Wigner distributions and OAM of quarks in light-front dressed quark model

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SPIN2014 October 20-24, 2014, Beijing, China
Outline

1. Introduction to Wigner Function
2. Wigner Distributions of quarks
3. Plots of the Numerical results
4. GTMDs and Orbital Angular Momentum
5. Conclusion
In 1932, E.P. Wigner in an attempt to study quantum corrections to classical statistical mechanics provided a probability distribution for $x$ and $p$ representing a quantum state.

It's not a genuine phase space distribution function because of HUP and the fact that it could also have negative values.
Introduction to Wigner Function

\[ W(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\zeta \exp\left(-\frac{i}{\hbar} p\zeta\right) \Psi^*(x - \frac{\zeta}{2}) \Psi(x + \frac{\zeta}{2}) \]

\( \psi(x) = e^{-x^2} \)

\( \psi_1(x) = e^{-x^2} H_1(x) \).
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Defination: Wigner Distributions of quarks

Wigner distributions for the quarks are defined as

\[ \rho^{[\Gamma]}(\vec{b}_\perp, \vec{k}_\perp, x, \vec{\sigma}) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i \Delta_\perp \cdot \vec{b}_\perp} W^{[\Gamma]}(\Delta_\perp, \vec{k}_\perp, x, \vec{\sigma}) \]

\( \Delta_\perp \Rightarrow \) momentum transfer of nucleon in transverse direction

\( \vec{b}_\perp \Rightarrow 2D \) vector in impact parameter space.

\( W^{[\Gamma]} \) is the quark-quark correlator given by

\[ W^{[\Gamma]}(\Delta_\perp, \vec{k}_\perp, x, \vec{\sigma}) = \frac{1}{2} \int \frac{dz^- d^2 z_\perp}{(2\pi)^3} e^{i(xp^+ z^- - \vec{k}_\perp \cdot \vec{z}_\perp)} \langle p^+, \frac{\Delta_\perp}{2}, \vec{\sigma} | \bar{\psi}(-\frac{z}{2}) \Omega \Gamma \psi(\frac{z}{2}) | p^+, -\frac{\Delta_\perp}{2}, \vec{\sigma} \rangle \bigg|_{z^+=0} \]

\( \Omega \Rightarrow \) gauge link for color gauge invariance \( \Rightarrow \) we use light-front gauge

and \( s \) the gauge link to be unity

\( \Gamma \Rightarrow \) Dirac matrix defining the types of quark densities
Wigner Distributions in dressed quark model

In this model we consider the target to be quark state dressed with a gluon

We calculate the quark-quark correlator by using the Fock space expansion of the dressed quark state

\[ |p^+, p_\perp, \sigma\rangle = \Phi^\sigma(p)b_\sigma^\dagger(p)|0\rangle + \sum_{\sigma_1\sigma_2} \int [dp_1] \int [dp_2] \sqrt{16\pi^3}p^+\delta^3(p - p_1 - p_2) \]
\[ \Phi^\sigma_{\sigma_1\sigma_2}(p; p_1, p_2)b_{\sigma_1}^\dagger(p_1)a_{\sigma_2}^\dagger(p_2)|0\rangle \]

The two particle LFWF are given by

\[ \Psi^\sigma_{\sigma_1\sigma_2}(x, q_\perp) = \frac{1}{m^2 - \frac{m^2 + (q_\perp)^2}{x} - \frac{(q_\perp)^2}{1-x}} \frac{g}{\sqrt{2(2\pi)^3}} T^a \chi_{\sigma_1}^\dagger \frac{1}{\sqrt{1-x}} \]
\[ \left[ -2x - \frac{q_\perp}{1-x} - \frac{(\sigma_\perp \cdot q_\perp)\sigma_\perp}{x} + \frac{im\sigma_\perp(1-x)}{x} \right] \chi_{\sigma}(\epsilon_{\sigma_2})^\ast \]
Wigner Distributions in dressed quark model

The leading order contribution to the quark-quark correlator is given by

\[ W[\gamma^+](\Delta_\perp, k_\perp, x, \sigma) = \frac{1}{(2\pi)^3} \sum_{\sigma_1, \sigma_2, \lambda_1} \Psi^{*\sigma}_{\lambda_1\sigma_2}(x, q'_\perp) \chi_{\lambda_1}^\dagger \chi_{\sigma_1} \Psi^{\sigma}_{\sigma_1\sigma_2}(x, q_\perp) \]

\[ W[\gamma^+\gamma_5](\Delta_\perp, k_\perp, x, \sigma) = \frac{1}{(2\pi)^3} \sum_{\sigma_1, \sigma_2, \lambda_1} \Psi^{*\sigma}_{\lambda_1\sigma_2}(x, q'_\perp) \chi_{\lambda_1}^\dagger \sigma_3 \chi_{\sigma_1} \Psi^{\sigma}_{\sigma_1\sigma_2}(x, q_\perp) \]

where the Jacobi relation for the transverse momenta in the symmetric frame is given by \[ q'_\perp = k_\perp + \frac{\Delta_\perp}{2} (1 - x) \] and \[ q_\perp = k_\perp - \frac{\Delta_\perp}{2} (1 - x) \].
Wigner Distributions in dressed quark model

Wigner distribution of quarks with longitudinal polarization $\lambda$ in a nucleon with longitudinal polarization $\Lambda$ can be decomposed as:

$$
\rho_{\Lambda,\lambda}(\vec{b}_\perp, \vec{k}_\perp, x) = \frac{1}{2} \left[ \rho_{UU}(\vec{b}_\perp, \vec{k}_\perp, x) + \Lambda \rho_{LU}(\vec{b}_\perp, \vec{k}_\perp, x) + \lambda \rho_{UL}(\vec{b}_\perp, \vec{k}_\perp, x) + \Lambda \lambda \rho_{LL}(\vec{b}_\perp, \vec{k}_\perp, x) \right] \quad \text{C.Lorce, B.Pasquini(2011)}
$$

<table>
<thead>
<tr>
<th>Wigner distribution</th>
<th>Represents distortion</th>
<th>$\Gamma$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{UU}$</td>
<td>unpolarized quarks in unpolarized target</td>
<td>$\gamma^+$</td>
<td>$\uparrow\uparrow + \downarrow\downarrow$</td>
</tr>
<tr>
<td>$\rho_{LU}$</td>
<td>longitudinal target polarization</td>
<td>$\gamma^+$</td>
<td>$\uparrow\uparrow - \downarrow\downarrow$</td>
</tr>
<tr>
<td>$\rho_{UL}$</td>
<td>longitudinal quark polarization</td>
<td>$\gamma^+\gamma_5$</td>
<td>$\uparrow\uparrow + \downarrow\downarrow$</td>
</tr>
<tr>
<td>$\rho_{LL}$</td>
<td>correlation between quark &amp; target longitudinal polarizations</td>
<td>$\gamma^+\gamma_5$</td>
<td>$\uparrow\uparrow - \downarrow\downarrow$</td>
</tr>
</tbody>
</table>
Final expression for the Wigner Distributions in dressed quark model

\( (\rho_{LU} = \rho_{UL}) \) So the three independent Wigner distribution are:

\[
\rho^{[\gamma^+]}_{UU}(b_\perp, k_\perp, x) = \int d\Delta_x \int d\Delta_y \frac{\cos(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[ I_1 + \frac{4m^2(1-x)}{x^2} \right]
\]

\[
\rho^{[\gamma^+]}_{LU}(b_\perp, k_\perp, x) = \int d\Delta_x \int d\Delta_y \frac{\sin(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[ 4(k_x \Delta_y - k_y \Delta_x) \frac{(1+x)}{x^2(1-x)} \right]
\]

\[
\rho^{[\gamma^+\gamma_5]}_{LL}(b_\perp, k_\perp, x) = \int d\Delta_x \int d\Delta_y \frac{\cos(\Delta_\perp \cdot b_\perp)}{D(q_\perp)D(q'_\perp)} \left[ I_1 - \frac{4m^2(1-x)}{x^2} \right]
\]

where \( A_x, A_y \) are \( x, y \) component of \( A_\perp \).
the plot has a peak centered at $b_x = b_y = 0$ decreasing in the outer regions of the $b$ space.

The distribution value increases with increasing $\Delta_{max}$. 

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3D plots for $\rho_{UU}$ in $k$ space

- The behavior in the $k$ space is similar to that in the $b$ space but the peaks have negative values.
3D plots for $\rho_{UU}$ in mixed space

For the mixed space plots we have integrated out the $k_x$ and $b_y$ dependence giving us the probability densities correlating $k_y$ and $b_x$.

- we observe a minima at $b_x = 0, k_y = 0$.
- As $\Delta_{max}$ increases the minima gets deeper.
3D plots for $\rho_{LU}$ in $b$ space

- we observe a dipole structure in these plots.
- dipole magnitude increases with increase in $\Delta_{max}$. 

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we observe a dipole structure in these plots.

dipole magnitude increases with increase in $\Delta_{max}$. 

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We observe the quadrupole structure in the mixed space.

- The pole values increase with increasing $\Delta_{\max}$. 

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The quark-quark correlator defining the Wigner distributions can be parameterized in terms of generalized transverse momentum dependent parton distributions (GTMDs).

\[ W_{\lambda,\lambda'}^{[\gamma^+] - \bar{u}(p',\lambda') \left[ F_{1,1} + \frac{i\sigma^i + k^i}{P^+} F_{1,2} + \frac{i\sigma^i + \Delta^i}{P^+} F_{1,3} + \frac{i\sigma^{ij} k^i \Delta^j}{M^2} F_{1,4} \right] u(p,\lambda) \]

\[ W_{\lambda,\lambda'}^{[\gamma^+\gamma_5]} = \frac{\bar{u}(p',\lambda')}{2M} \left[ -i\epsilon^{ij} k^i \Delta^j \frac{\Delta^j}{M^2} G_{1,1} + \frac{i\sigma^i + \gamma_5 k^i}{P^+} G_{1,2} + \frac{i\sigma^i + \gamma_5 \Delta^i}{P^+} G_{1,3} 
+ i\sigma^{+-} \gamma_5 G_{1,4} \right] u(p,\lambda) \]

we analytically calculate these GTMDs for the dressed quark model.
The final expression for the 8 GTMDs

\[ F_{11} = - \frac{N \left[ 4k_\perp^2 (1 + x^2) + (x - 1)^2 (4M^2(x - 1)^2 - (1 + x^2)\Delta_{\perp}^2) \right]}{D(q_\perp)D(q'_\perp)2x^2(x - 1)^3} \]

\[ F_{12} = \frac{2NM^2\Delta_{\perp}^2}{D(q_\perp)D(q'_\perp)x(k_y\Delta_x - k_x\Delta_y)} \]

\[ F_{13} = \frac{N}{D(q_\perp)D(q'_\perp)4x(k_y\Delta_x - k_x\Delta_y)} \left[ -8M^2(k_\perp\Delta_{\perp}) - (k_x\Delta_y - k_y\Delta_x)(4k_\perp^2(1 + x^2) + (x - 1)^2(4M^2(x - 1)^2 - (1 + x^2)\Delta_{\perp}^2)) \right] \frac{x(x - 1)^3}{x(x - 1)^3} \]

\[ F_{14} = \frac{2NM^2(1 + x)}{D(q_\perp)D(q'_\perp)x^2(x - 1)} \]
\[ G_{11} = -\frac{2NM^2(1+x)}{D(q^\perp)D(q'^\perp)x^2(x-1)} \]

\[ G_{12} = \frac{-N}{D(q^\perp)D(q'^\perp)x(x-1)} \left[ 4M^2 \frac{k_\perp . \Delta_\perp}{(k_y \Delta_x - k_x \Delta_y)} - \frac{(1+x)\Delta_\perp^2}{x} \right] \]

\[ G_{13} = \frac{N \left[ (1+x) \left( \Delta_2^2 - \Delta_1^2 + \Delta_1 \Delta_2 (k_2^2 - k_1^2) \right) + 4xM^2k_\perp^2 \right]}{D(q^\perp)D(q'^\perp)x^2(x-1)(k_y \Delta_x - k_x \Delta_y)} \]

\[ G_{14} = \frac{N \left[ -4k_\perp^2 (1+x^2) + (x-1)^2 \left( 4M^2(x-1)^2 - (1+x^2)\Delta_\perp^2 \right) \right]}{D(q^\perp)D(q'^\perp)2x^2(x-1)^3} \]

where \( N = \frac{1}{2(2\pi^3)} \) is the normalization constant.
The sum rule shown in the seminal paper by Ji gives the quark orbital angular momentum (OAM) in terms of the GPDs for longitudinal polarization

\[ L_z^q = \frac{1}{2} \int dx \{ x[H^q(x, 0, 0) + E^q(x, 0, 0)] - \tilde{H}^q(x, 0, 0) \} \quad \text{X.D.Ji}(1997) \]

The GPDs in the above equation are related to the GTMDs by the following relation:

\[
\begin{align*}
H(x, 0, t) &= \int d^2k_\perp F_{11} \\
E(x, 0, t) &= \int d^2k_\perp \left[ - F_{11} + 2 \left( \frac{k_\perp \cdot \Delta_\perp}{\Delta_\perp^2} F_{12} + F_{13} \right) \right] \\
\tilde{H}(x, 0, t) &= \int d^2k_\perp G_{14} \quad \text{S.Meissner, A.Metz, M.Schlegel}(2009)
\end{align*}
\]
Using the GTMDs calculated we have the following final expression for the kinetic orbital angular momentum of quarks in the dressed quark model

\[
L_q^z = \frac{N}{2} \int dx \left\{ I_1 f(x) + I_2 m_q^2 (1 - x)^2 \left( 2(1 - x)^2 - f(x) \right) \right\}
\]

where,

\[
I_1 = \int \frac{d^2 k_\perp}{m_q^2 (1 - x)^2 + (k_\perp)^2} = \pi \log \left[ \frac{Q^2 + m_q^2 (1 - x)^2}{\mu^2 + m_q^2 (1 - x)^2} \right]
\]

\[
I_2 = \int \frac{d^2 k_\perp}{\left( m_q^2 (1 - x)^2 + (k_\perp)^2 \right)^2} = \frac{\pi}{m_q^2 (1 - x)^2}
\]

\[
f(x) = \frac{2(1 + x^2)(1 + x)}{(x - 1)}
\]
$F_{1,4}$ appears purely at the level of the GTMDs and is related to the canonical OAM

$$l^q_z = - \int dx d^2k_\perp \frac{k^2_\perp}{m_q^2} F_{14}$$

Y. Hatta and S. Yoshida (2012)
C.Lorce, B.Pasquini (2011)

The final expression for the canonical quark OAM in the dressed quark model

$$l^q_z = -2N \int dx (x^2 - 1) \left[ I_1 - m_q^2 (x - 1)^2 I_2 \right]$$

In agreement with
Harindranath and Kundu (1999); Hikmat and Burkardt (2012),
Kanazawa, Lorce, Metz, Pasquini, Schlegel (2014)
Spin-orbit correlation

The GTMD $G_{11}$ is related to the spin-orbit correlation factor

$$C^q_{\bar{z}} = \int dx d^2 k_\perp \frac{k_\perp^2}{m_q^2} G_{11}$$

In the dressed quark model we find that $F_{14} = -G_{11}$

- So in the dressed quark model we get $l^q_{\bar{z}} = C^q_{\bar{z}}$

In agreement with ⇒ Kanazawa, Lorce, Metz, Pasquini, Schlegel (2014)
Numerical results for OAM

- $Q$ is the upper limit in the momentum integration.
- Both the OAM decreases in magnitude with increasing mass.
- The magnitude of the two OAM differs in our model

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Conclusion

- We calculated the Wigner distributions of quark in the dressed quark model.

- We calculated the Wigner distributions both for unpolarized and longitudinally polarized target state and showed the correlations in transverse momentum and position space.

- We calculated the quark kinetic OAM and the canonical OAM in the dressed quark model and showed that they are different in magnitude.

- Future work would involve calculating the Wigner distributions for the gluons in a dressed quark state and also consider transverse polarization of the target state.
THANK YOU FOR LISTENING
\[ D(k_\perp) = \left( m^2 - \frac{m^2 + (k_\perp)^2}{x} - \frac{(k_\perp)^2}{1-x} \right) \]

\[ I_1 = 4 \left( (k_\perp)^2 - \frac{\Delta^2_\perp (1-x)^2}{4} \right) \]