Working with Wilson Lines

Spin 2014, Beijing

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October 24, 2014

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Outline

Path-Ordered Exponentials

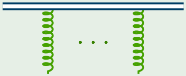
Wilson Line

$$\mathcal{U}[\mathcal{C}] = \mathcal{P}\,\mathrm{exp}\left(\mathrm{i}g\int_a^b \mathrm{d}\lambda\ \ (z^\mu)'\,A_\mu(\lambda)\right)$$

Path-Ordered Exponentials

Wilson Line

$$\mathcal{U}[\mathcal{C}] = \mathcal{P}\,\text{exp}\left(\mathrm{i}g \int_a^b \mathrm{d}\lambda \;\; \left(z^\mu\right)' A_\mu(\lambda) \right)$$



$$A(\lambda_m) \cdots A(\lambda_1)$$

Path-Ordering

Path-Ordering for Linear Lines

$$z^\mu = r^\mu + \hat{n}^\mu \lambda \qquad \lambda = a \ldots b \qquad \lambda_m \geq \cdots \geq \lambda_1$$

$$\mathcal{P} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \mathrm{d}z_1 \cdots \mathrm{d}z_m = m! \int_a^b \int_{\lambda_1}^b \cdots \int_{\lambda_{m-1}}^b \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m$$

$$= m! \int_a^b \int_a^{\lambda_m} \cdots \int_a^{\lambda_2} \mathrm{d}\lambda_m \cdots \mathrm{d}\lambda_1$$

Motivation

Interest in Wilson Lines

- Singularity structure of TMD governed by underlying Wilson line structure
- T-odd non-universality effects due to different Wilson line structures
- Investigation of Wilson loops, to describe geometric evolution of TMDs

Outline

Feynman Rules

Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty\,;\,r)} = \sum_{m=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(-k_m)\cdots\,\hat{n}\cdot A(-k_1)\times\cdots$$

$$\times \int_0^{\infty}\int_{\lambda_1}^{\infty}\cdots\int_{\lambda_{n-1}}^{\infty} \mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_m\;\mathrm{e}^{\mathrm{i}(r+\hat{n}\,\lambda_1)\cdot k_1}\ldots\mathrm{e}^{\mathrm{i}(r+\hat{n}\,\lambda_m)\cdot k_m}$$

Feynman Rules

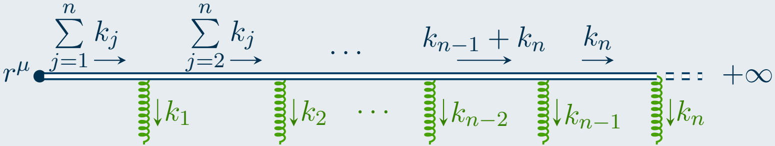
Wilson Line Bounded From Below

$$\mathcal{U}_{(+\infty\,;\,r)} = \sum_{m=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4} \right)^m \; \hat{n}\cdot A(-k_m) \cdots \hat{n}\cdot A(-k_1) \times \cdots$$

$$\cdots \times \, \mathrm{e}^{\mathrm{i}r\cdot K} \prod_{j=1}^{m} \frac{\mathrm{i}}{\hat{n}\cdot K^{\downarrow}(j) + \mathrm{i}\eta}$$

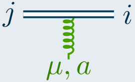
$$K^{\downarrow}(j) = \sum_{l=j}^{n} k_l$$

Feynman Diagram



Feynman Rules

Feynman Rules for Linear Wilson Lines

1) Wilson line propagator:		$=$	$\dfrac{\mathrm{i}}{\hat{n}\cdot k + \mathrm{i}\eta}$
2) external point:		$=$	$e^{\mathrm{i}r\cdot k}$
3) infinite point:		$=$	$1 \qquad (k=0)$
4) Wilson vertex:		$=$	$\mathrm{i}g\,\hat{n}^{\mu}\,(t^a)_{ij}$

Feynman Rules

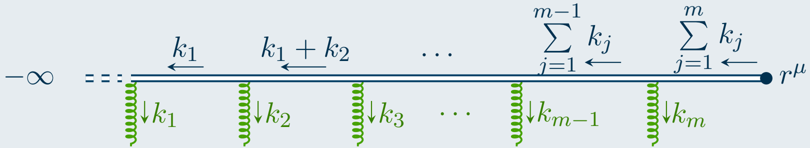
Wilson Line Bounded From Above

$$\mathcal{U}_{(r\,;\,-\infty)} = \sum_{m=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(-k_m) \cdots \hat{n}\cdot A(-k_1) \times$$

$$K^{\uparrow}(j) = \sum_{l=1}^{j} k_l$$

$$\times\, \mathrm{e}^{\mathrm{i}\,r\cdot K} \prod_{j=1}^{m} \frac{-\mathrm{i}}{\hat{n}\cdot K^{\uparrow}(j) - \mathrm{i}\eta}$$

Feynman Diagram



Different Types

Semi-Infinite Lines and Path Reversals

$$(\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{\mathrm{i}r\cdot K} \prod_j^m \frac{\mathrm{i}}{\hat{n}\cdot K^{\downarrow}(j) + \mathrm{i}\eta} \overset{\text{N}}{=} A^m(r,\hat{n})$$

$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{\mathrm{i}r\cdot K} \prod_j^m \frac{\mathrm{i}}{\hat{n}\cdot K^{\uparrow}(j) + \mathrm{i}\eta} \overset{\text{N}}{=} B^m(r,\hat{n})$$

$$(-\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot K^{\downarrow}(j) - \mathrm{i}\eta} = A^m(r,-\hat{n})$$

$$(\mathrm{i}g)^m A_m \cdots A_1\, \mathrm{e}^{\mathrm{i}r\cdot K} \prod_j^m \frac{-\mathrm{i}}{\hat{n}\cdot K^{\uparrow}(j) - \mathrm{i}\eta} = B^m(r,-\hat{n})$$

More Types

Finite Line

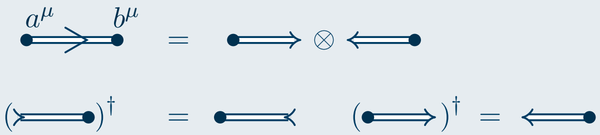
$$\mathcal{U}_{(b\,;\,a)} = \sum_{n=0}^{\infty} (\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(-k_m)\cdots\hat{n}\cdot A(-k_1) \times \cdots$$
$$\cdots \times \sum_{l=0}^{m} \mathrm{e}^{-\mathrm{i}a\cdot K(l)} \mathrm{e}^{-\mathrm{i}b\cdot K(m-l)} \prod_{j=1}^{l} \frac{-\mathrm{i}}{\hat{n}\cdot\widetilde{K}(j)} \prod_{j=l+1}^{m} \frac{\mathrm{i}}{n\cdot K(j)}$$

Hermitian Conjugate

$$\mathcal{U}^{\dagger}_{(r\,;\,-\infty)} = \sum_{n=0}^{\infty} (-\mathrm{i}g)^m \int \left(\frac{\mathrm{d}^4 k_i}{16\pi^4}\right)^m \hat{n}\cdot A(-k_m)\cdots\hat{n}\cdot A(-k_1) \times \cdots$$
$$\cdots \times \mathrm{e}^{\mathrm{i}b\cdot K} \prod_{j=1}^{m} \frac{-\mathrm{i}}{\hat{n}\cdot K^{\downarrow}(j) - \mathrm{i}\eta}$$

More Types

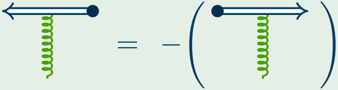
Finite Lines and Hermitian Conjugates



Relation Between A^m and B^m

Relation Between A^m and B^m

$$B^m(r,\hat{n}) = (-)^m A^m(r,\hat{n})\Big|_{(k_1,\ldots,k_m)\to(k_m,\ldots,k_1)}$$



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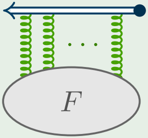
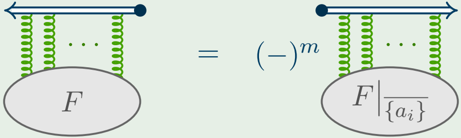


$$= (-)^m \int \left(\frac{\mathrm{d}k_i}{16\pi^4}\right)^m A^m(r,\hat{n}) F^{\mu_1\cdots\mu_m}_{a_1\cdots a_m}(k_m,\ldots,k_1)$$

(absorb gluon propagators in F)

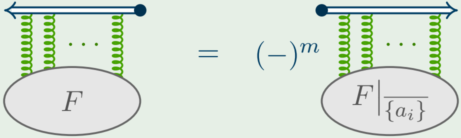
Relation Between A^m and B^m

After Symmetrising the Blob



Relation Between A^m and B^m

After Symmetrising the Blob



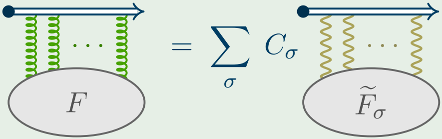
Easy Blob Example: 3-Gluon Vertex

Non-Trivial Colour Structure

Blob With Non-Trivial Colour Structure

$$F^{a_1 \cdots a_m}_{\mu_1 \cdots \mu_m}(k_1, \ldots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \cdots a_m)} \widetilde{F}_{\sigma(\mu_1 \cdots \mu_m)}(\sigma(k_1, \ldots, k_m))$$

Factorise Out Colour



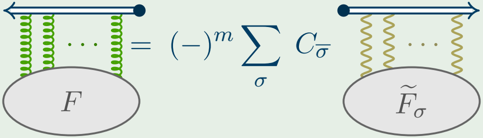
$$C_\sigma = t^{a_m} \ldots t^{a_1} C^{\sigma(a_1 \cdots a_m)}$$

Non-Trivial Colour Structure

Blob With Non-Trivial Colour Structure

$$F^{a_1 \cdots a_m}_{\mu_1 \cdots \mu_m}(k_1, \ldots, k_m) = \sum_{\text{perm}} C^{\sigma(a_1 \cdots a_m)} \widetilde{F}_{\sigma(\mu_1 \cdots \mu_m)}\left(\sigma(k_1, \ldots, k_m)\right)$$

Factorise Out Colour

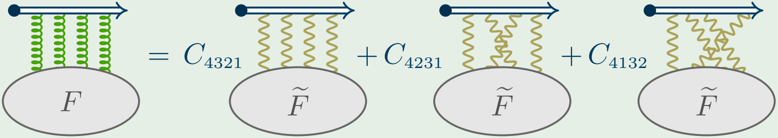


$$C_{\overline{\sigma}} = t^{a_1} \ldots t^{a_m} C^{\sigma(a_1 \cdots a_m)}$$

Non-Trivial Colour Structure

$m = 4$

$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$$



$$C_{ijkl} = t^{a_i} t^{a_j} t^{a_k} t^{a_l} \, f^{a_1 a_2 x} f^{x a_3 a_4}$$

Non-Trivial Colour Structure

$m = 4$

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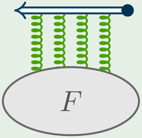
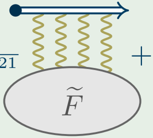
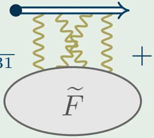


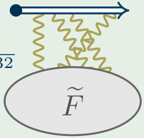
$$= C_{\overline{4321}}$$



$$+ C_{\overline{4231}}$$



$$+ C_{\overline{4132}}$$



$$C_{\overline{ijkl}} = C_{lkji} = t^{a_l} t^{a_k} t^{a_j} t^{a_i} f^{a_1 a_2 x} f^{x a_3 a_4} = C_{ijkl}$$

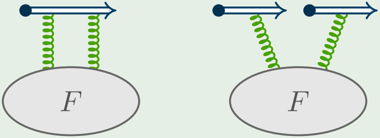
Non-Trivial Colour Structure

$m = 4$

Outline

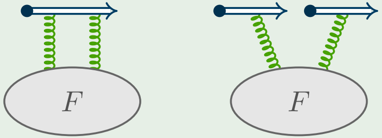
Basic Diagrams

$m = 2$

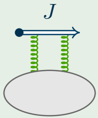


Basic Diagrams

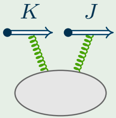
$m = 2$



$$\sum_{J=1}^{M}$$

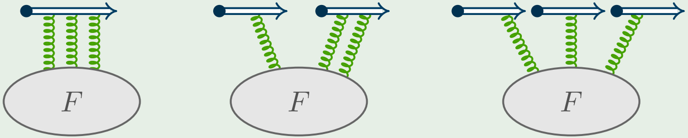


$$+ \sum_{K=2}^{M}\sum_{J=1}^{K-1}$$



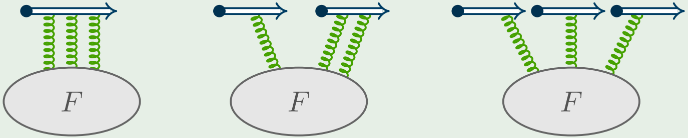
Basic Diagrams

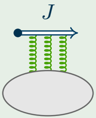
$m = 3$

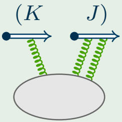


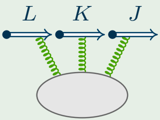
Basic Diagrams

$m = 3$



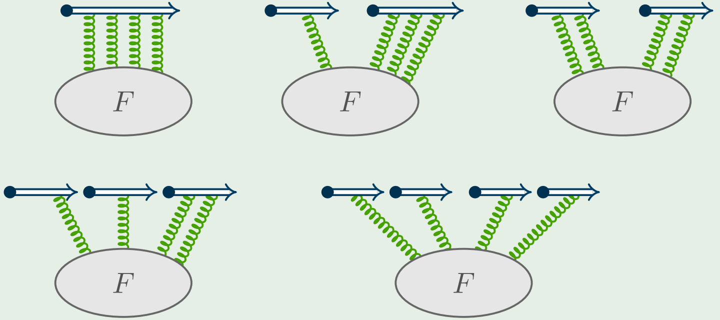
$\sum_{J=1}^{M}$ 

$+ \; 2 \sum_{K=2}^{M} \sum_{J=1}^{K-1}$ 

$+ \sum_{L=3}^{M} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1}$ 

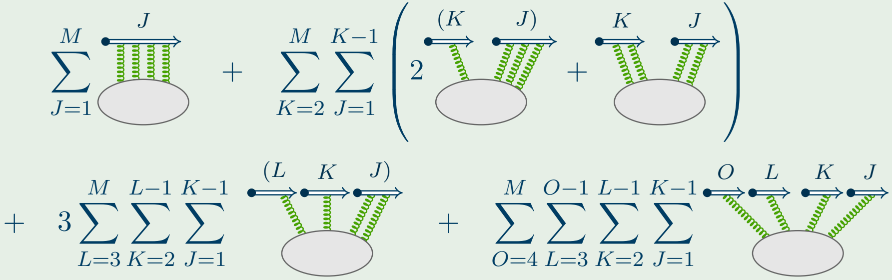
Basic Diagrams

$m = 4$



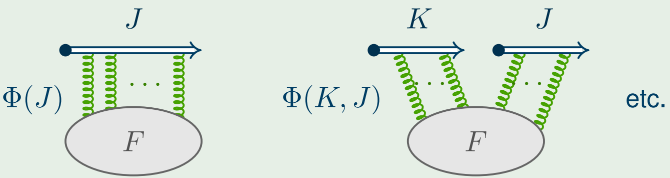
Basic Diagrams

$m = 4$

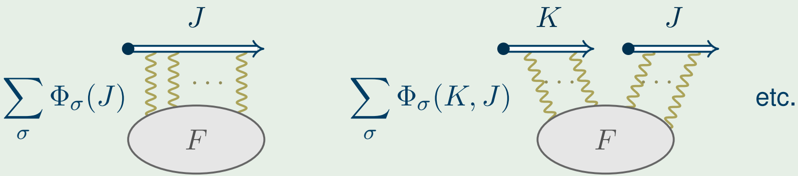


Path Constants

For a Blob With Trivial Colour Structure

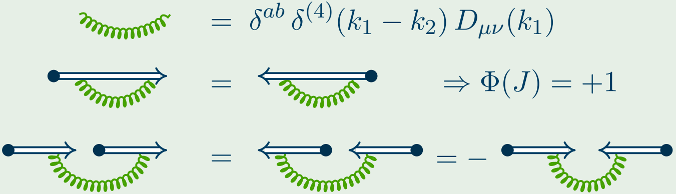


For a Blob With Non-Trivial Colour Structure



Blob Examples

$m = 2$



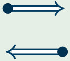
$$= \delta^{ab}\,\delta^{(4)}(k_1 - k_2)\,D_{\mu\nu}(k_1)$$

$$\Rightarrow \Phi(J) = +1$$

$$= -$$

$$\Rightarrow \Phi(K, J) = (-)^{\Phi_K + \Phi_J}$$

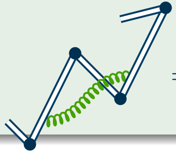
$$\phi_J = \begin{cases} 0 \\ 1 \end{cases}$$



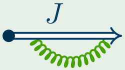
Blob Examples

$m = 2$

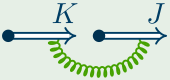
$$\mathcal{U}_2 = \sum_{J=1}^{M} \mathcal{U}_2^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \mathcal{U}_1^K \mathcal{U}_1^J$$



$$= \sum_{J}^{M}$$



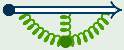
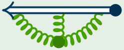
$$+ \sum_{K}^{M} \sum_{J}^{K-1} (-)^{\phi_K + \phi_J}$$



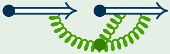
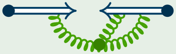
Blob Examples

$m = 3$

$$= g f^{abc} D^{k_1}_{\mu_1\nu_1} D^{k_2}_{\mu_2\nu_2} D^{k_3}_{\mu_3\nu_3} \, g^{\nu_1\nu_2} \, (k_1 - k_2)^{\nu_3} + \text{cross.}$$

 $=$ 

$$\Rightarrow \Phi(J) = +1$$

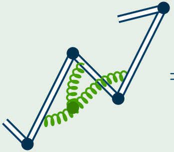
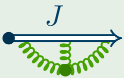
 $= -$ 

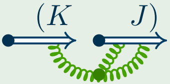
$$\Phi(K, J) = (-)^{\phi_K + \phi_J}$$

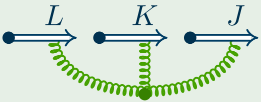
Blob Examples

$m = 3$

$$\mathcal{U}_3 = \sum_{J=1}^{M} \mathcal{U}_3^J + \sum_{K=2}^{M} \sum_{J=1}^{K-1} \left[\mathcal{U}_1^K \mathcal{U}_2^J + \mathcal{U}_2^K \mathcal{U}_1^J\right] + \sum_{L=3}^{M} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} \mathcal{U}_1^L \mathcal{U}_1^K \mathcal{U}_1^J$$


$= \sum_{J}^{M}$


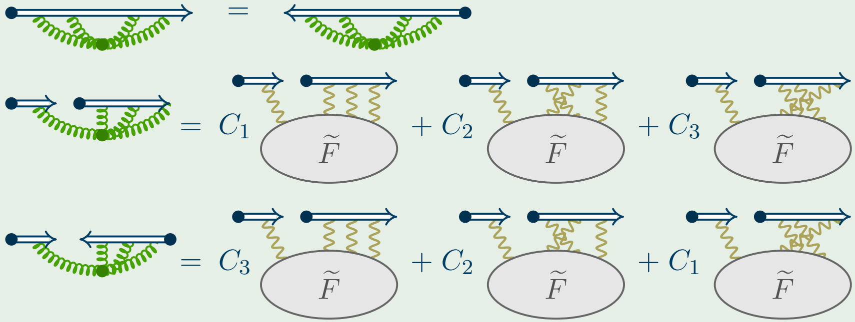
$+2 \sum_{K}^{M} \sum_{J}^{K-1} (-)^{\phi_K + \phi_J}$


$+ \sum_{L=3}^{M} \sum_{K=2}^{L-1} \sum_{J=1}^{K-1} (-)^{\phi_L + \phi_K + \phi_J}$


Blob Example With Non-Trivial Colour Structure

$m = 4$

$$= \sum_{\sigma} f^{a_1 a_2 x} f^{x a_3 a_4} \widetilde{F}_{\mu_1 \cdots \mu_4}(k_1, \ldots, k_4)$$

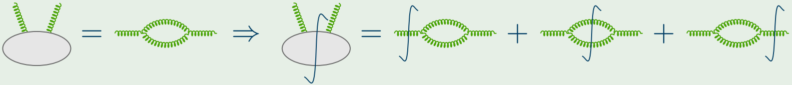


Final-State Cut

Cutting

- cutting happens at infinity $\Rightarrow$ no momentum cut in Wilson line
- cut line doesn't cross segment, only *between* segments

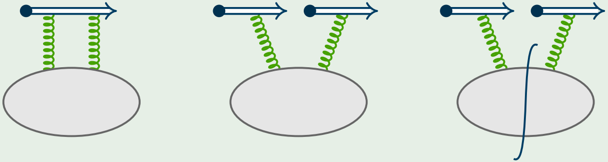
  $\Rightarrow$ only blob is cut
- define cut blob as sum of possible cuttings, e.g.:



- denote number of segments before the cut by M_c

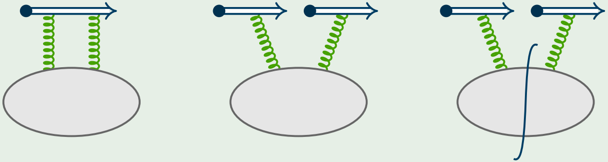
Expanding the Set of Basic Diagrams

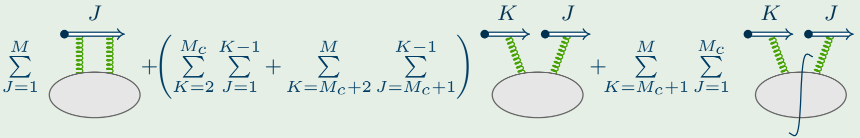
$m = 2$



Expanding the Set of Basic Diagrams

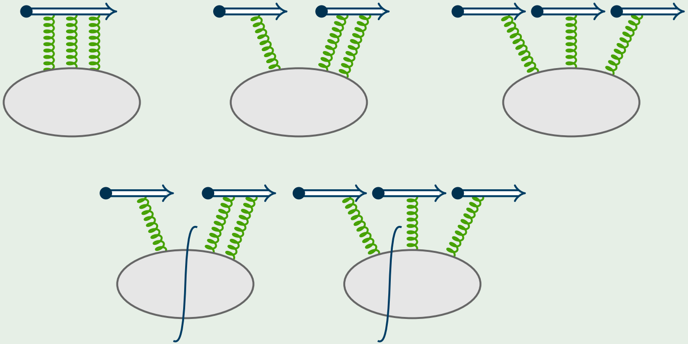
$m = 2$





Expanding the Set of Basic Diagrams

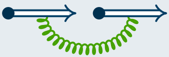
$m = 3$



Outline

Reusability

LO 2-GLuon Blob Connecting 2 Segments



$$\overset{\text{LC}}{=} \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon^2}+\frac{\pi}{3}\right)\left(\frac{n_K\cdot n_J}{2}\frac{\mu^2}{\eta^2}\right)^{\epsilon}$$

$$\overset{\text{off-LC}}{=} \frac{\alpha_s}{2\pi}\chi\coth\chi\left(\frac{1}{\epsilon}+\Upsilon\right)\left[\frac{1}{4}\left(n_K^2 n_J^2-(n_K\cdot n_J)^2\right)\frac{\mu^2}{\eta^2}\right]^{\epsilon}$$

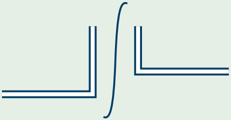
$$\left(\Upsilon = 2\ln 2 + \ln n_K^2 + 2\ln(1+\mathrm{e}^{\chi}) + \chi - \frac{1}{\chi}\left(\mathrm{Li}_2\mathrm{e}^{\chi} - \mathrm{Li}_2\mathrm{e}^{-\chi}\right)\right)$$

Reusability

$\Rightarrow$ difference only in path parameters $r_{J_i}, \hat{n}_{J_i}$ and path constants $\Phi(J_i)$!

TMD Example

TMD Wilson Line Self



$\Rightarrow$

SIDIS

$n^-, 0^-, \mathbf{0}_\perp$

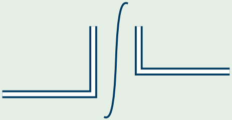
$\mathbf{n}_\perp, +\infty^-, \mathbf{0}_\perp$

$-\mathbf{n}_\perp, +\infty^-, \mathbf{r}_\perp$

$-n^-, r^-, \mathbf{r}_\perp$

TMD Example

TMD Wilson Line Self



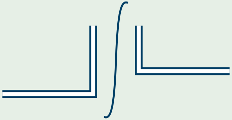
$\Rightarrow$

DY

$$\frac{\bullet\!\!\Longrightarrow\ -n^-, 0^-, \mathbf{0}_\perp \qquad \bullet\!\!\Longrightarrow\ \mathbf{n}_\perp, -\infty^-, \mathbf{0}_\perp}{\Longleftarrow\!\!\bullet\ \mathbf{n}_\perp, -\infty^-, \mathbf{r}_\perp \qquad \Longleftarrow\!\!\bullet\ -n^-, r^-, \mathbf{r}_\perp}$$

TMD Example

TMD Wilson Line Self



$\Rightarrow$

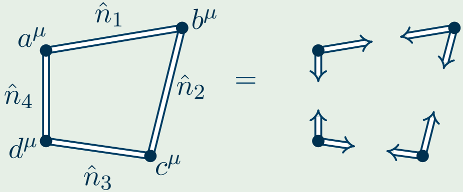
DY

$$\frac{\begin{array}{ll} \bullet\!\!\Longrightarrow & -n^-, 0^-, \mathbf{0}_\perp \\ \bullet\!\!\Longrightarrow & \mathbf{n}_\perp, -\infty^-, \mathbf{0}_\perp \end{array}}{\begin{array}{ll} \Longleftarrow\!\!\bullet & \mathbf{n}_\perp, -\infty^-, \mathbf{r}_\perp \\ \Longleftarrow\!\!\bullet & -n^-, r^-, \mathbf{r}_\perp \end{array}}$$

- Difference *only* in sign of n^- and $\pm\infty^-$. No difference in path constants!
- At this level still a gauge invariant, all-order statement.

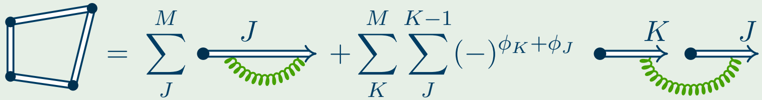
Quadrilateral Wilson Loop

Quadrilateral Wilson Loop



Quadrilateral Wilson Loop

First Order



Result for Light-Like Loop

$$\mathcal{U}_2 = \frac{\alpha_s C_F}{\pi} (-2\pi\mu^2)^\epsilon \, \Gamma(1-\epsilon) \times \cdots$$

$$\cdots \times \left[\frac{1}{\epsilon^2} \left((b-d)^2 - \mathrm{i}\eta \right)^\epsilon + \frac{1}{\epsilon^2} \left((c-a)^2 - \mathrm{i}\eta \right)^\epsilon \right]$$

Conjecture

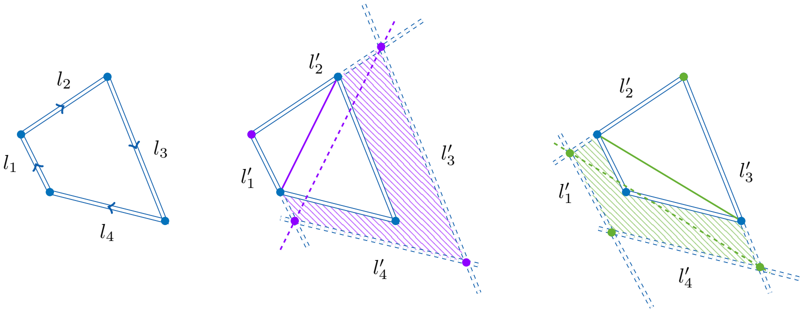
Geometric Evolution of Light-Like Quadrilateral

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\left\langle\frac{\delta}{\delta\ln\Sigma}\right\rangle\ln\mathcal{W}_\gamma=-\sum_{\text{cusps}}\Gamma_{\text{cusp}}$$

Gamma cusp at NLO:

$$\Gamma_{\text{cusp}}(g)=\frac{\alpha_s}{\pi}C_F+\left(\frac{\alpha_s}{\pi}\right)^2C_F\left(C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-N_f\frac{5}{18}\right)$$

Conjecture



Conclusions

Conclusions & Outlook

- framework to minimize number of diagrams for piecewise linear Wilson lines
- lesser diagrams in exchange for more general (and thus more complicated) integrals
- interesting for $M > 2$

- clean up result & calculate higher orders
- try framework for TMD Wilson line structure

Conclusions

Thank You!