Leading order symplectic dipole fringe field model

David Brett, University of Manchester

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Hamiltonian of a normal dipole fringe field as given in Forest's paper,

$$H = \frac{qB}{p_0(1+\delta)} \frac{b_0}{8} \left(4p_y xy + p_x x^2 - 3p_x y^2\right) \tag{1}$$

Define a_x and a_y as,

$$a_x = \frac{3y^2 - x^2}{2} \tag{2}$$

$$a_y = -2xy \tag{3}$$

this leads to a Hamiltonian of the form,

$$H = \frac{qB}{p_0(1+\delta)} \frac{b_0}{8} \left((p_y + 2xy)^2 - p_y^2 - 4x^2y^2 + \left(p_x - \frac{3y^2 - x^2}{2}\right)^2 - p_x^2 - \frac{(3y^2 - x^2)^2}{4} \right) \tag{4}$$

Splitting this Hamiltonian,

$$H_1 = \frac{qB}{p_0(1+\delta)} \frac{b_0}{8} (p_y + 2xy)^2 \tag{5}$$

$$H_2 = \frac{qB}{p_0(1+\delta)} \frac{b_0}{8} \left(p_x - \frac{3y^2 - x^2}{2}\right)^2 \tag{6}$$

$$H_3 = -\frac{qB}{p_0(1+\delta)} \frac{b_0}{8} \left(p_x^2 + p_y^2\right) \tag{7}$$

$$H_4 = -\frac{qB}{p_0(1+\delta)} \frac{b_0}{8} \frac{1}{4} \left(x^4 + 10x^2y^2 + 9y^4\right) \tag{8}$$

This results in the following maps for H_3 and H_4

$$e^{-:H_3:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \\ p_x \\ y - \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \\ p_y \\ z + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}(p_x^2+p_y^2) \\ \delta \end{pmatrix} \tag{9}$$

$$e^{-:H_4:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(x^3+5xy^2\right) \\ y \\ p_y + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(9y^3+5x^2y\right) \\ z - \frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}\frac{1}{4}\left(x^4+10x^2y^2+9y^4\right) \\ \delta \end{pmatrix} \tag{10}$$

The H_1 and H_2 must be split further.

$$e^{-:H_1:} = e^{:I_1:}e^{-:\tilde{H}_1(p_y):}e^{-:I_1:} \tag{11}$$

$$e^{-:H_2:} = e^{:I_2:}e^{-:\tilde{H}_2(p_x):}e^{-:I_2:} \tag{12}$$

where

$$\tilde{H}_1 = \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}p_y^2 \tag{13}$$

$$\tilde{H}_2 = \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}p_x^2 \tag{14}$$

$$I_1 = -x^2y \tag{15}$$

$$I_2 = \frac{y^3-x^2y}{2} \tag{16}$$

Leading to the following Lie transformations,

$$e^{-:\tilde{H}_1:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x \\ y + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \\ p_y \\ z - \frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_y^2 \\ \delta \end{pmatrix} \tag{17}$$

$$e^{-:\tilde{H}_2:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \\ p_x \\ y \\ p_y \\ z - \frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \\ \delta \end{pmatrix} \tag{18}$$

$$e^{:\tilde{I}_1:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x + 2xy \\ y \\ p_y + x^2 \\ z \\ \delta \end{pmatrix} \tag{19}$$

$$e^{-:\tilde{I}_1:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x - 2xy \\ y \\ p_y - x^2 \\ z \\ \delta \end{pmatrix} \tag{20}$$

$$e^{:\tilde{I}_2:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x + xy \\ y \\ p_y - \frac{3y^2-x^2}{2} \\ z \\ \delta \end{pmatrix} \tag{21}$$

$$e^{-:\tilde{I}_2:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x - xy \\ y \\ p_y + \frac{3y^2-x^2}{2} \\ z \\ \delta \end{pmatrix} \tag{22}$$

Using a second order splitting the transfer map is given by,

$$e^{-:H_1+H_2+H_3+H_4:}$$

$$= e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{4}:H_4:}e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{2}:H_2:}e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{4}:H_4:}e^{-\frac{1}{8}:H_3:} \tag{23}$$

$$e^{-:H_1:}e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{4}:H_4:}e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{2}:H_2:}e^{-\frac{1}{8}:H_3:}e^{-\frac{1}{4}:H_4:}e^{-\frac{1}{8}:H_3:} \tag{24}$$

where the splitting is optimised to reduce the number of computations. The list of operations is given by,

$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{1}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{2}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{3}$$
$$p_x \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(x^3 + 5xy^2\right) \tag{4}$$
$$p_y \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(9y^3 + 5x^2y\right) \tag{5}$$
$$z \mapsto z - \frac{1}{4}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{6}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{7}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{8}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{9}$$
$$p_x \mapsto p_x + xy \tag{10}$$
$$p_y \mapsto p_y - \frac{3y^2 - x^2}{2} \tag{11}$$
$$x \mapsto x + \frac{1}{2}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{12}$$
$$z \mapsto z - \frac{1}{2}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{13}$$
$$p_x \mapsto p_x - xy \tag{14}$$
$$p_y \mapsto p_y + \frac{3y^2 - x^2}{2} \tag{15}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{16}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{17}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{18}$$
$$p_x \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(x^3 + 5xy^2\right) \tag{19}$$
$$p_y \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(9y^3 + 5x^2y\right) \tag{20}$$
$$z \mapsto z - \frac{1}{4}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{21}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{22}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{23}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{24}$$
$$p_x \mapsto p_x + 2xy \tag{25}$$
$$p_y \mapsto p_y + x^2 \tag{26}$$
$$y \mapsto y + \frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{27}$$
$$z \mapsto z - \frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_y^2 \tag{28}$$
$$p_x \mapsto p_x - 2xy \tag{29}$$
$$p_y \mapsto p_y - x^2 \tag{30}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{31}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{32}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{33}$$
$$p_x \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(x^3 + 5xy^2\right) \tag{34}$$
$$p_y \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(9y^3 + 5x^2y\right) \tag{35}$$
$$z \mapsto z - \frac{1}{4}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{36}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{37}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{38}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{39}$$
$$p_x \mapsto p_x + xy \tag{40}$$
$$p_y \mapsto p_y - \frac{3y^2 - x^2}{2} \tag{41}$$
$$x \mapsto x + \frac{1}{2}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{42}$$
$$z \mapsto z - \frac{1}{2}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{43}$$
$$p_x \mapsto p_x - xy \tag{44}$$
$$p_y \mapsto p_y + \frac{3y^2 - x^2}{2} \tag{45}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{46}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{47}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{48}$$
$$p_x \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(x^3 + 5xy^2\right) \tag{49}$$
$$p_y \mapsto p_x + \frac{1}{4}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}\left(9y^3 + 5x^2y\right) \tag{50}$$
$$z \mapsto z - \frac{1}{4}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}p_x^2 \tag{51}$$
$$x \mapsto x - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_x \tag{52}$$
$$y \mapsto y - \frac{1}{8}\frac{qB}{p_0(1+\delta)}\frac{b_0}{8}2p_y \tag{53}$$
$$z \mapsto z + \frac{1}{8}\frac{qB}{p_0(1+\delta)^2}\frac{b_0}{8}(p_x^2 + p_y^2) \tag{54}$$