Phenomenology of renormalons in heavy quark physics and lattice
(or numerical evidence of renormalons)

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CERN TH-Institute: Resurgence and Transseries in Quantum, Gauge and String Theories, 2014

Renormalons

Originally (Lautrup, 't Hooft).
Renormalon: summation of "bubbles": $\beta_0 = -\frac{4}{3}T_F n_l \rightarrow \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_l$
naive non-abelianization. Running of α.

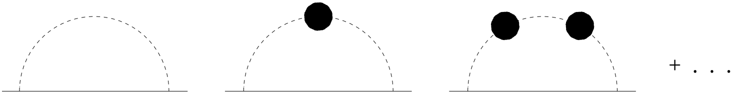

Figure: Sum of the bubbles in the quark propagator.

Pole mass (Bigi, Shifman, Uraltsev, Vainshtein; Beneke, Braun)

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \qquad B_n \sim n!$$

Beyond bubbles $\rightarrow$ Parisi; Beneke, ...

Modern view (... or not $\rightarrow$ NP OPE (Novikov, Shifman, Vainshtein, Zakharov))
Renormalon: Asymptotic behavior of the perturbative series associated to the Operator Product Expansion (OPE)/factorization/effective field theories (in asymptotically free gauge theories with marginal operators).

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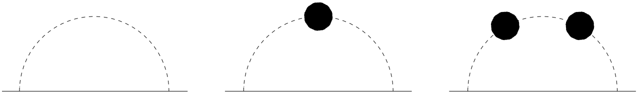

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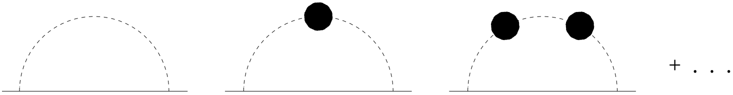

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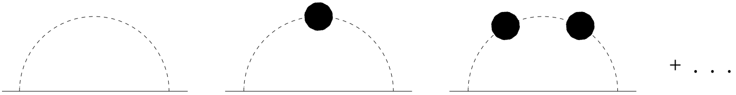

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Effective field theories (HQET, NRQCD, pNRQCD, ...)

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n$$

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$

IF SO such behavior should comply with the Operator Product Expansion.

Effective-field-theory/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.

Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})\,, \quad m_{\tilde{G}} = m_{\tilde{g},{\rm OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},{\rm OS}}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

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$$\delta_{np}^{\rm (pert.)} m_{\rm OS} = \delta_{np}^{\rm (pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

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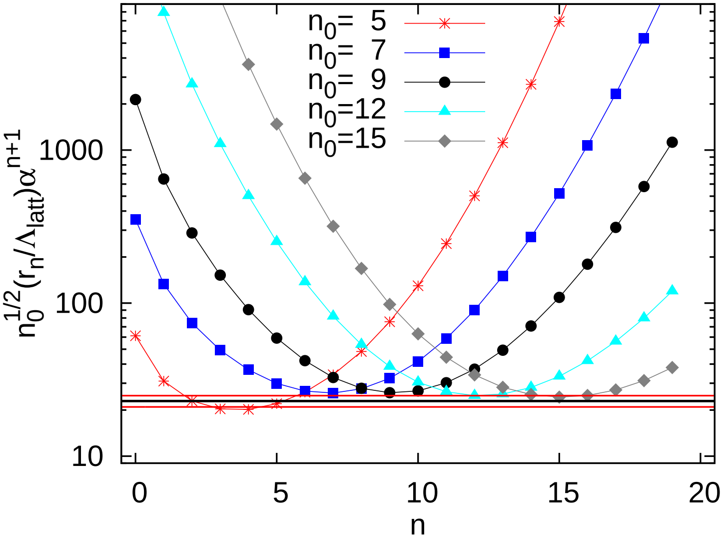

Figure: *c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx$ 0.096 ($n_0 = 5$) to $\alpha(\nu) \approx$ 0.036 ($n_0 = 15$). Bali, Bauer, AP, Torrero, 1303.3279.*

The maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*} ,$$

where $n^* \sim \frac{a}{\alpha_s}$. If a is positive c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{\rm QCD})^{\frac{|a|\beta_0}{2\pi}} .$$

The Borel transform of $c(\nu)$ reads

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} dt\, e^{-t/\alpha_s} B[c](t).$$

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Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \to c_n \sim n!$):

$$\int d^4x e^{iqx}\langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

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Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
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$$m_{\rm OS} = m_{\overline{\rm MS}} + \int\limits_0^\infty {\rm d}t\, e^{-t/\alpha_s}\, B[m_{\rm OS}](t)\,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}.$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots \right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n^{asym} = N_m\,\nu\, \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots \right).$$

$$b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\, b\beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right), \qquad \cdots$$

Determination of N_m

Lee → use a new function that kill the singularity:

$$D_m(u) = \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\rm OS}](t(u))$$
$$= N_m \nu \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots \right) + (1-2u)^{1+b} (\text{analytic term})$$

$$N_m = \frac{1}{\nu} D_m(u=1/2) \simeq \frac{1}{\nu} \sum_{k=0}^{N} D_m^{(k)} \left(\frac{1}{2}\right)^k$$
$$N_m = \frac{r_n}{(r_n^{asym}/N_m)}$$

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1 + (2u-1)\ln\frac{\nu}{m} + \cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

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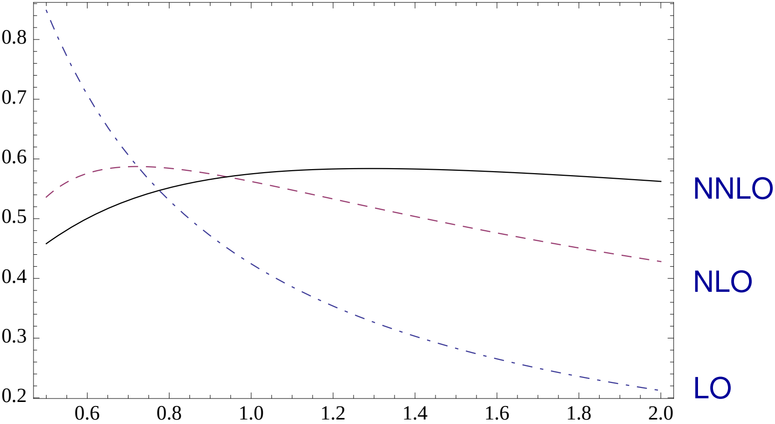

Figure: N_m for $n_l = 3$, as a function of $x \equiv \mu/m_b$, obtained from $D_m(u = 1/2)$. We name the different lines as NLO (dashed-dotted), NLO (dashed) and NNLO (solid) for $n = 0, 1, 2$, respectively.

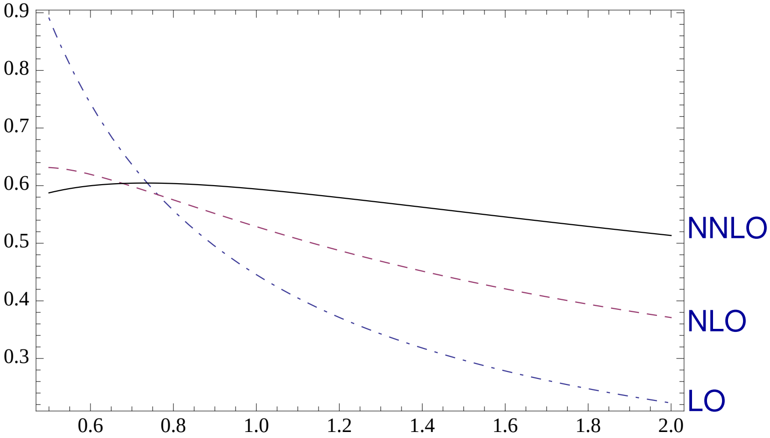

Figure: N_m for $n_l = 3$, as a function of $x \equiv \mu/m_b$, obtained from r_n/r_n^{asym} with r_n^{asym} truncated at $\mathcal{O}(1/n^3)$. We name the different lines as NLO (dashed-dotted), NLO (dashed) and NNLO (solid) for $n = 0, 1, 2$, respectively.

The static potential

$$V(r;\nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$$V_n^{asym} = N_V\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right)$$

$2m_{\rm OS} + V_s$ can be understood as an observable up to $O(r^2\Lambda^3_{\rm QCD}, \Lambda^2_{\rm QCD}/m)$ contributions $\rightarrow 2N_m + N_V = 0$

$$D_V(u) = \sum_{n=0}^{\infty} D_V^{(n)} u^n = (1-2u)^{1+b} B[V_s](t(u))$$
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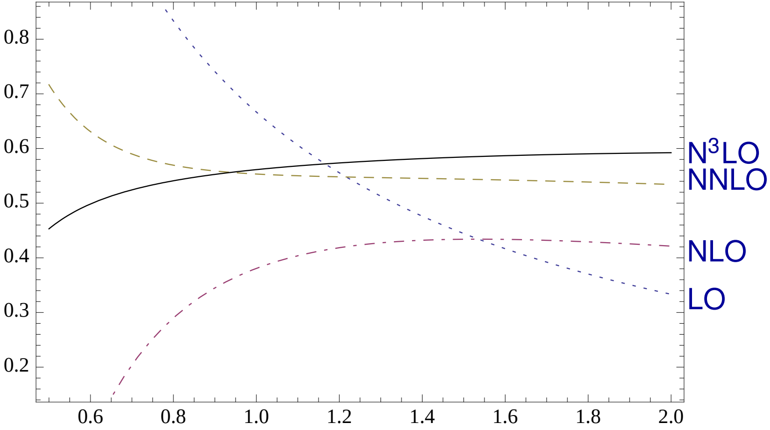


Figure: $-N_V/2$ obtained using $N_V\nu = D_V(u = 1/2)$ for $N_l = 3$, as a function of $x \equiv \nu r$, truncated at $N = 0, 1, 2, 3$, which we name as LO (dotted), NLO (dashed-dotted), NNLO (dashed) and NNNLO (solid) respectively.

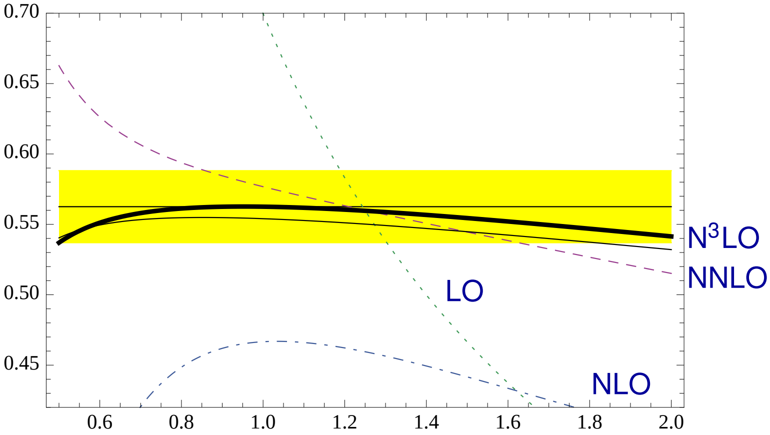


Figure: $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$N_m(n_l = 0) = 0.600(29)\,, \qquad N_m(n_l = 3) = 0.563(26)\,.$$

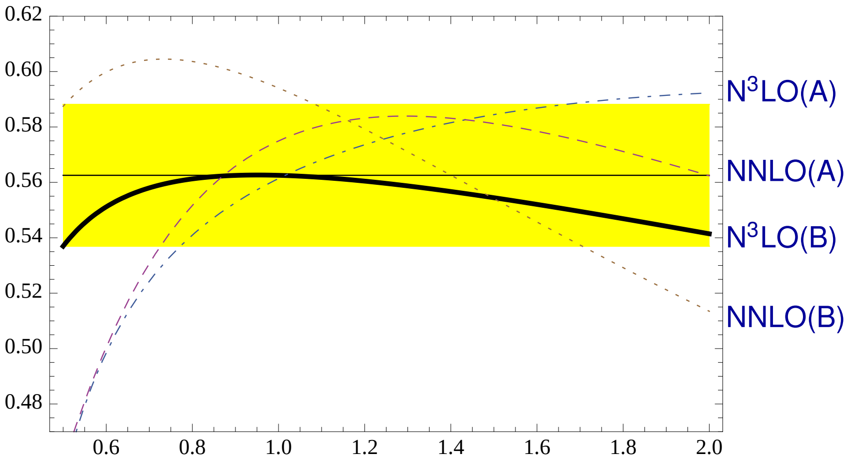


Figure: N_m obtained applying methods A) (dashed-dotted line) and B) (solid thick curve) to the the static potential at NNNLO for $N_l = 3$ as a function of $x \equiv \nu r$. For comparison we also include the NNLO evaluation from method A) and B) applied to the pole mass (dashed and dotted line respectively). The horizontal central line and bands correspond to our final central value and error.

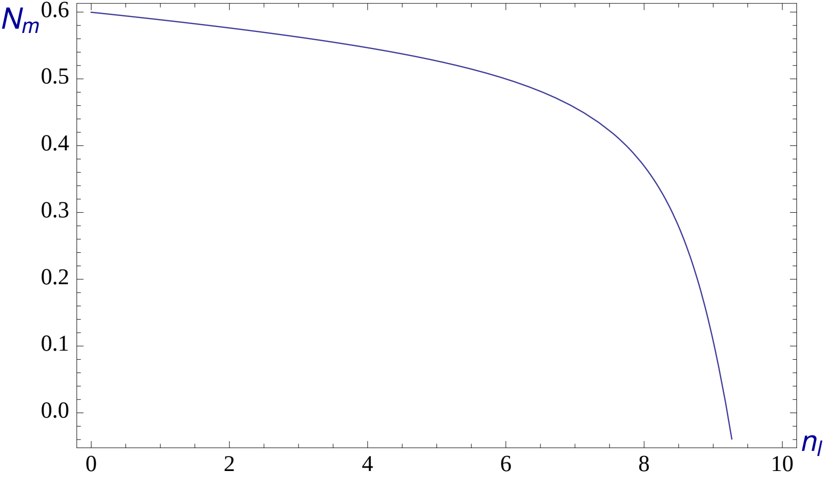
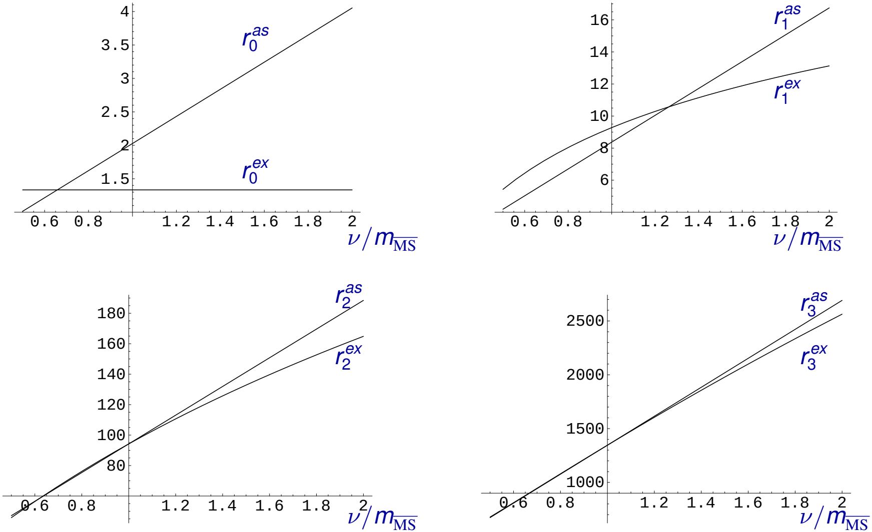


Figure: $N_m(x=1)$ *obtained from* $-(N_V/2)v_n/v_n^{asym}$ *($n=3$) from the static potential (NNNLO) as a function of* n_l.

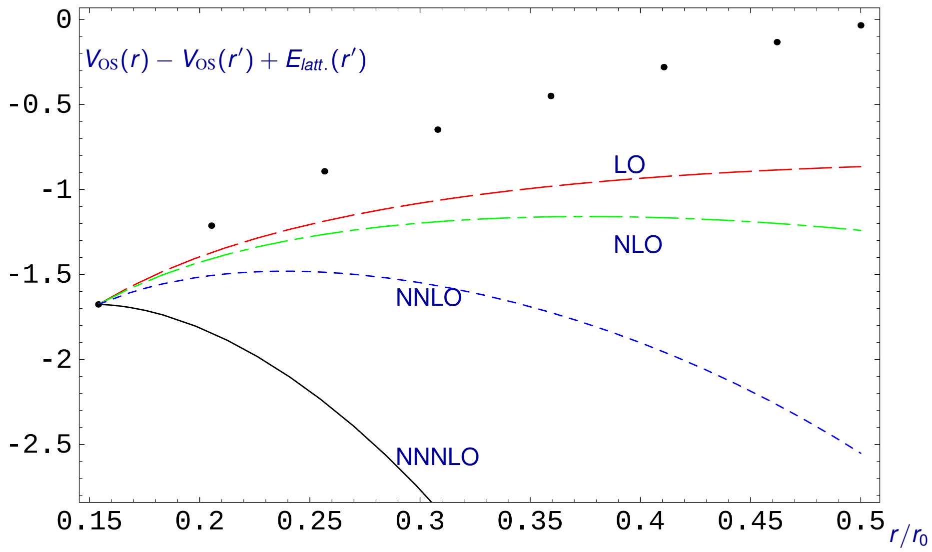
Check : $$r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!} \sim \nu$$

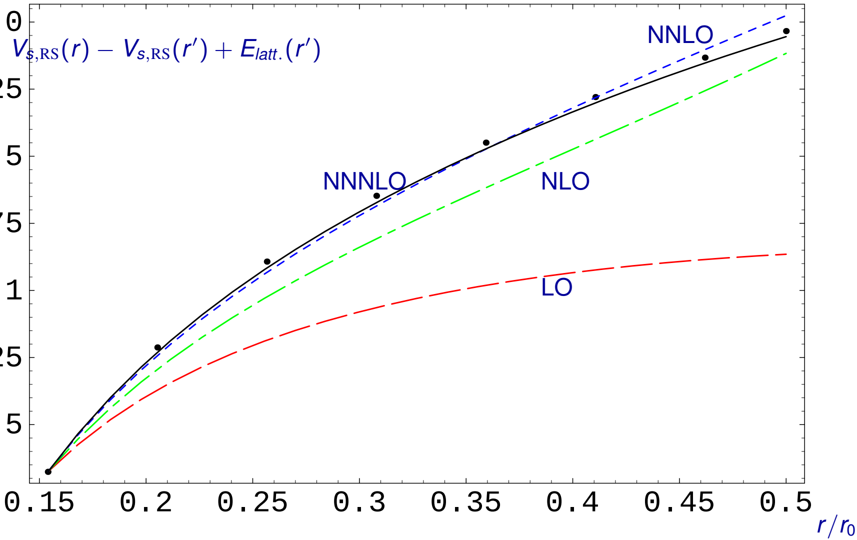


Some applications

Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

- $$2m_{\rm OS} + E_s(r) = 2m_{\rm OS} + V_s(r) + \mathcal{O}(r^2)$$
  $2m_{\rm OS} + V_s = 2m_{\rm RS} + V_{s,\rm RS}$ is renormalon free. Good description of the singlet static potential at short distances.
- $$2m_{\rm OS} + E_H(r) = 2m_{\rm OS} + V_o(r) + \Lambda_H + \mathcal{O}(r^2)$$
  $2m_{\rm OS} + V_o + \Lambda_H = 2m_{\rm RS} + V_{o,\rm RS} + \Lambda_{H,\rm RS}$ is renormalon free. Good description of the octet static potential at short distances.
- Good description of heavy quarkonium properties: low-lying bound states, non-relativistic sum rules,...
- ...


$V_{\rm OS}(r) - V_{\rm OS}(r') + E_{latt.}(r')$
LO
NLO
NNLO
NNNLO
0
-0.5
-1
-1.5
-2
-2.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
r/r_0


$V_{s,\mathrm{RS}}(r) - V_{s,\mathrm{RS}}(r') + E_{latt.}(r')$
NNLO
NNNLO
NLO
LO
0
-0.25
-0.5
-0.75
-1
-1.25
-1.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
r/r_0

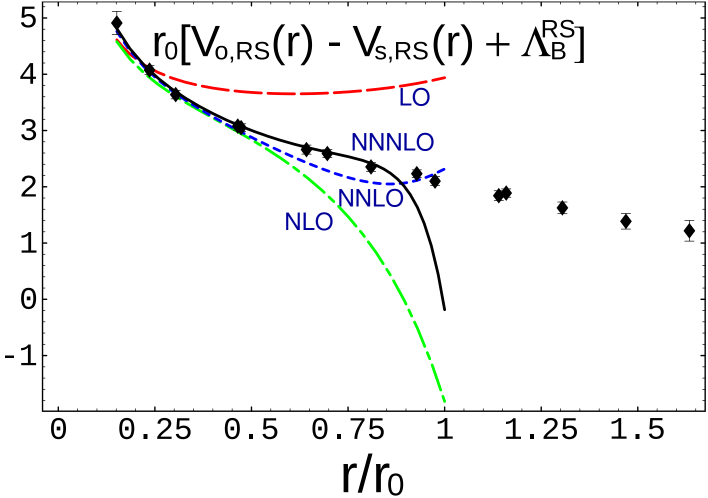


Figure: Splitting between the Π_u and the Σ_g^+ potentials and the comparison with the theoretical prediction.

$2m + V_o + \Lambda_B$ is renormalon free.

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon dominance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.

We use Numerical Stochastic Perturbation Theory in lattice regularization.

$$\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(3,\rho)}\alpha^{n+1}(1/a)\,(\text{fundamental}), \quad \delta m_{\tilde{g}} = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(8,\rho)}\alpha^{n+1}(1/a)\,(\text{adjoint})$$

$$\lim_{n\to\infty} c_n^{(3,\rho)} = r_n(\nu)/\nu$$

$$L^{(R)}(N_S,N_T) = \frac{1}{N_S^3}\sum_{\mathbf{n}}\frac{1}{d_R}\text{tr}\left[\prod_{n_4=0}^{N_T-1} U_4^R(n)\right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$P^{(R,\rho)}(N_S,N_T) = -\frac{\ln\langle L^{(R,\rho)}(N_S,N_T)\rangle}{aN_T} = \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S,N_T)\alpha^{n+1},$$

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Perturbative OPE (Zimmermann) at finite volume

$$\delta m(N_S) = \lim_{N_T\to\infty} P(N_S, N_T) \qquad \text{and} \qquad c_n(N_S) = \lim_{N_T\to\infty} c_n(N_S, N_T)\,.$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S)\,,$$

$f_n^{(0)} = f_n$ and the coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m with $m < n$ and β_j with $j \leq n-1$.

$$f_1(N_S) = f_1 + f_0\frac{\beta_0}{2\pi}\ln(N_S)\,,$$

$$f_2(N_S) = f_2 + \left[2f_1\frac{\beta_0}{2\pi} + f_0\frac{\beta_1}{8\pi^2}\right]\ln(N_S) + f_0\left(\frac{\beta_0}{2\pi}\right)^2\ln^2(N_S)\,,$$

and so on.

"Physical interpretation"

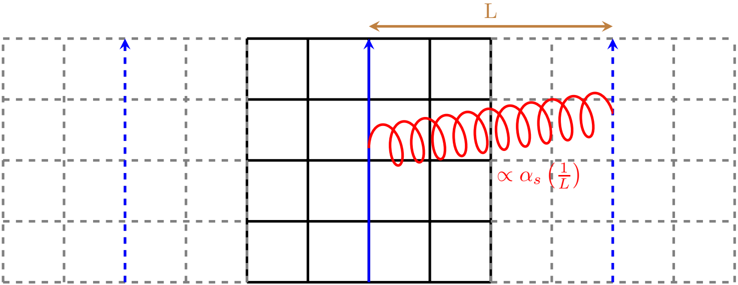

Figure: *Self-interactions with replicas producing* $1/L = 1/(aN_S)$ *Coulomb terms.*

$$P \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right),$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n!\,, \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!}\,.$$

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

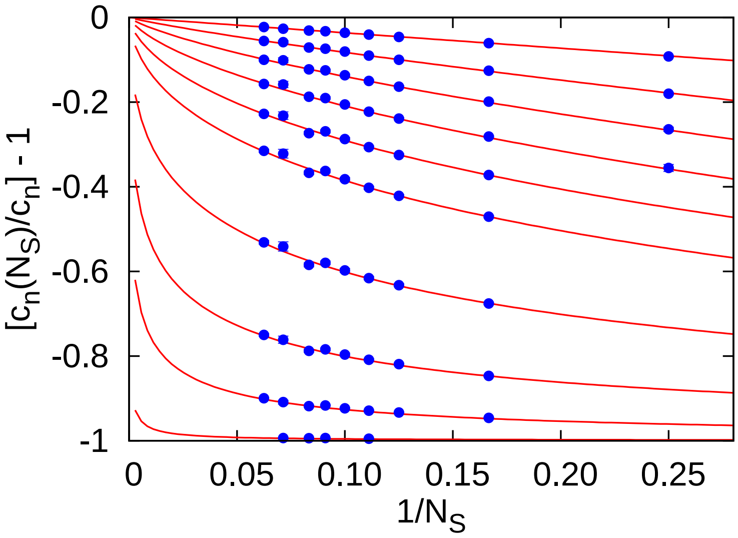

Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S) f_{0,\mathrm{DLPT}}^{(3,0)} / c_{0,\mathrm{DLPT}}^{(3,0)}$ *is shown for* $n = 0$.

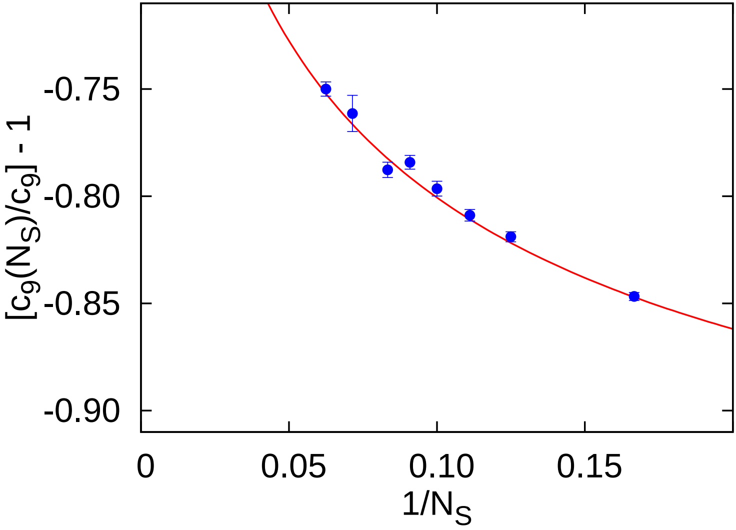

Figure: *Zoom of previous Figure for* $n = 9$.

Ratios

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n} = \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}$$

$$= \frac{\beta_0}{2\pi}\left\{1 + \frac{b}{n} - \frac{bs_1}{n^2} + \frac{1}{n^3}\left[b^2 s_1^2 + b(b-1)(s_1 - 2s_2)\right] + \mathcal{O}\left(\frac{1}{n^4}\right)\right\} \quad .$$

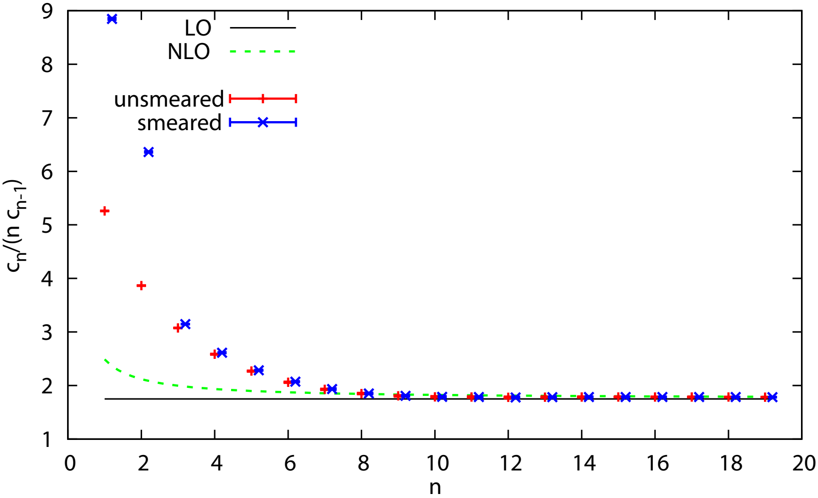

Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

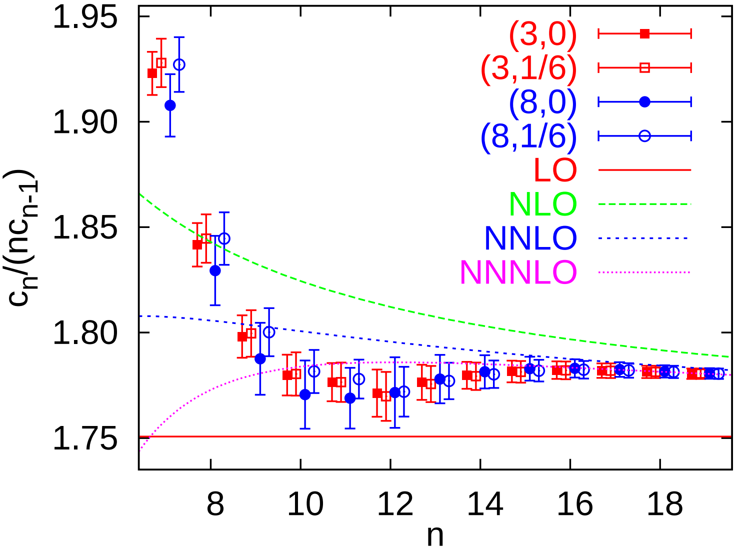

Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

$$f_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

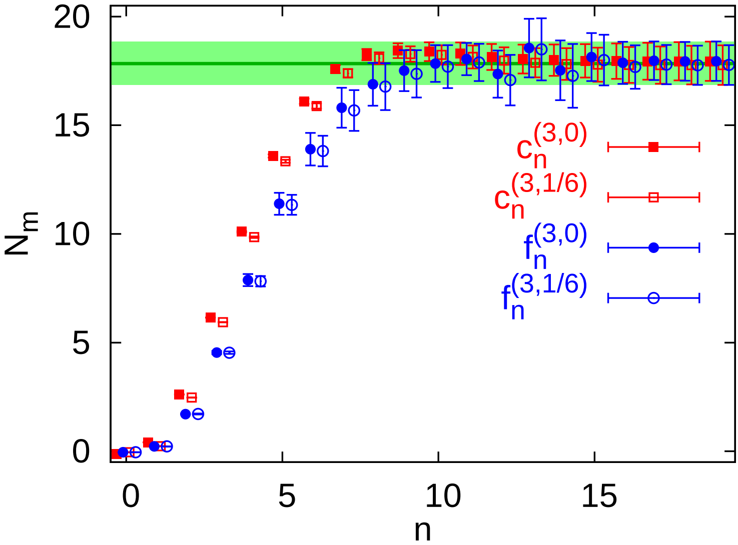

Figure: N_m^{latt}, determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result.

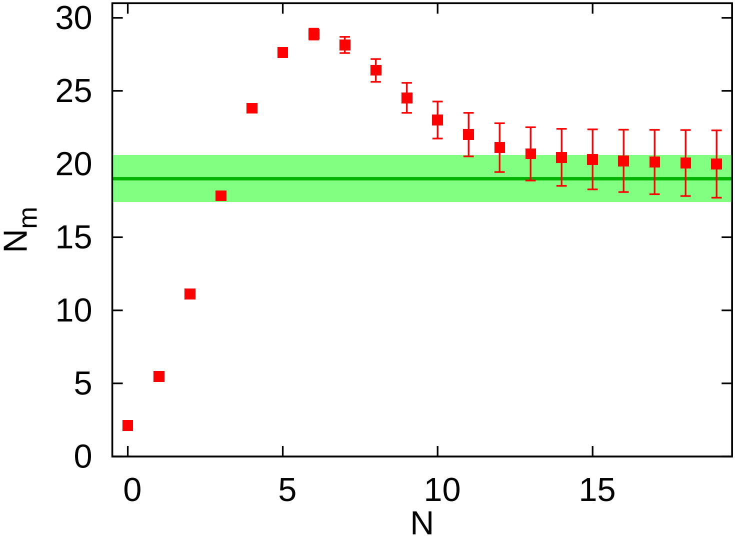

Figure: N_m^{latt}, determined from the coefficients $c_n^{(3,0)}$, $n \leq N$, using $(1-2u)^{1+b}B^{(N)}[\delta m](t(u))$.

From lattice to $\overline{\rm MS}$ scheme

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right),$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}, \quad \text{where} \quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}.$$

This yields the numerical values

$$N_m^{\overline{\rm MS}} = 0.620(35), \quad C_F/C_A\,N^{\overline{\rm MS}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\rm MS}}_{\Lambda} = 0.610(41).$$

Other combinations of interest are

$$N^{\overline{\rm MS}}_{V_s} = -1.240(69), \quad N^{\overline{\rm MS}}_{V_o} = 0.13(12).$$

Assuming that

$$c_{3,\overline{\rm MS}} \simeq N_m^{\overline{\rm MS}}\left(\frac{\beta_0}{2\pi}\right)^3\frac{\Gamma(4+b)}{\Gamma(1+b)}\left(1 + \frac{b}{(3+b)}s_1 + \frac{b(b-1)}{(3+b)(2+b)}s_2 + \cdots\right),$$

and using our central value $c^{(3,0)}_{3,\rm latt} = 794.5$, we obtain

$$d_3 \simeq 352(3), \qquad \beta_3^{\rm latt} = -1.16(12)\times 10^6.$$

From lattice to $\overline{\text{MS}}$ scheme

$$\alpha_{\overline{\text{MS}}}(\mu) = \alpha_{\text{latt}}(\mu)\left(1 + d_1\alpha_{\text{latt}}(\mu) + d_2\alpha^2_{\text{latt}}(\mu) + d_3\alpha^3_{\text{latt}}(\mu) + \mathcal{O}(\alpha^4_{\text{latt}})\right) ,$$

$$N^{\overline{\text{MS}}}_{m,m_{\tilde{g}}} = N^{\text{latt}}_{m,m_{\tilde{g}}}\Lambda_{\text{latt}}/\Lambda_{\overline{\text{MS}}} , \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\text{latt}} \approx 28.809338139488\,\Lambda_{\text{latt}} .$$

This yields the numerical values

$$N^{\overline{\text{MS}}}_m = 0.620(35) , \quad C_F/C_A\,N^{\overline{\text{MS}}}_{m_{\tilde{g}}} = -C_F/C_A\,N^{\overline{\text{MS}}}_{\Lambda} = 0.610(41) .$$

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and using our central value $c^{(3,0)}_{3,\text{latt}} = 794.5$, we obtain

$$d_3 \simeq 352(3) , \qquad \beta^{\text{latt}}_3 = -1.16(12)\times 10^6 .$$

From lattice to $\overline{\rm MS}$ scheme

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right)\,,$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}\,, \quad \text{where} \quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}\,.$$

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$$d_3 \simeq 352(3)\,, \qquad \beta^{\rm latt}_3 = -1.16(12)\times 10^6\,.$$

Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]}\, P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \rightarrow \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

where

$$P_{\rm pert} = \sum_{n\geq 0} p_n \alpha^{n+1},\ C_{\rm G} = 1 + \sum_{k\geq 0} c_k \alpha^{k+1},\ \frac{\pi^2}{36} a^4 \langle G^2\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n \alpha^{n+1}((Na)^{-1})$$

$$\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$$

$$= \sum_{n\geq 0} p_n \alpha^{n+1} - \frac{1}{N^4}\left(1 + \sum_{k\geq 0} c_k \alpha^{k+1}(a^{-1})\right) \times \sum_{n\geq 0} f_n \alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

$$\left(\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right)\right)$$

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$$\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$$

$$= \sum_{n\geq 0} p_n\alpha^{n+1} - \frac{1}{N^4}\left(1+\sum_{k\geq 0} c_k\alpha^{k+1}(a^{-1})\right)\times\sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

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$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

$$p_n^{\rm latt} \overset{n\to\infty}{=} N_P^{\rm latt} \left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \times \left\{1 + \frac{20.08931\ldots}{n+db} + \frac{505 \pm 33}{(n+db)(n+db-1)} + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}.$$

$$\frac{p_n}{np_{n-1}} = \frac{\beta_0}{2\pi d}\left\{1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{db\left[1 - 3ds_1 + d^2 b(s_1 + 2s_2)\right]}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}.$$

$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

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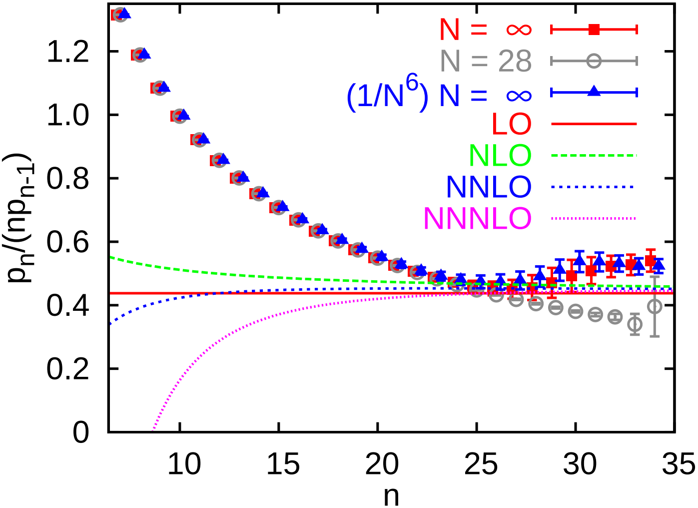


Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

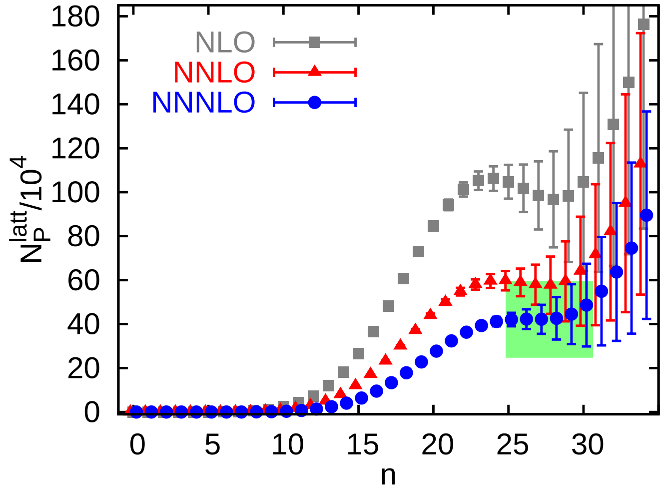
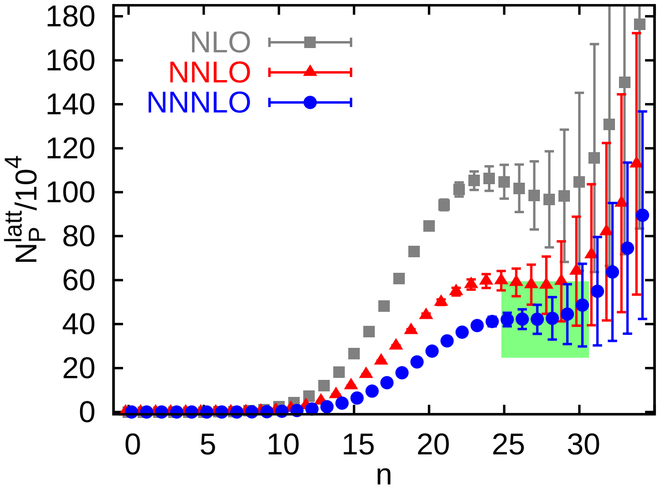


Figure: N_P, determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result.

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_G^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$



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Beyond perturbation theory

$$\langle P\rangle_{\rm pert} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle O_{\rm G}\rangle_{\rm soft} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na}$$

Beyond perturbation theory

$$\langle P\rangle_{\rm MC} = \frac{1}{Z}\int[dU_{x,\mu}]\,e^{-S[U]}P[U]\Big|_{\rm MC} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36}C_{\rm G}(\alpha)\,a^4\langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right).$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \to \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1+\mathcal{O}(\Lambda_{\rm QCD}^2(Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \to \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1+\mathcal{O}\left(\frac{1}{\Lambda_{\rm QCD}^2(Na)^2}\right)\right],$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda_{\rm QCD}^4$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

$$\langle G^2\rangle = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)}\left[\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)\right] + \mathcal{O}(a^2\Lambda_{\rm QCD}^2).$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha), \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j\alpha^{j+1}.$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0}\alpha^{n_0+1}$ is minimal.

Beyond perturbation theory

$$\langle P\rangle_{\rm MC} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm MC} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1 + \mathcal{O}(\Lambda^2_{\rm QCD}(Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1 + \mathcal{O}\left(\frac{1}{\Lambda^2_{\rm QCD}(Na)^2}\right)\right] ,$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda^4_{\rm QCD}$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

$$\langle G^2\rangle = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)}\left[\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)\right] + \mathcal{O}(a^2\Lambda^2_{\rm QCD}) .$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha) , \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1} .$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0}\alpha^{n_0+1}$ is minimal.

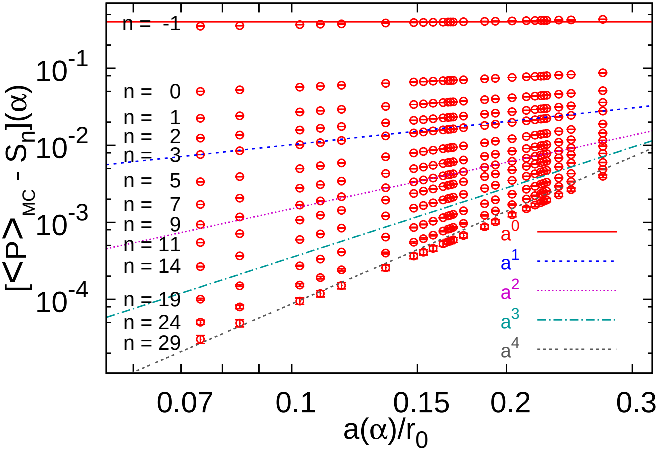

Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs. $a(\alpha)/r_0$. The lines $\propto a^j$ are drawn to guide the eye.

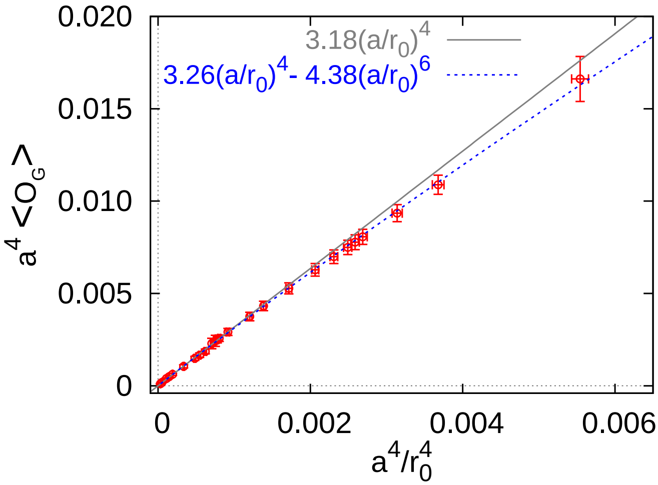

Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)$. The linear fit is to $a^4 < 0.0013\, r_0^4$ points only.

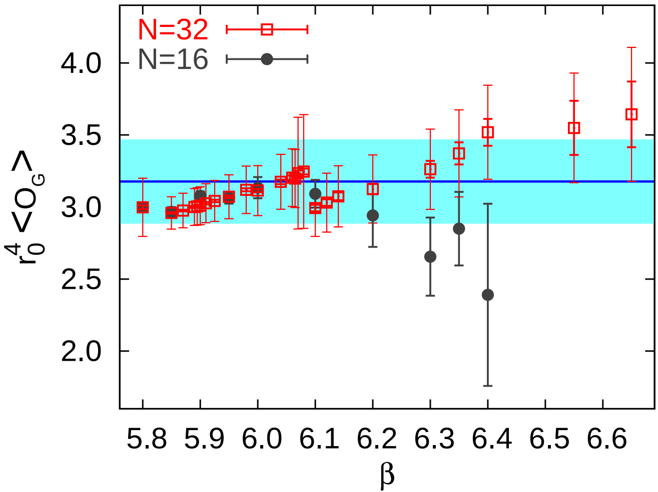

Figure: $\langle G^2 \rangle$ evaluated using the $N = 16$ and $N = 32$ MC data of Boyd et al. The error band is our prediction for $\langle G^2 \rangle$.

$$\langle G^2 \rangle = 3.18(29) r_0^{-4} = 24.2(8.0) \Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4 \,.$$

Uncertainty of the sum due to the truncation

$$\delta S_P = \sqrt{n_0}\, p_{n_0} \alpha^{n_0+1} \approx \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\, \Gamma(1+db)} N_P(\Lambda a)^4 \approx 12.06\, N_P(\Lambda a)^4 \,.$$

This object is scheme- and scale-independent (to $1/n$-precision)

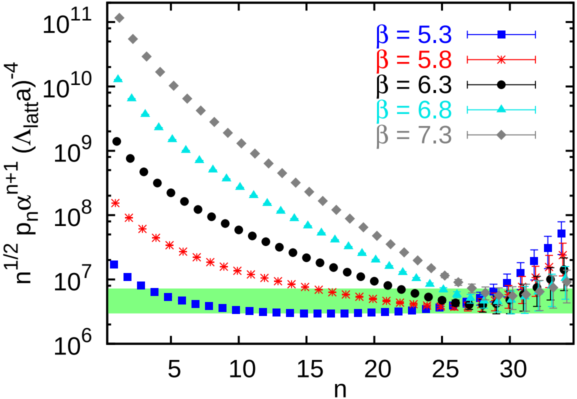

Figure: The combination $\sqrt{n}\, p_n \alpha^{n+1}/(\Lambda_{\rm latt} a)^4$, as a function of n for $\beta = 5.3, 5.8, 6.3, 6.8$ and 7.3. The error band corresponds to the theoretical expectation $12.06\, N_P = 5.1(2.1) \times 10^6$.

$$\sqrt{n_0}\frac{|r_{n_0}|}{\Lambda_{\rm latt}}\alpha^{n_0+1}(\nu) = \frac{2^{3/2-b}\pi^{3/2}}{\beta_0\Gamma(1+b)}|N_m| \approx 1.206\,|N_m|\,, \tag{1}$$

where $N_m = 19.0 \pm 1.6$.

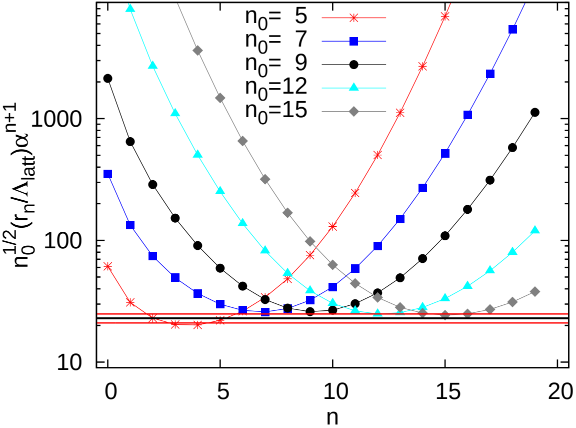

Figure: c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$).

$$\delta\langle G^2\rangle_{\rm NP} \simeq \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\,\Gamma(1+db)}\, N_{\rm G}^{\overline{\rm MS}}\Big|_{n_l=0} \Lambda^4_{\overline{\rm MS}} = 27(11)\,\Lambda^4_{\overline{\rm MS}} \sim 0.087\,{\rm GeV}^4\,.$$

$$\langle G^2\rangle = 3.18(29) r_0^{-4} = 24.2(8.0)\Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4\,.$$

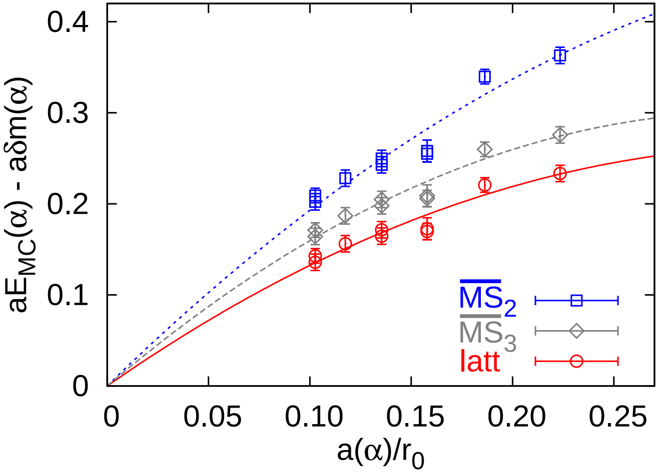

Figure: $aE_{\rm MC} - a\delta m$ vs. a/r_0. The expansion of $a\delta m$ was also converted into the $\overline{\rm MS}$ scheme at two ($\overline{\rm MS}_2$) and three ($\overline{\rm MS}_3$) loops. The curves are fits to $\overline{\Lambda} a + ca^2$.

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26)\,.$$

It is compulsory to take into account renormalon effects in order to do precision computations in heavy quark physics.

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

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$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$

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$$\delta\langle G^2\rangle_{\rm NP} = 27(11)\,\Lambda^4_{\overline{\rm MS}} \simeq 0.087\,{\rm GeV}^4\,. \qquad \langle G^2\rangle = 24.2(8.0)\Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4\,.$$

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Semi-analytic control of the n_l dependence??

Control of the scheme dependence??

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Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

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Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.
The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}}\left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right),$$

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m\, \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}\,.$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1}\,,$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.
Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m \nu_f \left(1 + c_1 + c_2 + \cdots\right)\,.$$

	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^{20})$	$\mathcal{O}(\alpha^{32})$
$N_S(N_T)$	$4(4)$	$8(8, 10, 12, 14)$	$4(8)$

Table: *The first arrow states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.*

$\mathcal{O}(\alpha^3)$	$N_S(N_T)$	$5(5, 6, 7, 8, 10)$			
$\mathcal{O}(\alpha^4)$	$N_S(N_T)$	$4(5, 6, 7, 8, 10, 12, 16, 20, 24)$	$12(16, 20)$		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	$6(6, 8, 10, 12, 16)$	$8(12, 16)$		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	$10(8, 12, 16, 20)$	$16(12, 16, 20)$		
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	$7(7, 8)$	$8(8, 10)$	$9(12)$	$10(10)$
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	$11(16)$	$12(12)$	$14(14)$	

Table: *The first column states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.*

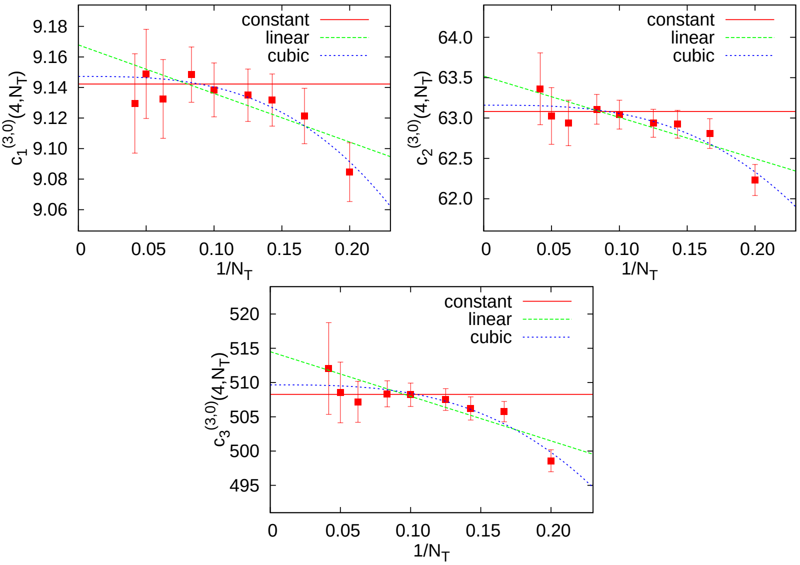

Figure: $c_{1,2,3}^{(3,0)}(4, N_T)$ *as a function of* $1/N_T$*, in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the* $N_T > 10$ *points.*