Introduction
SIDIS: a typical TMD process

Semi-inclusive DIS is a process sensitive to transverse momentum of quarks

\[ P_{h\perp} = \text{the observed transverse momentum of the produced hadron} = z_h Q_T \]
\[ Q_T = \text{the transverse momentum of the virtual photon w.r.t. } p \text{ and } h \]

Many transverse momentum dependent *angular distributions* have been measured in SIDIS by HERMES, COMPASS, and JLab experiments

Evolution is needed to compare these results, factorization dictates the evolution
Transverse Momentum of Quarks

Including transverse momentum of quarks involves much more than replacing $f_1(x) \rightarrow f_1(x, k_T^2)$ in collinear factorization expressions.

One deals with less inclusive processes and with TMD factorization.

TMD = \textit{transverse momentum dependent} parton distribution.

Here the transverse momentum dependence can be correlated with the spin, e.g.

\[ \Phi(x, k_T) = \frac{M}{2} \left\{ f_1(x, k_T^2) \frac{P}{M} + f_{1T}(x, k_T^2) \frac{\epsilon_{\mu\nu\rho\sigma} \gamma^\mu P^\nu k_\rho^T S_\sigma^T}{M^2} + g_{1s}(x, k_T^2) \frac{\gamma_5 P}{M} \right\} \]

\[ + h_{1T}(x, k_T^2) \frac{\gamma_5 S_T P}{M} + h_{1s}(x, k_T^2) \frac{\gamma_5 k_T P}{M^2} + h_{1T}(x, k_T^2) \frac{i k_T P}{M^2} \]

[Ralston, Soper '79; Sivers '90; Collins '93; Kotzinian '95; Mulders, Tangerman '95; D.B., Mulders '98]
TMD factorization
“Evolution” of TMD Factorization

- Collins & Soper, 1981: $e^+e^- \rightarrow h_1 h_2 X$ [NPB 193 (1981) 381]
- Collins (JCC), 2011: “Foundations of perturbative QCD” [Cambridge Univ. Press]
- P. Sun, B.-W. Xiao & F. Yuan, 2011: Higgs prod. (gluon TMDs) [PRD 84 (2011) 094005]

Main differences among the various approaches:
- treatment of rapidity/LC divergences, in order to make each factor well-defined
- redistribution of terms to avoid large logarithms
TMD factorization

TMD factorization proven for SIDIS, $e^+e^- \rightarrow h_1 \ h_2 \ \text{X and Drell-Yan (DY)}$

Schematic form of (new) TMD factorization “JCC” [Collins 2011]:

\[ d\sigma = H \times \text{convolution of } AB + \text{high}-q_T \text{ correction } (Y) + \text{power-suppressed} \]

A & B are TMD pdfs or FFs
(a soft factor has been absorbed in them)

Convolution in terms of A and B best deconvoluted by Fourier transform

Details in book by J.C. Collins
Summarized in arXiv:1107.4123

Foundations of Perturbative QCD

JOHN COLLINS
New TMD factorization expressions

$$\frac{d\sigma}{d\Omega d^4q} = \int d^2b \, e^{-ib\cdot q_T} \tilde{W}(b, Q; x, y, z) + \mathcal{O}(Q_T^2/Q^2)$$

$$\tilde{W}(b, Q; x, y, z) = \sum_a \tilde{f}_1^a(x, b^2; \zeta_F, \mu) \tilde{D}_1^a(z, b^2; \zeta_D, \mu) H(y, Q; \mu)$$

Fourier transforms of the TMDs are functions of the momentum fraction $x$ (or $z$), the transverse coordinate $b$, a rapidity variable $\zeta$, and the renormalization scale $\mu$

$$\zeta_F = M^2 x^2 e^{2(y_P - y_s)} \quad \zeta_D = M_h^2 e^{2(y_s - y_h)} / z^2$$

$y_s$ is an arbitrary rapidity that drops out of the final answer

$$\zeta_F \zeta_D \approx Q^4 \quad \zeta_F \approx \zeta_D \approx Q^2$$

The TMDs in principle also depend on the Wilson line $U$

$$\tilde{f}[U](x, b_T^2; \zeta, \mu)$$
New TMD factorization expressions

\[
\frac{d\sigma}{d\Omega d^4q} = \int d^2 b \ e^{-ib\cdot q_T} \tilde{W}(b, Q; x, y, z) + O\left(\frac{Q_T^2}{Q^2}\right)
\]

\[
\tilde{W}(b, Q; x, y, z) = \sum_a \tilde{f}_1^a(x, b^2; \zeta_F, \mu) \tilde{D}_1^a(z, b^2; \zeta_D, \mu) H(y, Q; \mu)
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\[
\tilde{f}[U](x, b^2_T; \zeta, \mu)
\]
Gauge invariance of TMD correlators

\[ \mathcal{L}_C[0, \xi] = \mathcal{P} \exp \left( -ig \int_{C[0, \xi]} ds_\mu A^\mu(s) \right) \]

\[ \Phi \propto \langle P \bar{\psi}(0) \mathcal{L}_C[0, \xi] \psi(\xi) \rangle_P \]

Efremov & Radyushkin, Theor. Math. Phys. 44 ('81) 774

Resulting Wilson lines depend on whether the color is incoming or outgoing

[Collins & Soper, 1983; DB & Mulders, 2000; Brodsky, Hwang & Schmidt, 2002; Collins, 2002; Belitsky, X. Ji & F. Yuan, 2003; DB, Mulders & Pijlman, 2003]

This does not automatically imply that this affects observables, but it turns out that it does in certain cases, for example, Sivers asymmetries

[Brodsky, Hwang & Schmidt, 2002; Collins, 2002; Belitsky, Ji & Yuan, 2003]
Process dependence of Sivers TMD

Gauge invariant definition of TMDs in semi-inclusive DIS contains a future pointing Wilson line, whereas in Drell-Yan (DY) it is past pointing [Belitsky, X. Ji & F. Yuan '03]

\[ \gamma^* p \rightarrow h X \text{ (SIDIS)} \]

\[ pp \rightarrow \gamma^* X \text{ (Drell-Yan)} \]

Lightcone infinity $\xi^+ \rightarrow -\infty^-$

One can use parity and time reversal invariance to relate the Sivers functions:

\[ f_{1T}^{\perp \text{[SIDIS]}} = - f_{1T}^{\perp \text{[DY]}} \]

[Collins '02]
Gauge invariant definition of TMDs in semi-inclusive DIS contains a future pointing Wilson line, whereas in Drell-Yan (DY) it is past pointing [Belitsky, X. Ji & F. Yuan '03]

\[ \xi - \xi_T \]

\[ f_{1T}^{[\text{SIDIS}]} = -f_{1T}^{[\text{DY}]} \] [Collins '02]

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The more hadrons are observed in a process, the more complicated the end result: more complicated $N_c$-dependent prefactors [Bomhof, Mulders & Pijlman '04; Buffing, Mulders '14]
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When color flow is in too many directions: factorization breaking [Collins & J. Qiu '07; Collins '07; Rogers & Mulders '10]
Scale dependence of TMDs

QCD corrections will also attach to the Wilson line, which needs renormalization

Wilson lines not smooth: cusp anomalous dimension
[Polyakov '80; Dotsenko & Vergeles '80; Brandt, Neri, Sato '81; Korchemsky, Radyushkin '87]

This determines the change with $\mu$

As a regularization of LC divergences, in JCC’s TMD factorization the path is taken off the lightfront, the variation in rapidity determines the change with $\zeta$

$$\tilde{f}[\mathcal{U}](x, b_T^2; \zeta, \mu)$$
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Two important consequences:

- yields energy evolution of TMD observables
- allows for calculation of the Sivers and Boer-Mulders effect on the lattice

Musch, Hägler, Engelhardt, Negele & Schäfer, 2012
New TMD factorization expressions

\[
\frac{d\sigma}{d\Omega d^4q} = \int d^2b \ e^{-ib\cdot q_T} \tilde{W}(b, Q; x, y, z) + \mathcal{O}(Q_T^2/Q^2)
\]

\[
\tilde{W}(b, Q; x, y, z) = \sum_a \tilde{f}_1^a(x, b^2; \zeta_F, \mu) \tilde{D}_1^a(z, b^2; \zeta_D, \mu) H(y, Q; \mu)
\]

Take \( \mu = Q \)

\[
H(Q; \alpha_s(Q)) \propto e_a^2 \left( 1 + \alpha_s(Q^2) F_1 + \mathcal{O}(\alpha_s^2) \right)
\]

Avoids large logarithms in H, but now they do appear in the TMDs

Use renormalization group equations to evolve the TMDs to the scale:

\[
\mu_b = C_1 / b = 2e^{-\gamma_E} / b \quad (C_1 \approx 1.123)
\]

Or to a fixed low (but still perturbative) scale \( Q_0 \), although that only works for not too large \( Q \)
RG and CS equations

\[
\frac{d \ln \tilde{f}(x, b; \zeta, \mu)}{d \ln \sqrt{\zeta}} = \tilde{K}(b; \mu) \quad \text{Collins-Soper equation}
\]

\[
\frac{d \ln \tilde{f}(x, b; \zeta, \mu)}{d \ln \mu} = \gamma_F(g(\mu); \zeta/\mu^2) \quad \text{RG equation}
\]

\[
d\tilde{K}/d \ln \mu = -\gamma_K(g(\mu))
\]

\[
\gamma_F(g(\mu); \zeta/\mu^2) = \gamma_F(g(\mu); 1) - \frac{1}{2} \gamma_K(g(\mu)) \ln(\zeta/\mu^2)
\]

Using these equations one can evolve the TMDs to the scale \(\mu_b\)

\[
\tilde{f}_1^a(x, b^2; \zeta_F, \mu) \tilde{D}^b_1(z, b^2; \zeta_D, \mu) = e^{-S(b, Q)} \tilde{f}_1^a(x, b^2; \mu^2_b, \mu_b) \tilde{D}^b_1(z, b^2; \mu^2_b, \mu_b)
\]

with Sudakov factor

\[
S(b, Q) = -\ln \left( \frac{Q^2}{\mu^2_b} \right) \tilde{K}(b, \mu_b) - \int_{\mu^2_b}^{Q^2} \frac{d\mu^2}{\mu^2} \left[ \gamma_F(g(\mu); 1) - \frac{1}{2} \ln \left( \frac{Q^2}{\mu^2} \right) \gamma_K(g(\mu)) \right]
\]
Perturbative expressions

At leading order in $\alpha_s$

\[
\tilde{K}(b, \mu) = -\alpha_s(\mu) \frac{C_F}{\pi} \ln(\mu^2 b^2 / C_1^2) + O(\alpha_s^2)
\]

\[
\gamma_K(g(\mu)) = 2\alpha_s(\mu) \frac{C_F}{\pi} + O(\alpha_s^2)
\]

\[
\gamma_F(g(\mu), \zeta/\mu^2) = \alpha_s(\mu) \frac{C_F}{\pi} \left( \frac{3}{2} - \ln \left( \frac{\zeta}{\mu^2} \right) \right) + O(\alpha_s^2)
\]

Such that the perturbative expression for the Sudakov factor becomes:

\[
S_p(b, Q) = \frac{C_F}{\pi} \int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \left( \ln \frac{Q^2}{\mu^2} - \frac{3}{2} \right) + O(\alpha_s^2)
\]

It can be used whenever the restriction $b^2 << 1/\Lambda^2$ is justified (e.g. at very large $Q^2$)

If also larger $b$ contributions are important, at moderate $Q$ and small $Q_T$ for instance, then one needs to include a nonperturbative Sudakov factor
Nonperturbative Sudakov factor

\[ \tilde{W}(b) \equiv \tilde{W}(b_*) e^{-S_{NP}(b)} \quad b_* = b/\sqrt{1 + b^2/b_{\text{max}}^2} \leq b_{\text{max}} \]

\[ b_{\text{max}} = 1.5 \ \text{GeV}^{-1} \Rightarrow \alpha_s(b_0/b_{\text{max}}) = 0.62 \]

such that \( W(b_*) \) can be calculated within perturbation theory

In general the nonperturbative Sudakov factor is \( Q \) dependent and of the form:

\[ S_{NP}(b, Q) = \ln(Q^2/Q_0^2)g_1(b) + g_A(x_A, b) + g_B(x_B, b) \quad Q_0 = \frac{1}{b_{\text{max}}} \]

Collins, Soper & Sterman, NPB 250 (1985) 199

The \( g_\cdot \) functions need to be fitted to data

Until recently \( S_{NP} \) typically chosen as a Gaussian, e.g. Aybat & Rogers (\( x=0.1 \)):

\[ S_{NP}(b, Q, Q_0) = \left[ 0.184 \ln \frac{Q}{2Q_0} + 0.332 \right] b^2 \]

Recently alternatives considered in: P. Sun & F. Yuan, PRD 88 (2013) 034016


New form suggested by Collins (QCD evolution workshop 2013):

\[ e^{-m\left(\sqrt{b^2 + b_0^2} - b_0\right)} \]
Problem is to find one single universal $S_{NP}$ that describes both SIDIS and DY/Z data.
Further resummations

\[ \tilde{F}(x, b_T; \zeta_f, \mu_f) = \tilde{R}(b_T; \zeta_i, \mu_i, \zeta_f, \mu_f) \tilde{F}(x, b_T; \zeta_i, \mu_i) \]

\[ \tilde{R}(b_T; \zeta_i, \mu_i, \zeta_f, \mu_f) = \exp \left\{ \int_{\mu_i}^{\mu_f} \frac{d\mu}{\mu} \gamma_F \left( \alpha_s(\mu), \ln \frac{\zeta_f}{\mu^2} \right) \right\} \left( \frac{\zeta_f}{\zeta_i} \right)^{-D(b_T; \mu_i)} \]

\[ D(b_T, \mu) = -\frac{1}{2} \tilde{K}(b_T, \mu) \]

\[ \frac{dD(b_T, \mu)}{d \ln \mu} = \Gamma_{\text{cusp}} = \frac{1}{2} \gamma_K \]

\[ D^R(b_T; \mu) = -\frac{\Gamma_0}{2\beta_0} \ln(1 - X) + \frac{1}{2} \left( \frac{a_s}{1 - X} \right) \left[ \frac{\beta_1 \Gamma_0}{\beta_0^2} (X + \ln(1 - X)) + \frac{\Gamma_1}{\beta_0} X \right] \]

\[ + \frac{1}{2} \left( \frac{a_s}{1 - X} \right)^2 \left[ 2d_2(0) + \frac{\Gamma_2}{2\beta_0} (X(2 - X)) + \frac{\beta_1 \Gamma_1}{2\beta_0^2} (X(X - 2) - 2\ln(1 - X)) + \frac{\beta_2 \Gamma_1}{2\beta_0^2} X^2 \right] \]

\[ + \frac{\beta_1^2 \Gamma_0}{2\beta_0^3} (\ln^2(1 - X) - X^2) \]

where we have used the notation

\[ a_s = \frac{\alpha_s(\mu)}{4\pi}, \quad X = a_s \beta_0 L_T, \quad L_T = \ln \frac{\mu^2 b_T^2}{4e^{-2\gamma_E}} = \ln \frac{\mu^2}{\mu_b^2} \]
Convergence fails as \( b \) approaches \( b_X \) which to leading order is

\[
b_X = \frac{C_1}{\mu_i} \exp\left(\frac{2\pi}{\beta_0 \alpha_s(\mu_i)}\right)
\]

\[Q_i = \sqrt{2.4} \text{ GeV}\]

**Fig. 1** Resummed \( D \) at \( Q_i = \sqrt{2.4} \text{ GeV} \) with \( n_f = 4 \) (a) and \( Q_i = 5 \text{ GeV} \) with \( n_f = 5 \) (b)

\[b_X \approx 7 \text{ GeV}^{-1}\]

\[b_X \approx 11 \text{ GeV}^{-1}\]

\[D^{\text{CSS}}(\text{LL})\]

\[D^{\text{R}}(\text{LL})\]

**Fig. 3** Resummed \( D(b; Q_i = \sqrt{2.4}) \) at LL of Eqs. (25), (a), and (26), (b), with the running of the strong coupling at various orders and decoupling coefficients included
This approach favors $b_{\text{max}} = 1.5 \text{ GeV}^{-1}$
Further resummations

Resummed TMD at low scales very small at large $b_T$ where $\alpha_s(\mu_b)$ is very large
New approach to Landau pole problem

\[ u(x=0.01, b_T) \]

\[ Q_i = 2 \text{ GeV} \]

Sensitivity to Landau pole minimized by using \( Q_i = Q_0 + q_T \) rather than \( \mu_b \)

Correspondingly a new \( F^{NP} \) form is considered

High \( Q \) data (DY/Z) need only \( \lambda_1 \) & \( \lambda_2 \)

Low \( Q \) (SIDIS) needs modification (\( \lambda_3 \))

\[ \tilde{F}_{q/N}^{NP}(x, b_T; Q) = e^{-\lambda_1 b_T} \left( 1 + \lambda_2 b_T^2 \right) \left( \frac{Q^2}{Q_0^2} \right)^{-\frac{\lambda_3}{2} b_T^2} \]

TMD evolution
Large $p_T$ tail

Factorization dictates the evolution:

Under evolution TMDs develop a power law tail

Up Quark TMD PDF, $x = .09, Q = 91.19$ GeV

\[ F_{u p}(x=0.09,k_T) (\text{GeV}^{-2}) \]

\[ b_{T,max} = 0.5 \text{ GeV}^{-1} \]

\[ b_{T,max} = 1.5 \text{ GeV}^{-1} \]

\[ Q = 91.19 \text{ GeV} \]

Gaussian Fit

Aybat & Rogers, PRD 83 (2011) 114042
Evolution of Sivers function

TMDs and their asymmetries become broader and smaller with increasing energy

Power law tail
Comparing TMD and DGLAP evolution

Anselmino, Boglione, Melis
PRD 86 (2012) 014028

All curves evolved from $Q^2 = 1 \text{ GeV}^2$

Makes quite a difference in this limited range of $Q$: from 1.5 to 4.5 GeV

$S_{NP}$ dominates evolution
TMD evolution of azimuthal asymmetries

• Sivers effect in SIDIS and DY
  [Idilbi, Ji, Ma & Yuan, 2004; Aybat, Prokudin & Rogers, 2012; Anselmino, Boglione, Melis, 2012;

• Collins effect in e^+e^- and SIDIS

• Sivers effect in J/ψ production
  [Godbole, Misra, Mukherjee, Rawoot, 2013; Godbole, Kaushik, Misra, Rawoot, 2014]

Main differences among the various approaches:
- treatment of nonperturbative Sudakov factor
- treatment of leading logarithms, i.e. the level of perturbative accuracy
TMD evolution of the Sivers asymmetry
Sivers Asymmetry

HERMES data \((Q^2) \sim 2.4 \text{ GeV}^2\) mostly above COMPASS data \((Q^2) \sim 3.8 \text{ GeV}^2\)
Evolution of the Sivers Asymmetry

Evolution from HERMES to COMPASS energy scale seems to work well

Aybat, Prokudin & Rogers, PRL 108 (2012) 242003

This is obtained using the 2011 TMD factorization, including some approximations that should be applicable at small $Q$:

- Y term is dropped (or equivalently the perturbative tail)
- evolve from a fixed starting $Q_0$ rather than $\mu_b$
- Gaussian TMDs at starting scale $Q_0$
TMD evolution of the Sivers asymmetry

Under very similar assumptions, the $Q$ dependence of the Sivers asymmetry resides in an overall factor:

$$A_{UT}^{\sin(\phi_h - \phi_S)} \propto A(Q_T, Q)$$

Observations:
- the peak of the Sivers asymmetry decreases as $1/Q^{0.7 \pm 0.1}$ (“Sudakov suppression”)
- the peak of the asymmetry shifts slowly towards higher $Q_T$, also offers a test

Testing these features needs a larger $Q$ range, requiring a high-energy EIC
TMD evolution of the Sivers asymmetry

Both approaches use the same formalism (2011 TMD factorization), very similar approximations and ingredients, the key difference is in the integration over $x, z, p_{h\perp}$.

The *integrated* asymmetry falls off fast, not of form $1/Q^\alpha$, but in the considered range it falls off faster than $1/Q$ but slower than $1/Q^2$.
At low $Q^2$ (up to $\sim$20 GeV$^2$), the $Q^2$ evolution is dominated by $S_{NP}$

[Anselmino, Boglione, Melis, PRD 86 (2012) 014028]

Precise low $Q^2$ data can help to determine the form and size of $S_{NP}$

Uncertainty in $S_{NP}$ determines the $\pm0.1$ in $1/Q^{0.7\pm0.1}$
TMD evolution of Collins asymmetries
Collins Effect

Collins effect is described by a TMD fragmentation function:

\[ H_1^\perp = T - S_T \]

[NPB 396 (1993) 161]
Collins Effect

Collins effect is described by a TMD fragmentation function:

\[ d \sigma(e p^\uparrow \rightarrow e' \pi X) \propto \left\{ 1 + |S_T| \sin(\phi_\pi - \phi_S) f_{1T} D_1 + |S_T| \sin(\phi_\pi + \phi_S) h_1 H_1 \right\} \]
Collins effect is described by a TMD fragmentation function:

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It gives rise to a \( \sin(\varphi_h + \varphi_S) \) asymmetry in SIDIS:

\[ H^\perp_1 = \pi \rightarrow k_T \]

transversity \( \otimes \) Collins function
Collins Asymmetry in SIDIS

No clear need for TMD evolution from HERMES to COMPASS
Double Collins Effect

The Collins fragmentation function provides a way to probe transversity ($h_1$), if measured independently in another process.

Double Collins effect gives rise to a $\cos 2\varphi$ asymmetry in $e^+e^- \rightarrow h_1 h_2 X$
[D.B., Jakob, Mulders, NPB 504 (1997) 345]

Clearly observed in experiment by BELLE (R. Seidl et al., PRL '06; PRD '08) and BaBar (I. Garzia at Transversity 2011 & J.P. Lees et al., arXiv:1309.527)
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Double Collins Asymmetry

\[
\frac{d\sigma(e^+e^- \rightarrow h_1h_2X)}{dz_1 dz_2 d\Omega d^2q_T} \propto \{1 + \cos 2\phi_1 A(q_T)\}
\]

Under similar assumptions as for the Sivers asymmetry:

\[
A(Q_T) = \frac{\sum_a e_a^2 \sin^2 \theta H_1^{(1)a}(z_1; Q_0) \overline{H}_1^{(1)a}(z_2; Q_0)}{\sum_b e_b^2 (1 + \cos^2 \theta) D_1^b(z_1; Q_0) \overline{D}_1^b(z_2; Q_0)} A(Q_T)
\]

Considerable Sudakov suppression \( \sim 1/Q \) (effectively twist-3)

Next steps

Peak of the asymmetry shifts slowly towards higher $Q_T$, offers a test

Data from charm factory (BEPC) important by providing data around $Q \approx 4$ GeV
Next steps

Peak of the asymmetry shifts slowly towards higher $Q_T$, offers a test

Data from charm factory (BEPC) important by providing data around $Q \approx 4 \text{ GeV}$

Peak at BELLE/BaBar around 2.1 GeV
Next steps

Peak of the asymmetry shifts slowly towards higher $Q_T$, offers a test

Data from charm factory (BEPC) important by providing data around $Q \approx 4$ GeV

The $1/Q$ behavior should modify the transversity extraction using Collins effect, full TMD evolution still to be implemented (for $Q \sim 10$ GeV $S_{pert}$ is important)

Need to check the TMD evolution of the Collins asymmetry in SIDIS, which is slower than that of the double Collins asymmetry (Jefferson Lab & possibly EIC)
Double Collins Asymmetry

Data from BES important by providing data at lower $Q$

One does have to worry about $1/Q^2$ corrections (analogue of the Cahn effect), which can be bounded by study simultaneously the $1/Q \cos \phi$ asymmetry.

P. Sun & F. Yuan, PRD 88 (2013) 034016

Higgs transverse momentum distribution
The transverse momentum distribution in Higgs production at LHC is also a TMD factorizing process

P. Sun, B.-W. Xiao & F. Yuan, PRD 84 (2011) 094005

In this case starting the evolution from a fixed scale $Q_0$ is not appropriate due to the large $Q/Q_0$ ratio

The linear polarization of gluons inside the unpolarized protons plays a role

[Catani & Grazzini, ’10; D.B., Den Dunnen, Pisano, Schlegel, Vogelsang, ’12]
TMD factorization expressions

\[
\frac{d\sigma}{dx_A dx_B d\Omega d^2 q_T} = \int d^2 b \, e^{-i b \cdot q_T} \tilde{W}(b, Q; x_A, x_B) + \mathcal{O} \left( \frac{Q_T^2}{Q^2} \right)
\]

\[
\tilde{W}(b, Q; x_A, x_B) = \tilde{f}_1^g(x_A, b^2; \zeta_A, \mu) \tilde{f}_1^g(x_B, b^2; \zeta_B, \mu) H(Q; \mu)
\]
TMD factorization expressions

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\]

\[
\tilde{W} (b, Q; x_A, x_B) = f_1^g (x_A, b^2; \zeta_A, \mu) f_1^g (x_B, b^2; \zeta_B, \mu) H (Q; \mu)
\]

This is a naive expression, since gluons can be polarized inside unpolarized protons [Mulders, Rodrigues '01]

\[
\Phi^\mu_\nu (x, p_T) = \frac{n_\rho n_\sigma}{(p \cdot n)^2} \int \frac{d(\xi \cdot P) \, d^2 \xi_T}{(2\pi)^3} \, e^{ip \cdot \xi} \langle P | \text{Tr} \left[ F^{\mu\rho}(0) F^{\nu\sigma}(\xi) \right] | P \rangle \bigg|_{\text{LF}}
\]

\[
= - \frac{1}{2x} \left\{ g^\mu_\nu f_1^g - \left( \frac{p^\mu_\rho p^\nu_\sigma}{M^2} + g^\mu_\nu \, \frac{p_T^2}{2M^2} \right) h_1^g \right\}
\]

Second term requires nonzero k_T, but is k_T even, chiral even and T even

\[
\tilde{\Phi}_g^{ij} (x, b) = \frac{1}{2x} \left\{ \delta^{ij} \tilde{f}_1^g (x, b^2) - \left( \frac{2b^i b^j}{b^2} - \delta^{ij} \right) \tilde{h}_1^g (x, b^2) \right\}
\]
Cross section

\[
\left. \frac{E \, d\sigma^{pp \rightarrow HX}}{d^3 \vec{q}} \right|_{q_T \ll m_H} = \frac{\pi \sqrt{2} G_F}{128 m_H^2 s} \left( \frac{\alpha_s}{4\pi} \right)^2 |A_H(\tau)|^2 \\
\times \left( C \left[ f_1^g f_1^g \right] + C \left[ w_H h_1^g h_1^g \right] \right) + \mathcal{O} \left( \frac{q_T}{m_H} \right)
\]

\[
w_H = \frac{(p_T \cdot k_T)^2 - \frac{1}{2} p_T^2 k_T^2}{2 M^4}
\]

\[
\tau = \frac{m_H^2}{(4 m_t^2)}
\]

The relative effect of linearly polarized gluons:

\[
\mathcal{R}(Q_T) \equiv \frac{C[w_H h_1^g h_1^g]}{C[f_1^g f_1^g]}
\]

\[
\mathcal{R}(Q_T) = \frac{\int d^2 b \, e^{i b \cdot q_T} e^{-S_A(b_*, Q) - S_{NP}(b, Q)} \tilde{h}_1^g(x_A, b_2^*; \mu_{b*}^2, \mu_{b*}) \tilde{h}_1^g(x_B, b_2^*; \mu_{b*}^2, \mu_{b*})}{\int d^2 b \, e^{i b \cdot q_T} e^{-S_A(b_*, Q) - S_{NP}(b, Q)} \tilde{f}_1^g(x_A, b_2^*; \mu_{b*}^2, \mu_{b*}) \tilde{f}_1^g(x_B, b_2^*; \mu_{b*}^2, \mu_{b*})}
\]
Consider now only the perturbative tails:

\[ \tilde{f}_1^g(x, b^2; \mu_b^2, \mu_b) = f_{g/P}(x; \mu_b) + \mathcal{O}(\alpha_s) \]

\[ \tilde{h}_1^{\perp g}(x, b^2; \mu_b^2, \mu_b) = \frac{\alpha_s(\mu_b) C_A}{2\pi} \int_1^x \frac{d\hat{x}}{\hat{x}} \left( \frac{\hat{x}}{x} - 1 \right) f_{g/P}(\hat{x}; \mu_b) + \mathcal{O}(\alpha_s^2) \]

This coincides with the CSS approach

[Nadolsky, Balazs, Berger, C.-P. Yuan, '07; Catani, Grazzini, '10; P. Sun, B.-W. Xiao, F. Yuan, '11]

**Improved resummation prediction on Higgs boson production at hadron colliders**

Jian Wang,¹ Chong Sheng Li,¹,²,* Hai Tao Li,¹ Zhao Li,³,† and C.-P. Yuan²,3,‡

They find permille level effects at the Higgs scale, but using the TMD approach at the LL level yields percent level effects

D.B. & den Dunnen, NPB 886 (2014) 421

Wang et al. also use a different $S_{NP}$
TMD/CSS evolution effects

\[ R(Q_T) \]

\[ Q_T \text{ [GeV]} \]

\[ x_A = x_B = Q/(8\text{TeV}) \]

MSTW08 LO gluon distribution

\[ \text{D.B. & den Dunnen, NPB 886 (2014) 421} \]
Beyond CSS

In the TMD factorized expression there may be nonperturbative contributions from small $p_T$ which mainly affect large $b$

CSS only allows NP contribution via $S_{NP}$ and does not allow all possibilities of the TMD approach

To illustrate this we consider a model which is approximately Gaussian at low $p_T$ and has the correct tail at high $p_T$ or small $b$
Comparison

Tail-only (CSS)

Gauss+tail

Gaussian+tail evolves much more slowly than tail-only (CSS) expression
At low $Q$ there is quite some uncertainty from the very small $b$ region ($b \ll 1/Q$) where the perturbative expressions for $S_A$ are all incorrect (don’t satisfy $S(0)=0$).

**Standard regularization:**

\[
Q^2/\mu_b^2 = b^2 Q^2/b_0^2 \rightarrow Q^2/\mu_b'^2 \equiv (bQ/b_0 + 1)^2
\]
Very small $b$ region

For very small $b$ region ($b \ll 1/Q$) the perturbative expressions for $S_A$ are all incorrect

$$S_A(b, Q) = \frac{C_A}{\pi} \int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \ldots \overset{b \ll 1/Q}{\rightarrow} - \frac{C_A}{\pi} \int_{Q^2}^{\mu_b^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \ldots$$

Sudakov suppression ($e^{-\#}$) becomes an unphysical Sudakov enhancement ($e^{+\#}$)
For very small $b$ region ($b \ll 1/Q$) the perturbative expressions for $S_A$ are all incorrect

$$S_A(b, Q) = \frac{C_A}{\pi} \int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \left[ \ldots \right] \longrightarrow - \frac{C_A}{\pi} \int_{Q^2}^{\mu_b^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \left[ \ldots \right]$$

Sudakov suppression ($e^{-\#}$) becomes an unphysical Sudakov enhancement ($e^{+\#}$)

$$\frac{d\sigma}{dq_T^2} = Y(q_T^2) + \int \frac{d^2b}{4\pi} e^{-iq \cdot b} \sigma_0(1 + A) \exp S(b)$$

$$A_T^2 = A_T^2(y) = \frac{(S + Q^2)^2}{4S \cosh^2 y} - Q^2.$$
Higher twist
**P_T and Q^2-dep Higher Twist** $A_{LU}^{\sin \phi}$

Study for SSA transition from non-perturbative to perturbative regime.

Study for $Q^2$ dependence of beam SSA allows to check the higher twist nature and access quark-gluon correlations.
Conclusions
Conclusions

• Significant recent developments on TMD factorization and evolution:
  • New TMD factorization expressions by JCC (2011) & EIS (2012)
  • Improvements through additional resummations (Echevarria et al.) lifts analyses to the NNLL level (2013/4)
  • Progress towards describing SIDIS, DY & Z production data by a universal non-perturbative function (2013/4)

• Consequences of TMD evolution studied (in varying levels of accuracy) for:
  • Sivers & (single and double) Collins effect asymmetries
  • Higgs production including the effect of linear gluon polarization

• Future data from JLab12 and BES and perhaps a high-energy EIC can help to map out the Q dependence of Sivers and Collins asymmetries in greater detail

• Future data from LHC on Higgs and $\chi_{c/b0}$ production could do the same for gluon dominated TMD processes

• TMD (non-)factorization at next-to-leading twist remains entirely unexplored
Back-up slides
Further resummations

For the TMD at small $b$ one often considers the perturbative tail, which is calculable

$$\tilde{f}_{g/P}(x, b^2; \mu, \zeta) = \sum_{i=g,q} \int_x^1 \frac{d\hat{x}}{\hat{x}} C_{i/g}(x/\hat{x}, b^2; g(\mu), \mu, \zeta) f_{i/P}(\hat{x}; \mu) + \mathcal{O}((\Lambda_{QCD} b)^a)$$

To extend it to be valid at larger $b$ values one can perform further resummation:

$$\tilde{F}_{q/N}^{\text{pert}}(x, b_T; \zeta, \mu) = \left( \frac{\zeta b_T^2}{4 e^{-2\gamma_E}} \right)^{-D_R(b_T;\mu)} e^{h_R(b_T;\mu) - h_R(b_T;\mu)} \sum_j \int_x^1 \frac{dz}{z} \hat{C}_{q-j}(x/z, b_T; \mu) f_{j/N}(z; \mu)$$

$$\tilde{F}_{q/N}(x, b_T; Q_i^2, \mu_i) = \tilde{F}_{q/N}^{\text{pert}}(x, b_T; Q_i^2, \mu_i) \tilde{F}_{q/N}^{\text{NP}}(x, b_T; Q_i)$$

$$\tilde{F}_{q/N}^{\text{NP}}(x, b_T; Q_i) \equiv \tilde{F}_{q/N}^{\text{NP}}(x, b_T) \left( \frac{Q_i^2}{Q_0^2} \right)^{-D_{NP}(b_T)}$$

Tool to compare different methods: The $L$ function

(JCC & Rogers, in preparation)

- Shape change of transverse momentum distribution comes only from $b_T$-dependence of $\tilde{K}$

- So define scheme independent

$$L(b_T) = -\frac{\partial}{\partial \ln b_T^2} \frac{\partial}{\partial \ln Q^2} \ln \tilde{W}(b_T, Q, x_A, x_B)^{\text{CSS}} = -\frac{\partial}{\partial \ln b_T^2} \tilde{K}(b_T, \mu)$$

- QCD predicts it is
  - independent of $Q, x_A, x_B$
  - independent of light-quark flavor
  - RG invariant
  - perturbatively calculable at small $b_T$
  - non-perturbative at large $b_T$

Collins, QCD Evolution workshop, May 12, 2014

$L$ is called $A$ in Collins, 1409.5408
Comparing different results using the $L$ function

(Preliminary)

<table>
<thead>
<tr>
<th>$Q$</th>
<th>Typical $b_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 GeV</td>
<td>3 GeV$^{-1}$</td>
</tr>
<tr>
<td>10 GeV</td>
<td>1.2 GeV$^{-1}$</td>
</tr>
<tr>
<td>$m_Z$</td>
<td>0.5 GeV$^{-1}$</td>
</tr>
</tbody>
</table>

SY = Sun & Yuan (PRD 88, 114012 (2013)):

$$L_{SY} = C_F \frac{\alpha_s(Q)}{\pi}$$

Depends on $Q$: contrary to QCD
Sivers asymmetry expression

\[ A_{ab}(x, z, Q_T) = \frac{\int db b^2 J_1(bQ_T) \tilde{f}^a_{1T}(x, b_2^2; Q_0^2, Q_0) \tilde{D}_1^a(z, b_2^2; Q_0^2, Q_0) \exp(-S_p(b_*, Q, Q_0) - S_{NP}(b, Q/Q_0))}{MQ_T \int db J_0(bQ_T) f^b_1(x, b_2^2; Q_0^2, Q_0) D^b_1(z, b_2^2; Q_0^2, Q_0) \exp(-S_p(b_*, Q, Q_0) - S_{NP}(b, Q/Q_0))} \]

Assume that the TMDs of \( b^* \) are slowly varying functions of \( b \) in the dominant \( b \) region (\( b \sim 1/Q_T \gg 1/Q \), hence \( b^* \approx b_{\text{max}} = 1/Q_0 \)): \( \Phi(x, b^*) \approx \Phi(z, 1/Q_0) \)

This approximation means dropping the perturbative tail of TMDs and leads to a decoupling of \( x \) and \( b \) dependence.

\[ A_{ab}(x, z, Q_T) = \frac{\int db b^2 J_1(bQ_T) \tilde{f}^a_{1T}(x; Q_0) D^a_1(z; Q_0)}{M^2 f^b_1(x; Q_0) D^b_1(z; Q_0) A(Q_T)} \]
TMD factorization expressions

Differential cross section of $e^+ e^- \rightarrow h_1 h_2 X$ process at small $Q_T$:

$$\frac{d\sigma}{dz_1 dz_2 d\Omega d^2 q_T} = \int d^2 b \ e^{-ib \cdot q_T} \tilde{W}(b, Q; z_1, z_2) + O \left( \frac{Q_T^2}{Q^2} \right)$$

$$\tilde{W}(b, Q; z_1, z_2) = \sum_{a,b} \tilde{D}_1^a(z_1, b; Q_0, \alpha_s(Q_0)) \tilde{D}_1^b(z_2, b; Q_0, \alpha_s(Q_0))$$

$$\times e^{-S(b, Q, Q_0)} \ H_{ab}(Q; \alpha_s(Q))$$

TMDs are taken at a fixed scale $Q_0$, the smallest perturbative scale

$$\tilde{\Delta}(z, b) = \frac{M}{4} \left\{ \tilde{D}_1(z, b^2) \frac{P}{M} + \left( \frac{\partial}{\partial b^2} \tilde{H}_1^+(z, b^2) \right) \frac{2}{M^2} \right\}$$

Assume that the TMDs of $b^*$ are slowly varying functions of $b$ in the dominant $b$ region ($b \sim 1/Q_T \gg 1/Q$, hence $b^* \approx b_{max} = 1/Q_0$): $\Delta(z, b^*) \approx \Delta(z, 1/Q_0)$

This approximation means dropping the perturbative tails and leads to a decoupling of $z$ and $b$ dependence (gives same result for SIDIS & DY)
Double Collins Asymmetry

\[
\frac{d\sigma(e^+e^- \to h_1 h_2 X)}{dz_1 dz_2 d\Omega d^2q_T} \propto \{1 + \cos 2\phi_1 A(q_T)\}
\]

\[
A(Q_T) = \sum_a e^2_a \sin^2 \theta \frac{H_{1\perp}^{a}(z_1; Q_0) \bar{H}_{1\perp}^{(1)a}(z_2; Q_0)}{\sqrt{\sum_b e^2_b (1 + \cos^2 \theta) D^b_1(z_1; Q_0) \bar{D}^b_1(z_2; Q_0)}} A(Q_T)
\]

\[
A(Q_T) = M^2 \int db b^3 J_2(bQ_T) \exp \left(-S_p(b_*, Q, Q_0) - S_{NP}(b, Q/Q_0)\right) \\
\int db b J_0(bQ_T) \exp \left(-S_p(b_*, Q, Q_0) - S_{NP}(b, Q/Q_0)\right)
\]

Considerable Sudakov suppression $\sim 1/Q$ (effectively twist-3)

Tree level expression

\[
\frac{E \, d\sigma^{pp \to HX}}{d^3 \vec{q}} \bigg|_{q_T \ll m_H} = \frac{\pi \sqrt{2} G_F}{128 m_H^2 s} \left( \frac{\alpha_s}{4\pi} \right)^2 |A_H(\tau)|^2 \\
\times \left( \mathcal{C} \left[ f_1^g f_1^g \right] + \mathcal{C} \left[ w_H h_1^g h_1^g \right] \right) + \mathcal{O} \left( \frac{q_T}{m_H} \right)
\]

The gluon TMDs enter in convolutions:

\[
\mathcal{C}[w f f] \equiv \int d^2 p_T \int d^2 k_T \, \delta^2(p_T + k_T - q_T) \, w(p_T, k_T) f(x_A, p_T^2) f(x_B, k_T^2)
\]

\[
w_H = \frac{(p_T \cdot k_T)^2 - \frac{1}{2} p_T^2 k_T^2}{2 M^4}
\]

\[
\tau = m_H^2 / (4 m_t^2)
\]

The relative effect of linearly polarized gluons:

\[
\mathcal{R}(Q_T) \equiv \frac{\mathcal{C}[w_H h_1^g h_1^g]}{\mathcal{C}[f_1^g f_1^g]}
\]
Beyond tree level

\[ C \left[ f_1^g f_1^g \right] = \int \frac{d^2 b}{(2\pi)^2} e^{i b \cdot q_T} \tilde{f}_1^g (x_A, b^2; \zeta_A, \mu) \tilde{f}_1^g (x_B, b^2; \zeta_B, \mu) \]

\[ = \int \frac{d^2 b}{(2\pi)^2} e^{i b \cdot q_T} e^{-S_A(b, Q)} \tilde{f}_1^g (x_A, b^2; \mu_b^2, \mu_b) \tilde{f}_1^g (x_B, b^2; \mu_b^2, \mu_b) \]

Perturbative Sudakov factor:

\[ S_A(b, Q) = \frac{C_A}{\pi} \int_{\mu_b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \alpha_s(\mu) \left[ \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{11 - 2n_f/C_A}{6} \right] + \mathcal{O}(\alpha_s^2) \]

\[ = -\frac{36}{33 - 2n_f} \left[ \ln \left( \frac{Q^2}{\mu_b^2} \right) + \ln \left( \frac{Q^2}{\Lambda^2} \right) \ln \left( 1 - \frac{\ln (Q^2/\mu_b^2)}{\ln (Q^2/\Lambda^2)} \right) \right. \]

\[ + \left. \frac{11 - 2n_f/C_A}{6} \ln \left( \frac{\ln (Q^2/\Lambda^2)}{\ln (\mu_b^2/\Lambda^2)} \right) \right] \]
\[
\frac{d\sigma}{dx_A dx_B d\Omega d^2q_T} = \int d^2b \ e^{-ib \cdot q_T} \tilde{W}(b, Q; x_A, x_B) + O\left(\frac{Q_T^2}{Q^2}\right)
\]

The integral is over all \(b\), including nonperturbatively large \(b\)

\[\tilde{W}(b) \equiv \tilde{W}(b_\star) \ e^{-S_{NP}(b)}\]

\[b_\star = b / \sqrt{1 + b^2 / b_{\text{max}}^2} \leq b_{\text{max}}\]

\[b_{\text{max}} = 1.5 \ \text{GeV}^{-1} \Rightarrow \alpha_s(b_0 / b_{\text{max}}) = 0.62\]

No extraction of \(S_{NP}\) exists, e.g. use a modified Aybat-Rogers \(S_{NP}\)

\[
S_{NP}(b, Q, Q_0) = \frac{C_A}{C_F} \left[0.184 \ln \frac{Q}{2Q_0} + 0.332\right] b^2
\]
\[ \mathcal{R}(Q_T) = \frac{\int d^2b \, e^{ib \cdot q_T} e^{-S_A(b_*,Q) - S_{NP}(b,Q)} \, \tilde{h}_1^g(x_A, b_2^2, \mu_{b_\ast}^2, \mu_{b_\ast}) \, \tilde{h}_1^g(x_B, b_2^2, \mu_{b_\ast}^2, \mu_{b_\ast})}{\int d^2b \, e^{ib \cdot q_T} e^{-S_A(b_*,Q) - S_{NP}(b,Q)} \, \tilde{f}_1^g(x_A, b_2^2, \mu_{b_\ast}^2, \mu_{b_\ast}) \, \tilde{f}_1^g(x_B, b_2^2, \mu_{b_\ast}^2, \mu_{b_\ast})} \]

\[ \tilde{h}_1^g(x, b^2) = \int d^2p_T \, \frac{(b \cdot p_T)^2 - \frac{1}{2} b^2 p_T^2}{b^2 M^2} \, e^{-ib \cdot p_T} \, h_1^g(x, p_T^2) \]

\[ = -\pi \int dp_T^2 \, \frac{p_T^2}{2M^2} \, J_2(bp_T) h_1^g(x, p_T^2) \]

Consider now only the perturbative tails:

\[ \tilde{f}_1^g(x, b^2; \mu_b^2, \mu_b) = f_{g/P}(x; \mu_b) + \mathcal{O}(\alpha_s) \]

\[ \tilde{h}_1^g(x, b^2; \mu_b^2, \mu_b) = \frac{\alpha_s(\mu_b) C_A}{2\pi} \int_x^1 \frac{d\hat{x}}{\hat{x}} \left( \frac{\hat{x}}{x} - 1 \right) f_{g/P}(\hat{x}; \mu_b) + \mathcal{O}(\alpha_s^2) \]

This coincides with the CSS approach

[Nadolsky, Balazs, Berger, C.-P. Yuan, ’07; Catani, Grazzini, ’10; P. Sun, B.-W. Xiao, F. Yuan, ’11]
FIG. 3. The ratios between the transverse momentum distributions with and without $G$ functions at the Tevatron (1.96 TeV) and the LHC (7, 8, and 14 TeV). The oscillations of the ratio curves in the figure are due to numerical uncertainties.
They include third factor, but not second
May explain suppression partly

Wang et al. use also different $S_{NP}$
Beyond CSS

In the TMD factorized expression there may be nonperturbative contributions from small $p_T$ which mainly affect large $b$

The perturbative tail holds for small $b$ which is dominated by large $p_T$, but there is an intermediate region

CSS only allows NP contribution via $S_{NP}$ and does not allow all possibilities of the TMD approach

To illustrate this we consider a model which is approximately Gaussian at low $p_T$ and has the correct tail at high $p_T$ or small $b$:

$$ f_1^g(x, p_T^2) = f_1^g(x) \frac{R^2}{2\pi} \frac{1}{1 + p_T^2 R^2} \quad R = 2 \text{ GeV}^{-1} $$

$$ h_1^g(x, p_T^2) = c f_1^g(x) \frac{M^2 R_h^4}{2\pi} \frac{1}{(1 + p_T^2 R_h^2)^2} $$

To satisfy Soffer-like bound:

$$ R_h^2 = \frac{3R^2}{2} \quad c = 2 $$
Gaussian+tail model

\[\tilde{f}_1^g(x, b^2; \mu_b^2, \mu_b) = f_1^g(x; \mu_b) K_0(b/R) / \ln(Rb_0/b + 1)\]

\[\tilde{h}_1^{1/2}g(x, b^2; \mu_b^2, \mu_b) = \frac{c}{4} f_1^g(x; \mu_b) \frac{b}{R_h} K_1(b/R_h) / \ln(R_h b_0/b + 1)\]