RESURGENCE, THE 1/N EXPANSION, AND STRING THEORY

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The resurgence theory of Ecalle gives the most systematic way of organizing the information encoded in asymptotic expansions and their non-perturbative corrections.

It has been applied successfully in non-linear ODEs [Costin, Delabaere, ...] and Quantum Mechanics [Zinn-Justin, Jentschura, Voros, Delabaere, Dillinger, Pham, ...]. There have been some recent applications in QFT [Argyres, Dunne, Unsal, Cherman...] which you have heard about in this conference

These are however examples with *one coupling constant*, where one looks at series of form

$$S(g) = \sum_{k \geq 0} a_k g^k$$

$a_k \to$ numbers!

In this talk I want to look at problems with *two coupling constants.* They provide generalizations of the problems which have been studied so far:

non-linear ODEs → matrix models at large N

QM/QFT → $1/N$ expansion

QM/QFT → string theory

These generalizations have been comparatively much less studied

Matrix models

We will define the matrix integral partition function by:

$$Z(N, g_s) = \int \prod_{i=1}^{N} \mathrm{d}x_i \, \mathrm{e}^{-\frac{1}{g_s} V(x_i)} \Delta^2(x)$$

V(x) polynomial — Vandermonde

Its free energy has an asymptotic expansion (*1/N expansion* or *'t Hooft expansion*) of the form

$$F(N, g_s) = \log Z(N, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

functions!

$t = g_s N$ 't Hooft parameter

This is a non-trivial generalization of the saddle-point expansion of conventional integrals

The problem of explicitly computing the genus *g* free energies was first addressed in the classic paper of [Brezin-Itzykson-Parisi-Zuber]. An *a priori* complete solution of the problem was worked out by B. Eynard and collaborators by using algebro-geometric methods.

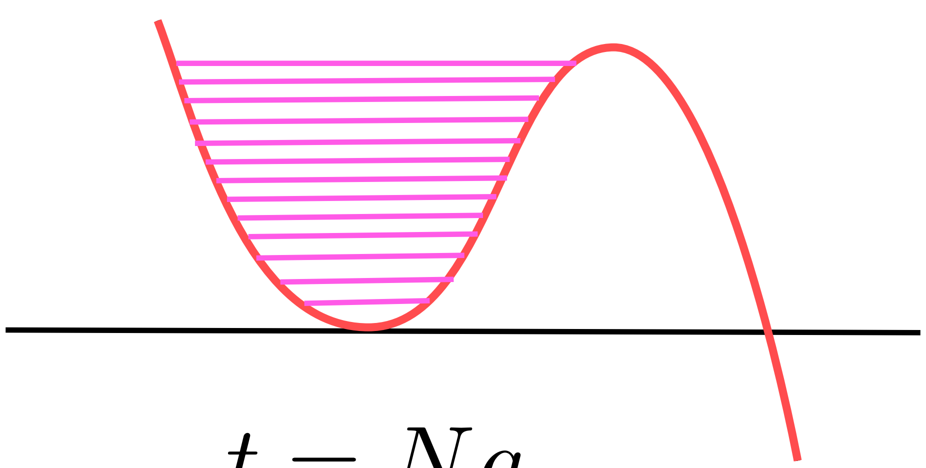
The genus g free energies are analytic at $t=0$. However, for fixed 't Hooft parameter, the $1/N$ expansion diverges as

$$F_g(t) \sim (2g)!(A(t))^{-2g}$$

Therefore, this raises the type of questions for which resurgence is a useful tool

In fact, matrix models are generalizations of problems which have been already adressed by resurgence: non-linear ODEs

I will present a simple example of this, which is one of my favorites. Let us consider a model where *V(x)* is a cubic polynomial. There is a critical value of the 't Hooft parameter for which the genus g free energies have a critical behavior



$$t = N g_s$$

$$F_g(t) \sim c_g (t - t_c)^{5/2(g-1)}$$

One can then define a "double-scaled free energy"

$$F_{\rm ds}(z) = \sum_{g \geq 0} c_g z^{-\frac{5}{2}(g-1)}$$

and $\quad u(z) = F''_{\rm ds}(z) \quad$ satisfies Painleve I

$$-\frac{1}{6}u''(z) + u^2 = z \qquad u(z) \sim \sqrt{z}, \qquad z \to \infty$$

The asymptotic divergence of the *1/N* expansion becomes now the standard divergence of a formal solution to an ODE

$$c_g \sim (2g)! A^{-2g}, \quad A = \frac{8\sqrt{3}}{5}$$

[Eynard-Zinn-Justin, Shenker, …]

Physically this describes the partition function of quantum gravity in two dimensions, as a function of the "cosmological constant"

[Brezin-Kazakov, Douglas-Shenker, Gross-Migdal]

The resurgent analysis of PI involves a trans-series solution of the form

$$u = u^{(0)}(z) + \sum_{\ell \geq 1} C^{\ell} \mathrm{e}^{-\ell A z^{5/4}} u^{(\ell)}(z), \qquad z^{5/4} = \frac{1}{g_s}$$

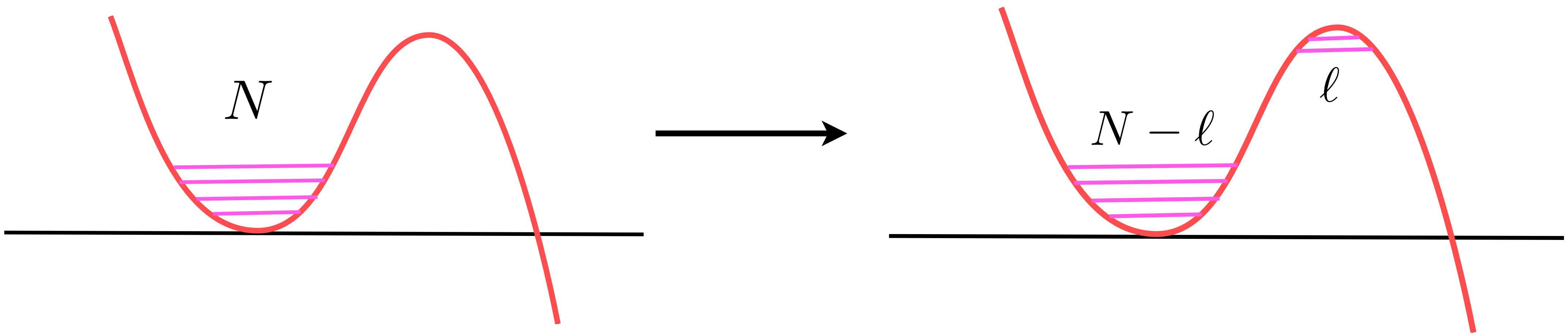
and from here one obtains a precise Dingle-type relation for the growth of the coefficients of the double-scaled free energy

$$c_g = S\, \Gamma\Big(2g - \frac{5}{2}\Big) A^{-2g} \Big\{ 1 + \frac{a_1}{g} + \frac{a_2}{g^2} + \cdots \Big\}$$

S ↑ Stokes constant (one instanton at one-loop)

a_1, a_2 ↑ coefficients in $u^{(1)}(z)$

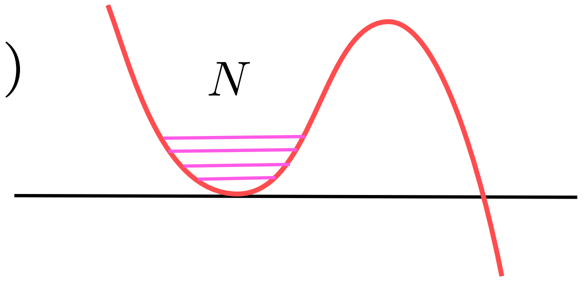
We can now go back to the matrix model and ask what is the "physical" origin of the trans-series sectors. They are given by the double-scaling limit of the large *N* instantons of the matrix model, described by *eigenvalue tunneling* [David]. In 2d gravity, they correspond to D-brane amplitudes



Using these configurations we can then build up a trans-series for the full matrix model.

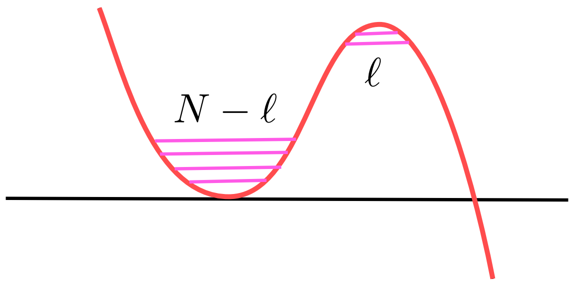
perturbative partition function

$$Z^{(0)}(t, g_s) = \int_{\text{one cut}} \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} \mathrm{Tr}\, V(M)}$$



The total partition function will include instanton corrections to the perturbative partition function, coming from eigenvalue tunneling:

$$Z(t, g_s) = Z^{(0)}(t, g_s) + \sum_{\ell \geq 1} \mathrm{e}^{-\ell A(t)/g_s} C^{\ell} Z^{(\ell)}(t, g_s)$$



This trans-series can be obtained from a difference equation which, in the double-scaling limit, becomes Painleve I [M.M. 2008]

The above trans-series for Painleve I is not the most general one. Ecalle's theory requires [Garoufalidis-Its-Kapaev-M.M]

$$\sum_{\ell,m} C_1^\ell C_2^m \mathrm{e}^{-(\ell-m)Az^{5/4}} u^{(\ell,m)}(z)$$

This general trans-series is needed to understand the asymptotics of the instanton solutions. A similar construction can be done for the full matrix model [Aniceto-Schiappa-Vonk]

Open question: what is the meaning of the general trans-series in the original matrix integral?

The ABJM matrix model

The ABJM matrix model is defined by

$$Z(N,k) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{8\pi k} \frac{1}{\cosh \frac{x_i}{2}} \prod_{i<j} \left(\tanh \left(\frac{x_i - x_j}{2k} \right) \right)^2$$

$g_s \sim 1/k$ 'tHooft parameter $\lambda = \frac{N}{k}$

It was shown, by using localization techniques [Kapustin-Willett-Yaakov] that this matrix integral computes the partition function on the three-sphere of ABJM theory, a superconformal field theory in three dimensions which is dual to M-theory on a certain AdS background [Maldacena]. It therefore describes quantum (super)gravity in high dimensions

‘t Hooft expansion

N large, λ fixed

‘t Hooft/genus expansion

$$F(N,k) = \log Z(N,k) = \sum_{g=0}^{\infty} F_g(\lambda) N^{2-2g}$$

The genus g free energies are computable for arbitrary g.
The resulting series is factorially divergent

$$F_g(\lambda) \sim (2g)!(A(\lambda))^{-2g}$$

This series seems to be Borel summable. However, the Borel resummation does *not* agree with the exact answer, and non-perturbative corrections are still needed

[Grassi-M.M.-Zakany]

M-theory expansion

N large, k fixed

M-theory expansion/
"thermodynamic" limit

to go from the 't Hooft expansion to the M-theory expansion: *resum* the genus expansion at fixed, strong 't Hooft coupling -it can be done! (Gopakumar-Vafa resummation)

$$F_g(\lambda) = \sum_{\ell \geq 1} a_{g,\ell}(\lambda)\, \mathrm{e}^{-2\pi\ell\sqrt{2\lambda}}$$

$$F_{\text{'t Hooft}}(N,k) = \sum_{\ell \geq 1} c_\ell(N,k) \mathrm{e}^{-2\pi\ell\sqrt{2N/k}}$$

↓

they have *poles* for all rational *k*!

The 't Hooft/genus expansion is incomplete

Of course, we knew this, since trans-series should be included. However, in this problem the resulting M-theory series is not divergent, and "most" of the coefficients in

$$F(N,k) = \sum_{\ell \geq 1} c_\ell(N,k) \mathrm{e}^{-\ell\sqrt{N/k}}$$

grow only exponentially

Therefore, we don't have to cure a divergent expansion. We have to cure a "mostly" convergent expansion, where some of the coefficients have poles

Beyond the 't Hooft expansion

We need a *new treatment of the matrix model* which gives the missing information in the 't Hooft expansion.

In the *Fermi gas approach* [M.M.-Putrov], *Z(N,k)* is interpreted as the canonical partition function of an one-dimensional *ideal* Fermi gas of *N* particles. The energy levels are determined by the spectral problem:

density matrix
$$\hat{\rho}|\psi_n\rangle = \mathrm{e}^{-E_n}|\psi_n\rangle \qquad n = 1, 2, \cdots$$

$$\rho(x_1, x_2) = \frac{1}{2\pi k} \frac{1}{\left(2\cosh\frac{x_1}{2}\right)^{1/2}} \frac{1}{\left(2\cosh\frac{x_2}{2}\right)^{1/2}} \frac{1}{2\cosh\left(\frac{x_1 - x_2}{2k}\right)}$$

k plays the role of *Planck's constant*: $\hbar = 2\pi k$

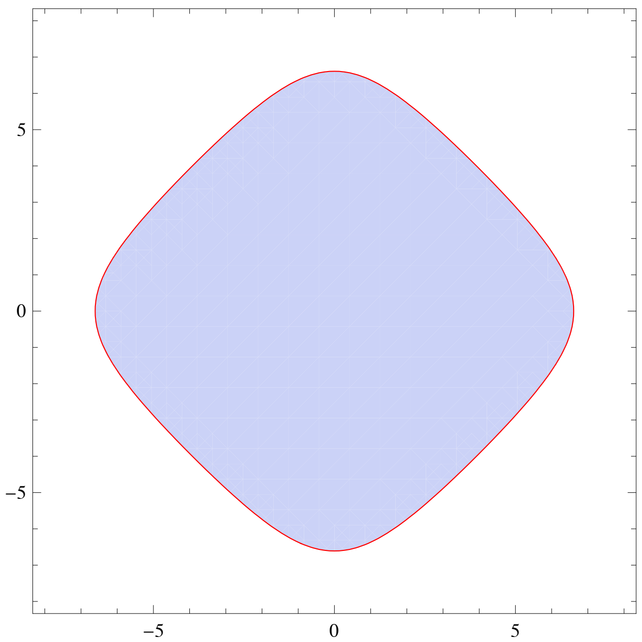
Semiclassical regime $\longleftrightarrow$ strong string coupling

This is an unconventional spectral problem (integral rather than differential equation). However, for large energies this is a gas of *N* fermions with Hamiltonian

$$H \approx \log\left(2\cosh\frac{p}{2}\right) + \log\left(2\cosh\frac{q}{2}\right) \approx \frac{|p|}{2} + \frac{|q|}{2}$$

semiclassically:

$$\mathrm{vol}(E;k) \approx 8E^2, \;\; E \gg 1$$



$$H(q,p) = E$$

To encode the information on the ideal gas, we can use the *quantum-corrected phase-space volume*:

$\mathrm{vol}(E;k)$

spectrum:

$$\mathrm{vol}(E;k) = 4\pi^2 k \left(n + \frac{1}{2}\right)$$

thermodynamics:

$$\frac{\mathrm{d}}{\mathrm{d}E}\mathrm{vol}(E;k) \propto \rho(E;k) \longrightarrow$$

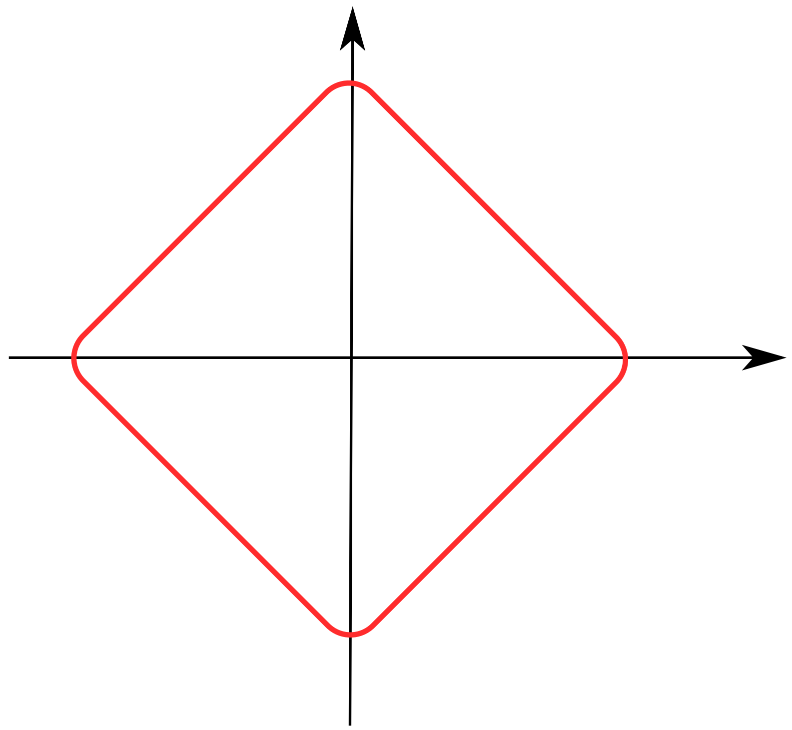
grand potential $J(\mu, k)$

$Z(N,k)$

In this problem,

large N $\longleftrightarrow$ large E $\longleftrightarrow$ large quantum numbers

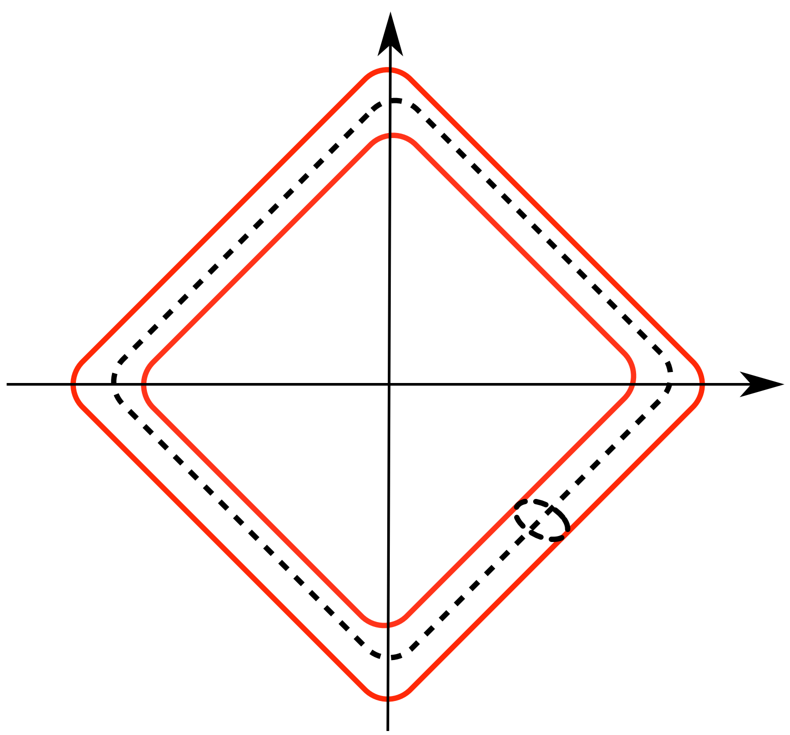
This suggests using the WKB expansion to calculate the quantum volume, working at *large E* but *fixed* Planck constant [Kallen-M.M]. Non-perturbative effects in N in $F(N,k)$ come from *exponentially small corrections in E*



perturbative WKB: *real* trajectories

$$\mathrm{vol_p}(E;k) = 8E^2 + \sum_{\ell \geq 1} \left(E a_\ell(k) + b_\ell(k)\right) \mathrm{e}^{-2\ell E}$$

analytic at $k = 0$ $\qquad b_1(k) = 4\pi k \csc\left(\frac{\pi k}{2}\right) \cos^2\left(\frac{\pi k}{2}\right)$



non-perturbative WKB: *complex* trajectories/instantons [Balian-Parisi-Voros]

$$\mathrm{vol_{np}}(E;k) = \sum_{\ell \geq 1} s_\ell(k)\, \mathrm{e}^{-4\ell E/k}$$

$$s_1(k) = 8\pi k \cot\left(\frac{2\pi}{k}\right)$$

equivalent to knowing all-genus free energies $F_g(\lambda)$

The perturbative WKB quantum volume leads again to an "almost" convergent series: when *k* is a fixed rational number, some of the coefficients have *poles*, but "most" of them are finite and grow only exponentially! In the free energy the perturbative WKB contribution computes terms of the form

$$\mathrm{e}^{-\sqrt{kN}}$$

which are non-perturbative from the point of view of the 't Hooft expansion. These correspond to membrane instantons in M-theory and I will call this the membrane contribution, or the M2-contribution

Of course, the spectrum is well-defined for any real value of *k*, therefore there must be something canceling the poles. This is the contribution of instantons! In fact, the total volume

$$\mathrm{vol}(E;k) = \mathrm{vol}_{\mathrm{np}}(E;k) + \mathrm{vol}_{\mathrm{p}}(E;k)$$

has *no* poles and determines the spectrum through an *exact quantization condition* [cf. Zinn-Justin, Voros]

$$\frac{\mathrm{vol}(E;k=1)}{4\pi^2} = \frac{2}{\pi^2}E^2 - \frac{7}{24} + \frac{8}{\pi^2}\mathrm{e}^{-4E} + \frac{1}{\pi^2}\mathrm{e}^{-4E} - \frac{52}{\pi^2}E\mathrm{e}^{-8E} - \frac{1}{4\pi^2}\mathrm{e}^{-8E} + \frac{1472}{3\pi^2}E\mathrm{e}^{-12E} - \frac{152}{9\pi^2}\mathrm{e}^{-12E} + \mathcal{O}(E\mathrm{e}^{-16E})$$

As a series in *exp(-4E)*, this seems to have a *finite* radius of convergence

From the exact, singularity-free quantum volume, we obtain the singularity-free ABJM free energy in the M-theory expansion, including the (resummed) 't Hooft expansion (which comes from non-perturbative WKB) *and* the contribution from membranes (which comes from perturbative WKB). Schematically,

$$F(N,k) = F_{\text{'t Hooft}}(N,k) + F_{\text{M2}}(N,k)$$

which has no poles: *HMO cancellation mechanism* [Hatsuda-Moriyama-Okuyama]

This will be a double expansion in

$$\mathrm{e}^{-\sqrt{N/k}} \qquad \text{and} \qquad \mathrm{e}^{-\sqrt{kN}}$$

and it seems also to have a finite radius of convergence

This leads to a very different picture of how to incorporate non-perturbative effects: the lack of non-perturbative information leads to poles, and then instantons remove the poles. The final result is a *convergent* series

So maybe in M-theory we don't need resurgence…

Open question: what are the instanton trans-series for this model? How do they relate to the convergent series appearing here?

Generalizations

The above phenomena occur in other matrix models related to string theory and M-theory

ABJ theory [Matsumoto-Moriyama, Honda-Okuyama, Kallen]

N_f matrix model [Mezei-Pufu,Grassi-M.M.,Hatsuda-Okuyama]

"M-theoretic matrix models"

These models can be often solved in terms of a quantum-mechanical spectral problem, but *we need new tools* to analyze the corresponding exact WKB quantization conditions, both perturbatively and non-perturbatively

Conclusions

• Matrix models can be analyzed in the context of resurgence. They are an ideal arena which goes beyond solvable one-parameter models like nonlinear ODEs, yet they are complicated enough to display new phenomena. They can be used to analyze some simple problems in string theory and M-theory.

•Resurgence requires instanton sectors with no known semiclassical interpretation in the matrix integral. What are they?

•In some examples related to M-theory, we end up with *convergent* expansions. We need new tools to understand them and to embed them in the general theory of resurgence.