1 Taylor map

Format *name*

name

Name of the Taylor map found in file "tmmap.dat".

type

Type identifier is 29 for all Taylor maps.

Remarks

The file containing the Taylor map is a large file with every Taylor map required for the lattice. It takes the following form with no line spacing between maps:

`NAME`

name

Number of coefficients per series

6 sections for the map to each of the 6 MADX coordinates with each section in the form of a list of exponents (Exp.) (integer) and coefficients (double) in SI units ($\mathrm{m}^i \cdot \mathrm{rad}^j$):

Exp. x, Exp. p_x, Exp. y, Exp. p_y, Exp. z, Exp. δ, Coefficient

$\ldots \times$ *Number of coefficients per series*

The author proposes two uses for the Taylor map element. One is for prototyping symplectic thin elements by setting the conjugate position maps to being maps to themselves effectively producing a momentum kick only. The six sections of the map must all be the same length so the map must be padded with a series of zero lines. The second use is as a thick element in the thin 6d tracking. For this to fully work in 6d thin tracking the user is required to produce reverse drifts half the length of the element and sandwich the element between these reverse drifts in the lattice. The maximum number of different maps that can be loaded is limited by the parameter *tmelem* (200) with the total length of all these combined limited by the parameter *tmmxterm* (10^6).

Method

Coordinate change

$$
\begin{aligned}
x &\mapsto x_6/1000 \\
p_x &\mapsto {x'}_6 \times (1+\delta)/1000 \\
y &\mapsto y_6/1000 \\
p_y &\mapsto {y'}_6 \times (1+\delta)/1000 \\
z &\mapsto z_6/\beta_0/1000 \\
\delta &\mapsto p_{\sigma 6}/\beta_0
\end{aligned}
$$

Taylor map iteration, for each variable $X = \{x, p_x, y, p_y, z, \delta\}$ the map is iterated over the list of exponents E and coefficient value V.

$$
\begin{aligned}
x_6 &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times 1000 \\
{x'}_6 &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times \frac{1000}{1+\delta} \\
y_6 &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times 1000 \\
{y'}_6 &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times \frac{1000}{1+\delta} \\
z_6 &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times 1000 \times \beta_0 \\
p_{\sigma 6} &\mapsto \sum_i x^{E_{ix}} p_x^{E_{ipx}} y^{E_{iy}} p_x^{E_{ipx}} z^{E_{iz}} \delta^{E_{i\delta}} V_i \times 1000 \times \beta_0
\end{aligned}
$$

2 Numerical field integrator

Format *name type*
name
Name of the vector file "[*name*].6pot".
type
Type identifier is 30 for all numerical integrations.
Remarks
Each potential has its own file containing the the potential expressed as an expansion of x, y and z at different s values at regular binning seperation of the integration step size such that the input file takes the form:

---

s_0, *Exp. x, Exp. y, Exp. z,* A_x *coefficient,* A_y *coefficient,* A_z *coefficient*
$\vdots$
$s_0 + \Delta s$, *Exp. x, Exp. y, Exp. z,* A_x *coefficient,* A_y *coefficient,* A_z *coefficient*
$\vdots\vdots$
L, *Exp. x, Exp. y, Exp. z,* A_x *coefficient,* A_y *coefficient,* A_z *coefficient*

---

where, s_0 is the initial s position in the reference frame of the element, Δs is the integration step length and L is the final position in the reference frame of the element. The exponents of x, y and z are integer values and $\vec{A}$ coefficients are to double precision and are of the non-normalised form of the vector potential in SI units ($\mathrm{V}\cdot\ \mathrm{s}\cdot\ \mathrm{m}^{-1}$). For magnets the fields might be fitted to the form of a series by solving the Laplace equation [1] and for RF elements the fields might be fitted by solving the Helmholtz equation [2]. For each type of element a single pass map is outputted in the form of files named "[*name*].1map". The integrator used is a second order explicit integration of the accelerator Hamiltonian in MADX coordinates [3]. Coordinates transformations are performed in the element as are reverse drifts half the length of the element either side of the integration of the element in order to create a thin element to work in 6d thin tracking. The size of the potential of one element is in theory limited to by the parameter *fwterms* (100,000) and number of different element types to *fwelem* (200) which can be extended if required by the user in the source code.
Method
Coordinate change

$$\begin{aligned}
x &\mapsto x_6/1000\\
p_x &\mapsto x'_6 \times (1+\delta_6)/1000\\
y &\mapsto y_6/1000\\
p_y &\mapsto y'_6 \times (1+\delta_6)/1000\\
z &\mapsto z_6/\beta_0/1000\\
\delta &\mapsto p_{\sigma 6}/\beta_0
\end{aligned}$$

Reverse drift over half the length

$$
\begin{aligned}
x \mapsto & x - L\frac{p_x}{2(1+\delta)} \\
y \mapsto & y - L\frac{p_y}{2(1+\delta)} \\
z \mapsto & z + L\left(\frac{p_x^2+p_y^2}{4(1+\delta)^2} + \frac{1}{4\gamma^2(1+\delta)^2}\right)
\end{aligned}
$$

Numerical integration with splitting:

$$
\begin{aligned}
& e^{-\Delta s:H_1+H_2+H_3+H_4:} = e^{-\frac{\Delta s}{2}:H_1+H_2+H_3:}e^{-\Delta s:H_4:}e^{-\frac{\Delta s}{2}:H_1+H_2+H_3:} \\
& = e^{-\frac{\Delta s}{4}:H_1+H_2:}e^{-\frac{\Delta s}{2}:H_3:}e^{-\frac{\Delta s}{4}:H_1+H_2:}e^{-\Delta s:H_4:}e^{-\frac{\Delta s}{4}:H_1+H_2:}e^{-\frac{\Delta s}{2}:H_3:}e^{-\frac{\Delta s}{4}:H_1+H_2:} \\
& = e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{4}:H_2:}e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{2}:H_3:}e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{4}:H_2:}e^{-\frac{\Delta s}{8}:H_1:} \\
& e^{-\Delta s:H_4:}e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{4}:H_2:}e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{2}:H_3:}e^{-\frac{\Delta s}{8}:H_1:}e^{-\frac{\Delta s}{4}:H_2:}e^{-\frac{\Delta s}{8}:H_1:}
\end{aligned}
$$

With Lie transformations,

$$
e^{-\Delta s:H_1:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \\ s \\ p_s \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z + \Delta s\left(\frac{1}{\beta_0} - 1 - \frac{1}{2\beta_0^2\gamma_0^2\left(\frac{1}{\beta_0}+\delta\right)^2}\right) \\ \delta \\ s + \Delta s \\ p_s \end{pmatrix} \tag{1}
$$

$$
e^{-\Delta s:H_4:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \\ s \\ p_s \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x + \Delta s\frac{\partial a_s}{\partial x} \\ y \\ p_y + \Delta s\frac{\partial a_s}{\partial y} \\ z \\ \delta + \Delta s\frac{\partial a_s}{\partial z} \\ s \\ p_s \end{pmatrix} \tag{2}
$$

with the mixed terms H_2 and H_3 defined,

$$
e^{:I_x:}e^{-\Delta s:\tilde{H}_2(p_x):}e^{-:I_x} = e^{\Delta s:H_2:} \tag{3}
$$

where,

$$e^{:I_x:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \\ s \\ p_s \end{pmatrix} \mapsto \begin{pmatrix} x \\ p_x - a_x \\ y \\ p_y - \int_0^x \frac{\partial}{\partial y} a_x(x,y,z,s)\ dx \\ z \\ \delta - \int_0^x \frac{\partial}{\partial z} a_x(x,y,z,s)\ dx \\ s \\ p_s \end{pmatrix} \tag{4}$$

$$e^{-\Delta s:\tilde{H}_2:}\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \\ s \\ p_s \end{pmatrix} \mapsto \begin{pmatrix} x + \Delta s \frac{p_x}{\left(\frac{1}{\beta_0}+\delta\right)} \\ p_x \\ y \\ p_y \\ z - \Delta s \frac{p_x^2}{2\left(\frac{1}{\beta_0}+\delta\right)^2} \\ \delta \\ s \\ p_s \end{pmatrix} \tag{5}$$

Reverse drift over half the length

$$\begin{aligned} x &\mapsto x - L\frac{p_x}{2(1+\delta)} \\ y &\mapsto y - L\frac{p_y}{2(1+\delta)} \\ z &\mapsto z + L\left(\frac{p_x^2 + p_y^2}{4(1+\delta)^2} + \frac{1}{4\gamma^2(1+\delta)^2}\right) \end{aligned}$$

Coordinate change

$$\begin{aligned} x_6 &\mapsto x \times 1000 \\ x'_6 &\mapsto \frac{p_x}{1+\delta} \times 1000 \\ y_6 &\mapsto y \times 1000 \\ y'_6 &\mapsto \frac{p_y}{1+\delta} \times 1000 \\ z_6 &\mapsto z \times 1000 \times \beta_0 \\ p_{\sigma 6} &\mapsto \delta \times \beta_0 \end{aligned}$$

3 Normal quadrupole fringe field

Format *name type* α
name
type
Type identifier is 31 for all fringe fields.
α
Fringe field strength. Entrance face and exit face with strengths $\pm k_0/6$ $[m^2]$ respectively.
Remarks
Implementation of the leading order fringe field component as described in the reference [4]. Describes only the lumped kick of the B_z component.
Method
Coordinate change

$$\begin{aligned} x &\mapsto x_6/1000 \\ p_x &\mapsto {x'}_6 \times (1+\delta)/1000 \\ y &\mapsto y_6/1000 \\ p_y &\mapsto {y'}_6 \times (1+\delta)/1000 \\ z &\mapsto z_6/\beta_0/1000 \\ \delta &\mapsto p_{\sigma 6}/\beta_0 \end{aligned}$$

Rotate -45°

$$\begin{aligned} x &\mapsto \frac{x-y}{\sqrt{2}} \\ p_x &\mapsto \frac{p_x-p_y}{\sqrt{2}} \\ y &\mapsto \frac{x+y}{\sqrt{2}} \\ p_y &\mapsto \frac{p_x+p_y}{\sqrt{2}} \end{aligned}$$

Leading order effect

$$\begin{aligned} x &\mapsto x - \alpha\frac{y^3}{1+\delta} \\ p_y &\mapsto p_y + 3\alpha\frac{p_x y^2}{1+\delta} \\ z &\mapsto z + \alpha\frac{p_x y^3}{(1+\delta)^2} \end{aligned}$$

$$\begin{aligned} y &\mapsto y - \alpha\frac{x^3}{1+\delta} \\ p_x &\mapsto p_x + 3\alpha\frac{p_y x^2}{1+\delta} \\ z &\mapsto z + \alpha\frac{p_y x^3}{(1+\delta)^2} \end{aligned}$$

Coordinate change

$$\begin{aligned} x_6 &\mapsto x \times 1000 \\ {x'}_6 &\mapsto \frac{p_x}{1+\delta} \times 1000 \\ y_6 &\mapsto y \times 1000 \\ {y'}_6 &\mapsto \frac{p_y}{1+\delta} \times 1000 \\ z_6 &\mapsto z \times 1000 \times \beta_0 \\ p_{\sigma 6} &\mapsto \delta \times \beta_0 \end{aligned}$$

Rotate 45°

$$\begin{aligned} x &\mapsto \frac{x+y}{\sqrt{2}} \\ x' &\mapsto \frac{x'+y'}{\sqrt{2}} \\ y &\mapsto \frac{-x+y}{\sqrt{2}} \\ y' &\mapsto \frac{-x'+y'}{\sqrt{2}} \end{aligned}$$

References

[1] C Mitchell. Accurate transfer maps for realistic beam-line elements: Straight elements *Phys. Rev. ST Accel. Beams*, **13**(6), 2010.

[2] D Abell. Numerical computation of high-order transfer maps for rf cavities. *Phys. Rev. ST Accel. Beams*, **9**(5), 2006.

[3] Y Wu, E Forest, and D Robin. Explicit symplectic integrator for s-dependent static magnetic field. *Phys. Rev. E*, **68**(4), 2003.

[4] E Forest, J Milutinović. Leading order hard edge fringe fields effects exact in $(1+\delta)$ and consistent with Maxwell's equations for rectilinear magnets *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, **269**(3), 1988.