

Working with Wilson Lines

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Outline

- 1 Wilson Line Exponentials
- 2 Linear Wilson Lines
- 3 Piecewise Linear Wilson Lines: Methodology
- 4 Example Calculations

Path-Ordered Exponentials

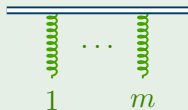
Wilson Line

$$\mathcal{U}[C] = \mathcal{P} \exp \left(ig \int_a^b d\lambda (z^\mu)' A_\mu(\lambda) \right)$$

Path-Ordered Exponentials

Wilson Line

$$U[C] = \mathcal{P} \exp \left(ig \int_a^b d\lambda (z^\mu)' A_\mu(\lambda) \right)$$



$$A(\lambda_m) \cdots A(\lambda_1)$$

Path-Ordering

Path-Ordering for Linear Lines

$$z^\mu = r^\mu + \hat{n}^\mu \lambda \quad \lambda = a \dots b \quad \lambda_m \geq \dots \geq \lambda_1$$

$$\begin{aligned} \mathcal{P} \int_c \dots \int_c dz_1 \dots dz_m &= m! \int_a^b \int_{\lambda_1}^b \dots \int_{\lambda_{m-1}}^b d\lambda_1 \dots d\lambda_m \\ &= m! \int_a^b \int_a^{\lambda_m} \dots \int_a^{\lambda_2} d\lambda_m \dots d\lambda_1 \end{aligned}$$

Motivation

Interest in Wilson Lines

- Singularity structure of TMD governed by underlying Wilson line structure
- T-odd non-universality effects due to different Wilson line structures
- Investigation of Wilson loops, to describe geometric evolution of TMDs



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Feynman Rules

Wilson Line Bounded From Below

$$\begin{aligned}
 \mathcal{U}_{(+\infty; r)} &= \sum_{m=0}^{\infty} (ig)^m \int \left(\frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(-k_m) \cdots \hat{n} \cdot A(-k_1) \times \cdots \\
 &\quad \times \int_0^{\infty} \int_{\lambda_1}^{\infty} \cdots \int_{\lambda_{n-1}}^{\infty} d\lambda_1 \cdots d\lambda_m e^{i(r+\hat{n} \lambda_1) \cdot k_1} \cdots e^{i(r+\hat{n} \lambda_m) \cdot k_m}
 \end{aligned}$$

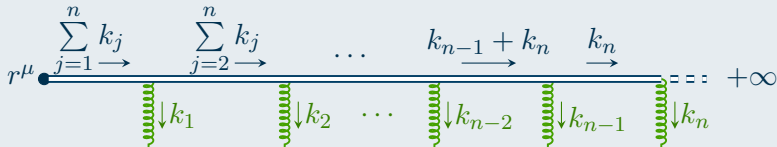
Feynman Rules

Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty; r)} = \sum_{m=0}^{\infty} (ig)^m \int \left(\frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(-k_m) \cdots \hat{n} \cdot A(-k_1) \times \cdots$$

$$K^\downarrow(j) = \sum_{l=j}^n k_l \quad \cdots \times e^{ir \cdot K} \prod_{j=1}^m \frac{i}{\hat{n} \cdot K^\downarrow(j) + i\eta}$$

Feynman Diagram



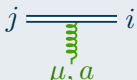
Feynman Rules

Feynman Rules for Linear Wilson Lines

1) Wilson line propagator: $\frac{\overrightarrow{k}}{\text{====}}$ = $\frac{i}{\hat{n} \cdot k + i\eta}$

2) external point: $r^\mu \bullet \frac{\overrightarrow{k}}{\text{====}}$ = $e^{ir \cdot k}$

3) infinite point: $\text{=====} + \infty$ = $1 \quad (k = 0)$

4) Wilson vertex: $j \text{====} i$ = $ig \hat{n}^\mu (t^a)_{ij}$


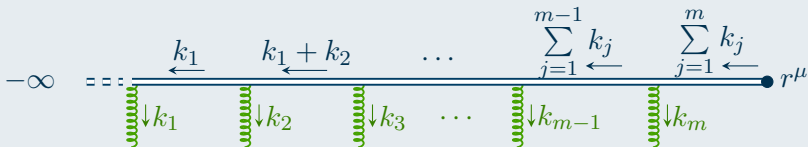
Feynman Rules

Wilson Line Bounded From Above

$$\mathcal{U}_{(r; -\infty)} = \sum_{m=0}^{\infty} (ig)^m \int \left(\frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(-k_m) \cdots \hat{n} \cdot A(-k_1) \times$$

$$K^\dagger(j) = \sum_{l=1}^j k_l \quad \times e^{ir \cdot K} \prod_{j=1}^m \frac{-i}{\hat{n} \cdot K^\dagger(j) - i\eta}$$

Feynman Diagram



Different Types

Semi-Infinite Lines and Path Reversals

$$\bullet \implies (ig)^m A_m \cdots A_1 e^{ir \cdot K} \prod_j^m \frac{i}{\hat{n} \cdot K^\downarrow(j) + i\eta} \stackrel{N}{=} A^m(r, \hat{n})$$

$$\longleftarrow \bullet (-ig)^m A_m \cdots A_1 e^{ir \cdot K} \prod_j^m \frac{i}{\hat{n} \cdot K^\uparrow(j) + i\eta} \stackrel{N}{=} B^m(r, \hat{n})$$

$$\bullet \longleftarrow (-ig)^m A_m \cdots A_1 e^{ir \cdot K} \prod_j^m \frac{-i}{\hat{n} \cdot K^\downarrow(j) - i\eta} = A^m(r, -\hat{n})$$

$$\longleftarrow \bullet (ig)^m A_m \cdots A_1 e^{ir \cdot K} \prod_j^m \frac{-i}{\hat{n} \cdot K^\uparrow(j) - i\eta} = B^m(r, -\hat{n})$$

More Types

Finite Line

$$\mathcal{U}_{(b;a)} = \sum_{n=0}^{\infty} (ig)^m \int \left(\frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(-k_m) \cdots \hat{n} \cdot A(-k_1) \times \cdots$$

$$\cdots \times \sum_{l=0}^m e^{-ia \cdot K(l)} e^{-ib \cdot K(m-l)} \prod_{j=1}^l \frac{-i}{\hat{n} \cdot \tilde{K}(j)} \prod_{j=l+1}^m \frac{i}{n \cdot K(j)}$$

Hermitian Conjugate

$$\mathcal{U}_{(r;-\infty)}^\dagger = \sum_{n=0}^{\infty} (-ig)^m \int \left(\frac{d^4 k_i}{16\pi^4} \right)^m \hat{n} \cdot A(-k_m) \cdots \hat{n} \cdot A(-k_1) \times \cdots$$

$$\cdots \times e^{ib \cdot K} \prod_{j=1}^m \frac{-i}{\hat{n} \cdot K^\downarrow(j) - i\eta}$$

More Types

Finite Lines and Hermitian Conjugates

$$\overset{\alpha^\mu}{\bullet} \longrightarrow \overset{b^\mu}{\bullet} = \bullet \longrightarrow \otimes \longleftarrow \bullet$$

$$(\longleftarrow \bullet)^\dagger = \bullet \longrightarrow \quad (\bullet \longrightarrow)^\dagger = \longleftarrow \bullet$$

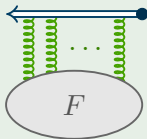
Relation Between A^m and B^m Relation Between A^m and B^m

$$B^m(r, \hat{n}) = (-)^m A^m(r, \hat{n}) \Big|_{(k_1, \dots, k_m) \rightarrow (k_m, \dots, k_1)}$$

$$\leftarrow \text{---} \bullet \begin{array}{c} \color{green} \text{wavy} \\ \downarrow \end{array} = - \left(\bullet \text{---} \rightarrow \begin{array}{c} \color{green} \text{wavy} \\ \downarrow \end{array} \right)$$

Relation Between A^m and B^m Relation Between A^m and B^m

$$B^m(r, \hat{n}) = (-)^m A^m(r, \hat{n}) \Big|_{(k_1, \dots, k_m) \rightarrow (k_m, \dots, k_1)}$$



$$= (-)^m \int \left(\frac{dk_i}{16\pi^4} \right)^m A^m(r, \hat{n}) F_{a_1 \dots a_m}^{\mu_1 \dots \mu_m}(k_m, \dots, k_1)$$

(absorb gluon propagators in F)

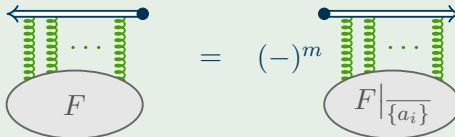
Relation Between A^m and B^m

After Symmetrising the Blob

$$\begin{array}{c} \leftarrow \\ \text{---} \\ \vdots \\ \text{---} \\ F \end{array} = (-)^m \begin{array}{c} \text{---} \\ \rightarrow \\ \vdots \\ \text{---} \\ F \Big|_{\{a_i\}} \end{array}$$

Relation Between A^m and B^m

After Symmetrising the Blob



Easy Blob Example: 3-Gluon Vertex

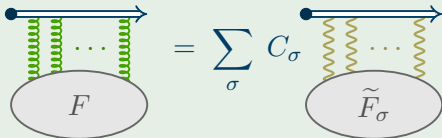


Non-Trivial Colour Structure

Blob With Non-Trivial Colour Structure

$$F_{\mu_1 \dots \mu_m}^{a_1 \dots a_m}(k_1, \dots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \dots a_m)} \tilde{F}_{\sigma(\mu_1 \dots \mu_m)}(\sigma(k_1, \dots, k_m))$$

Factorise Out Colour



$$C_{\sigma} = t^{a_m} \dots t^{a_1} C^{\sigma(a_1 \dots a_m)}$$

Non-Trivial Colour Structure

Blob With Non-Trivial Colour Structure

$$F_{\mu_1 \dots \mu_m}^{a_1 \dots a_m}(k_1, \dots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \dots a_m)} \tilde{F}_{\sigma(\mu_1 \dots \mu_m)}(\sigma(k_1, \dots, k_m))$$

Factorise Out Colour

$$\left(\text{blob } F \text{ with } m \text{ external lines} \right) = (-)^m \sum_{\sigma} C_{\bar{\sigma}} \left(\text{blob } \tilde{F}_{\sigma} \text{ with } m \text{ external lines} \right)$$

$$C_{\bar{\sigma}} = t^{a_1} \dots t^{a_m} C^{\sigma(a_1 \dots a_m)}$$

Non-Trivial Colour Structure

 $m = 4$

$$\begin{aligned}
 & \text{Diagram of a wavy line} = \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4) \\
 & \text{Diagram of a line with 4 wavy lines} = C_{4321} \text{Diagram of a line with 4 wavy lines} + C_{4231} \text{Diagram of a line with 4 wavy lines} + C_{4132} \text{Diagram of a line with 4 wavy lines} \\
 & C_{ijkl} = t^{a_i} t^{a_j} t^{a_k} t^{a_l} f^{a_1 a_2 x} f^{x a_3 a_4}
 \end{aligned}$$

Non-Trivial Colour Structure

 $m = 4$

$$\text{Diagram} = \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4)$$

$$F = C_{4321} \tilde{F} + C_{4231} \tilde{F} + C_{4132} \tilde{F}$$

$$C_{\overline{ijkl}} = C_{lkji} = t^{a_l} t^{a_k} t^{a_j} t^{a_i} f^{a_1 a_2 x} f^{x a_3 a_4} = C_{ijkl}$$

Non-Trivial Colour Structure

$$m = 4$$

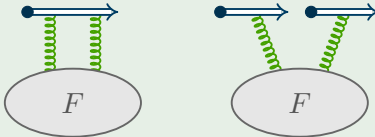




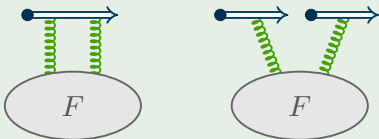
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Basic Diagrams

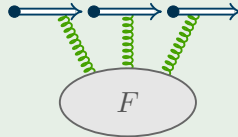
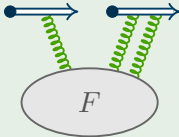
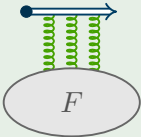
 $m = 2$ 

Basic Diagrams

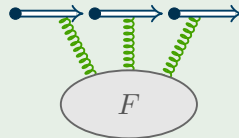
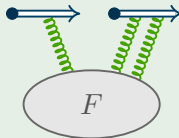
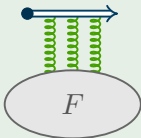
 $m = 2$ 

$$\sum_{J=1}^M \text{Diagram 1} + \sum_{K=2}^M \sum_{J=1}^{K-1} \text{Diagram 2}$$

Basic Diagrams

 $m = 3$ 

Basic Diagrams

 $m = 3$ 

$$\sum_{J=1}^M \text{Diagram 1}$$

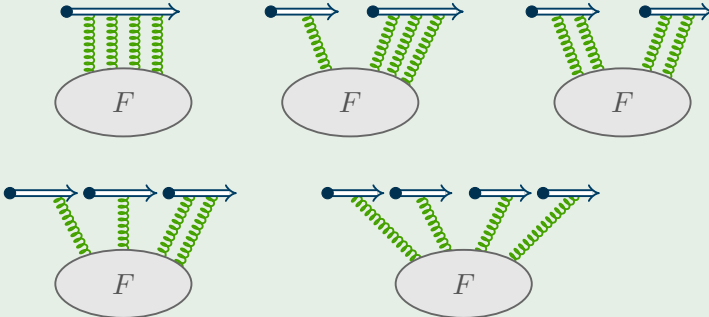
+

$$2 \sum_{K=2}^M \sum_{J=1}^{K-1} \text{Diagram 2}$$

+

$$\sum_{L=3}^M \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} \text{Diagram 3}$$

Basic Diagrams

 $m = 4$ 

Basic Diagrams

 $m = 4$

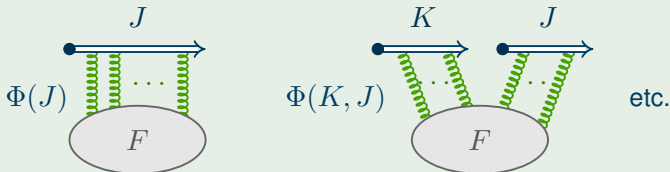
$$\begin{aligned}
& \sum_{J=1}^M \text{Diagram 1} + \sum_{K=2}^M \sum_{J=1}^{K-1} \left(2 \text{Diagram 2} + \text{Diagram 3} \right) \\
& + 3 \sum_{L=3}^M \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} \text{Diagram 4} + \sum_{O=4}^M \sum_{L=3}^{O-1} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} \text{Diagram 5}
\end{aligned}$$

The diagrams are:

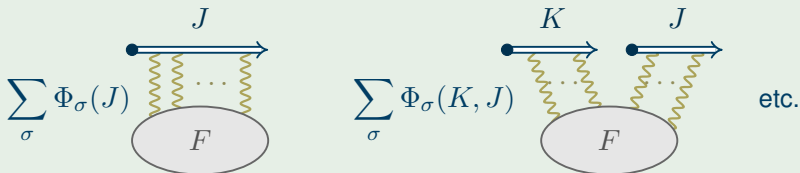
- Diagram 1:** A horizontal line with a dot at the left end and an arrow pointing right, labeled J . Below the line are three vertical wavy lines.
- Diagram 2:** A horizontal line with a dot at the left end and an arrow pointing right, labeled J . Below the line are two vertical wavy lines. Above the line, a bracket labeled (K) spans the two wavy lines.
- Diagram 3:** A horizontal line with a dot at the left end and an arrow pointing right, labeled J . Below the line are two vertical wavy lines.
- Diagram 4:** A horizontal line with a dot at the left end and an arrow pointing right, labeled J . Below the line are two vertical wavy lines. Above the line, a bracket labeled (L) spans the two wavy lines.
- Diagram 5:** A horizontal line with a dot at the left end and an arrow pointing right, labeled J . Below the line are two vertical wavy lines. Above the line, a bracket labeled (O) spans the two wavy lines.

Path Constants

For a Blob With Trivial Colour Structure




For a Blob With Non-Trivial Colour Structure



Blob Examples


 $m = 2$



$$= \delta^{ab} \delta^{(4)}(k_1 - k_2) D_{\mu\nu}(k_1)$$



$$= \Rightarrow \Phi(J) = +1$$



$$= -$$

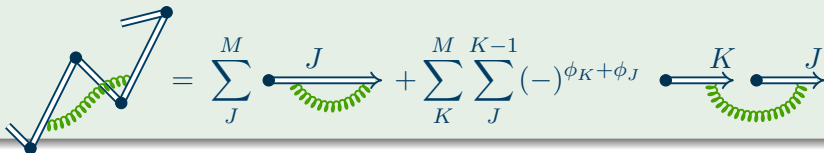
$$\Rightarrow \Phi(K, J) = (-)^{\Phi_K + \Phi_J}$$

$$\phi_J = \begin{cases} 0 & \bullet \longrightarrow \\ 1 & \longleftarrow \bullet \end{cases}$$

Blob Examples

 $m = 2$

$$u_2 = \sum_{J=1}^M u_2^J + \sum_{K=2}^M \sum_{J=1}^{K-1} u_1^K u_1^J$$



Blob Examples

 $m = 3$ 

$$= g f^{abc} D_{\mu_1 \nu_1}^{k_1} D_{\mu_2 \nu_2}^{k_2} D_{\mu_3 \nu_3}^{k_3} g^{\nu_1 \nu_2} (k_1 - k_2)^{\nu_3} + \text{cross.}$$



$$= \text{diagram} \Rightarrow \Phi(J) = +1$$



$$= - \text{diagram} \quad \Phi(K, J) = (-)^{\phi_K + \phi_J}$$

Blob Examples

 $m = 3$

$$u_3 = \sum_{J=1}^M u_3^J + \sum_{K=2}^M \sum_{J=1}^{K-1} [u_1^K u_2^J + u_2^K u_1^J] + \sum_{L=3}^M \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} u_1^L u_1^K u_1^J$$

$$= \sum_J^M \text{blob}(J) + 2 \sum_K^M \sum_J^{K-1} (-)^{\phi_K + \phi_J} \text{blob}(K, J) + \sum_{L=3}^M \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} (-)^{\phi_L + \phi_K + \phi_J} \text{blob}(L, K, J)$$

Blob Example With Non-Trivial Colour Structure

$m = 4$

$$\text{blob} = \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \tilde{F}_{\mu_1 \dots \mu_4}(k_1, \dots, k_4)$$

$$\text{blob} = \text{blob}$$

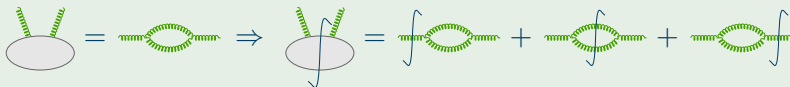
$$\text{blob} = C_1 \text{blob} + C_2 \text{blob} + C_3 \text{blob}$$

$$\text{blob} = C_3 \text{blob} + C_2 \text{blob} + C_1 \text{blob}$$

Final-State Cut

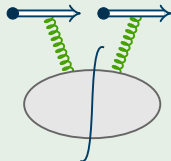
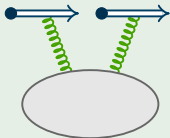
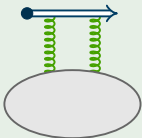
Cutting

- cutting happens at infinity \Rightarrow no momentum cut in Wilson line
- cut line doesn't cross segment, only *between* segments
 \Rightarrow only blob is cut
- define cut blob as sum of possible cuttings, e.g.:

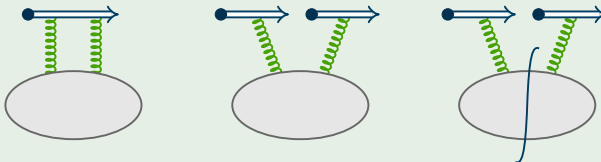


- denote number of segments before the cut by M_c

Expanding the Set of Basic Diagrams

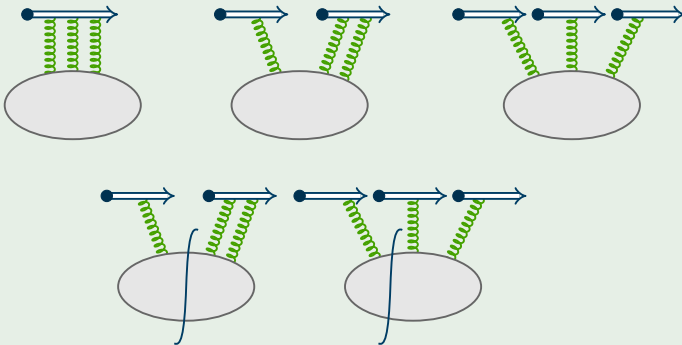
 $m = 2$ 

Expanding the Set of Basic Diagrams

 $m = 2$ 

$$\sum_{J=1}^M \text{Diagram 1} + \left(\sum_{K=2}^{M_C} \sum_{J=1}^{K-1} + \sum_{K=M_C+2}^M \sum_{J=M_C+1}^{K-1} \right) \text{Diagram 2} + \sum_{K=M_C+1}^M \sum_{J=1}^{M_C} \text{Diagram 3}$$

Expanding the Set of Basic Diagrams

 $m = 3$ 




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Reusability

LO 2-GLuon Blob Connecting 2 Segments



$$\begin{aligned} &\stackrel{\text{LC}}{=} \frac{\alpha_s}{2\pi} \left(\frac{1}{\epsilon^2} + \frac{\pi}{3} \right) \left(\frac{n_K \cdot n_J \mu^2}{2 \eta^2} \right)^\epsilon \\ &\stackrel{\text{off-LC}}{=} \frac{\alpha_s}{2\pi} \chi \coth \chi \left(\frac{1}{\epsilon} + \Upsilon \right) \left[\frac{1}{4} \left(n_K^2 n_J^2 - (n_K \cdot n_J)^2 \right) \frac{\mu^2}{\eta^2} \right]^\epsilon \\ &\left(\Upsilon = 2 \ln 2 + \ln n_K^2 + 2 \ln(1 + e^\chi) + \chi - \frac{1}{\chi} (\text{Li}_2 e^\chi - \text{Li}_2 e^{-\chi}) \right) \end{aligned}$$

Reusability

⇒ difference only in path parameters r_{J_i} , \hat{n}_{J_i} and path constants $\Phi(J_i)$!

TMD Example

TMD Wilson Line Self


 \Rightarrow

SIDIS


 $n^-, 0^-, \mathbf{0}_\perp$

 $\mathbf{n}_\perp, +\infty^-, \mathbf{0}_\perp$

 $-\mathbf{n}_\perp, +\infty^-, \mathbf{r}_\perp$

 $-n^-, r^-, \mathbf{r}_\perp$

TMD Example

TMD Wilson Line Self


 \Rightarrow

DY

$$\bullet \Rightarrow -n^-, 0^-, \mathbf{0}_\perp$$

$$\bullet \Rightarrow \mathbf{n}_\perp, -\infty^-, \mathbf{0}_\perp$$

$$\leftarrow \bullet \mathbf{n}_\perp, -\infty^-, \mathbf{r}_\perp$$

$$\leftarrow \bullet -n^-, r^-, \mathbf{r}_\perp$$

TMD Example

TMD Wilson Line Self


 \Rightarrow

DY

$$\bullet \rightleftarrows -n^-, 0^-, \mathbf{0}_\perp$$

$$\bullet \rightleftarrows \mathbf{n}_\perp, -\infty^-, \mathbf{0}_\perp$$

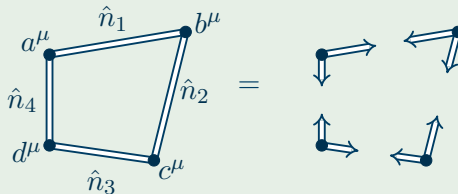
$$\leftleftarrows \bullet \mathbf{n}_\perp, -\infty^-, \mathbf{r}_\perp$$

$$\leftleftarrows \bullet -n^-, r^-, \mathbf{r}_\perp$$

- Difference *only* in sign of n^- and $\pm\infty^-$. No difference in path constants!
- At this level still a gauge invariant, all-order statement.

Quadrilateral Wilson Loop

Quadrilateral Wilson Loop



Quadrilateral Wilson Loop

First Order

The diagram shows a quadrilateral Wilson loop on the left, which is equal to the sum of two diagrams on the right. The first diagram on the right shows a single Wilson line segment labeled J with a gluon emission (represented by a green curly line) from its bottom side. The second diagram shows two Wilson line segments labeled K and J meeting at a vertex, with a gluon emission from the bottom side of the K segment. The two diagrams are summed with a plus sign, and the second diagram is multiplied by a phase factor $(-)^{\phi_K + \phi_J}$.

$$\text{Quadrilateral Loop} = \sum_J^M \text{Diagram 1} + \sum_K^M \sum_J^{K-1} (-)^{\phi_K + \phi_J} \text{Diagram 2}$$

Result for Light-Like Loop

$$\mathcal{U}_2 = \frac{\alpha_s C_F}{\pi} (-2\pi\mu^2)^\epsilon \Gamma(1 - \epsilon) \times \dots$$

$$\dots \times \left[\frac{1}{\epsilon^2} ((b - d)^2 - i\eta)^\epsilon + \frac{1}{\epsilon^2} ((c - a)^2 - i\eta)^\epsilon \right]$$

Conjecture

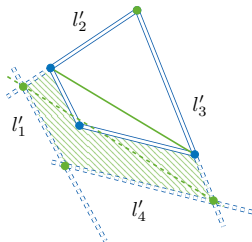
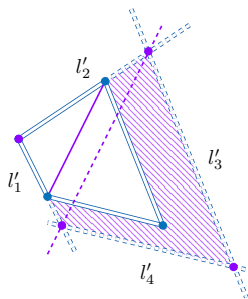
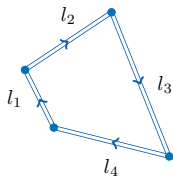
Geometric Evolution of Light-Like Quadrilateral

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \left\langle \frac{\delta}{\delta \ln \Sigma} \right\rangle \ln \mathcal{W}_\gamma = - \sum_{\text{cusps}} \Gamma_{\text{cusp}}$$

Gamma cusp at NLO:

$$\Gamma_{\text{cusp}}(g) = \frac{\alpha_s}{\pi} C_F + \left(\frac{\alpha_s}{\pi} \right)^2 C_F \left(C_A \left(\frac{67}{36} - \frac{\pi^2}{12} \right) - N_f \frac{5}{18} \right)$$

Conjecture



Conclusions

Conclusions & Outlook

- framework to minimize number of diagrams for piecewise linear Wilson lines
- lesser diagrams in exchange for more general (and thus more complicated) integrals
- interesting for $M > 2$

- clean up result & calculate higher orders
- try framework for TMD Wilson line structure

Conclusions

Thank You!