditions for its validity.