FROM MULTI-INSTANTON EXPANSION TO EXACT RESULTS

J. Zinn-Justin*

CEA/IRFU (irfu.cea.fr)

Centre de Saclay

91191 Gif-sur-Yvette (France)

---

*Email: jean.zinn-justin@cea.fr

Conference "Resurgence and Transseries in Quantum, Gauge and String Theories", CERN 30/06/2014–04/07/2014.

Abstract

In the late seventies, the large order behaviour of perturbation theory was determined, following Lipatov, by path or field integral methods (instanton calculus). It became rapidly clear that the issue was not *are perturbative series divergent?*, they generally are, but are they Borel summable? In this case one can hope to recover the true functions from the expansion.

We found that the simplest examples of non-Borel summable series was provided by quantum mechanics in the case of potentials with degenerate minima. The question then was what kind of additional information is needed to specify the exact functions?

In this talk, conjectures about the exact semi-classical expansion of low-lying energy levels for a few analytic potentials, like the quartic double-well or the periodic cosine potentials, with degenerate minima are reviewed. They take the form of generalized Bohr–Sommerfeld quantization formulae whose origin has been later clarified using the theory of resurgent functions.

These formulae involve an infinite number of perturbative series, but which can be generated by a few spectral functions. In the simplest cases only two functions appear and, recently, it has been pointed out that the two are simply related.

The conjectures were initially suggested by semi-classical evaluations of the partition function based on the path integral formalism.

The infinite number of saddle points, i.e., multi-instantons, generated by the steepest descent method yields contributions that can be summed exactly at leading order. The same strategy could still be useful in problems where our present understanding is more limited.

Finally, these properties have a direct interpretation within the framework of the complex WKB expansion of the solutions of the Schrödinger equation.

References

[1] E. Brézin, G. Parisi and J. Zinn-Justin, *Phys. Rev.* D16 (1977) 408; E.B. Bogomolny and V.A. Fateev, *Phys. Lett.* B71 (1977) 93.

[2] J. Zinn-Justin, *Nucl. Phys.* B192 (1981) 125; B218 (1983) 333; *J. Math. Phys.* 22 (1981) 511; 25 (1984) 549.

[3] Several results have also been reported in

J. Zinn-Justin, contribution to the Proceedings of the Franco-Japanese Colloquium *Analyse algébrique des perturbations singulières*, Marseille-Luminy, October 1991, L. Boutet de Monvel ed., Collection Travaux en cours, 47, Hermann (Paris 1994).

[4] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* chap. 43, Oxford Univ. Press (Oxford 1989), 4th ed. 2002.

[5] F. Pham, *Resurgence, Quantized Canonical Transformation and Multi-*

Instanton, Algebraic Analysis, vol. II (1988); *C.R. Acad. Sci. Paris* 309 (1989) 999;
E. Delabaere, H. Dillinger et P. Pham, *C.R. Acad. Sci. Paris* 310 (1990) 141;
E. Delabaere et H. Dillinger, thesis Université de Nice (Nice 1991).

[6] E. Delabaere, *C.R. Acad. Sci. Paris* 314 (1992) 807.

[7] R. Seznec and J. Zinn-Justin, *J. Math. Phys.* 20 (1979) 1398.

[8] A.A. Andrianov, *Ann. Phys. (NY)* 140 (1982) 82.

[9] R. Damburg, R. Propin and V. Martyshchenko, *J. Phys* A17 (1984) 3493.

[10] V. Buslaev, V. Grecchi, *J. Phys.* A26 (1993) 5541.

[11] For the complex WKB method, as discussed here, see also
A. Voros, *Ann. IHP* A 39 (1983) 211.

[12] The large order behaviour of the ground state has been calculated in
E. Brézin, J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* D15 (1977)

1544.

[13] U.D. Jentschura, and J. Zinn-Justin, *J. Phys.* A34 (2001) L1–L6; *Phys. Lett.* B 596 (2004) 138-144; *Ann. Phys. NY* 313 (2004) 197-267; *Ann. Phys. NY* 313 (2004) 269-325; *Annals of Physics* 326 (2011) 2186-2242.

[14] Ulrich D. Jentschura, Andrey Surzhykov, Jean Zinn-Justin, *Annals of Physics* 325 (2010) 1135–1172.

[15] G.V. Dunne and M. Unsal, hep-th peprint *Arxiv*: 1306.4405.

Introduction

Remark. The results we describe here, mainly apply to polynomial potentials and, on a case by case study, to some other analytic potentials like periodic potentials of cosine potential type.

Perturbative expansions are obtained by first approximating the potential by a harmonic potential near its minimum. This leads to (in general divergent) expansions in powers of $\hbar$ for energy eigenvalues of order $\hbar$.

For potentials with degenerate minima, expansions in powers of $\hbar$ are shown to be non-Borel summable. Moreover, additional contributions of order $\exp(-\text{const.}\,/\hbar)$, generated by quantum tunnelling, have to be added to the perturbative expansion. Therefore, the determination of eigenvalues starting from their expansion for $\hbar$ small becomes a non-trivial problem.

In this situation, our conjectures give a systematic procedure to calculate energy eigenvalues, for $\hbar$ finite, from expansions that are shown to contain powers of $\hbar$, $\ln\hbar$ and $\exp(-\text{const.}\,/\hbar)$.

Moreover, generalized Bohr–Sommerfeld formulae allow to infer the infinite number of series that appear in such formal expansions from a few WKB expansions.

This relation with the WKB expansion is not completely trivial. Indeed, the perturbative expansion corresponds from the viewpoint of the WKB approximation to a situation with confluent singularities and thus, for example, the usual WKB expressions for barrier penetration are not uniform when the energy goes to zero.

Finally, the origin of these conjectures have found a natural explanation in the framework of Ecalle's theory of resurgent functions, as has been shown by Pham's collaborators.

Generalized Bohr–Sommerfeld quantization formulae

We first explain the conjecture in the case of the so-called quartic double-well potential. The symbol g plays the role of $\hbar$ and energy eigenvalues are measured in units of $\hbar$, a normalization adapted to perturbative expansions.

The quartic double-well potential

The Hamiltonian corresponding to the double-well potential can be written as

$$H = -\frac{g}{2}\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{1}{g}V(q), \quad V(q) = \frac{1}{2}q^2(1-q)^2.$$

The potential is symmetric in $q \leftrightarrow (1-q)$ and thus the Hamiltonian commutes with the corresponding reflection operator,

$$P\psi(q) = \psi(1-q) \ \Rightarrow [H, P] = 0\,.$$

The potential has two symmetric degenerate minima.

Eigenfunctions and eigenvalues

The eigenfunctions of H satisfy

$$H\psi_{\epsilon,N}(q) = E_{\epsilon,N}(g)\psi_{\epsilon,N}(q), \quad P\psi_{\epsilon,N}(q) = \epsilon\psi_{\epsilon,N}(q),$$

where $\epsilon = \pm 1$ and $E_{\epsilon,N}(g) = N + 1/2 + O(g)$.

We have conjectured (Zinn-Justin 1983) that the eigenvalues $E_{\epsilon,N}(g)$ have an exact semi-classical expansion of the form

$$E_{\epsilon,N}(g) = \sum_{0}^{\infty} E^{(0)}_{N,l} g^l$$
$$+ \sum_{n=1}^{\infty} \left(\frac{2}{g}\right)^{Nn} \left(-\epsilon \frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^n \sum_{k=0}^{n-1} (\ln(-2/g))^k \sum_{l=0}^{\infty} e_{N,nkl} g^l .$$

The series $\sum e_{N,nkl} g^l$ in powers of g are not Borel summable for $g > 0$ and have to be summed for g negative first, where $\ln(-g)$ is also real. One then proceeds by analytic continuation to $g > 0$ consistently for the series and $\ln(-g)$.

In the analytic continuation, the Borel sums become complex with imaginary parts exponentially smaller by about a factor $\mathrm{e}^{-1/3g}$ than the real parts. These imaginary contributions are cancelled by the perturbative imaginary parts coming from the function $\ln(-2/g)$.

We have also conjectured that all the series are generated by an expansion for g small of a spectral equation or generalized Bohr–Sommerfeld quantization formula, which in the case of the double-well potential reads ($\epsilon = \pm 1$)

$$\frac{1}{\Gamma(\frac{1}{2} - B)} + \frac{\epsilon i}{\sqrt{2\pi}} \left(-\frac{2}{g}\right)^{B(E,g)} \mathrm{e}^{-A(E,g)/2} = 0$$

with

$$B(E,g) = -B(-E,-g) = E + \sum_{k=1} g^k b_{k+1}(E),$$

$$A(E,g) = -A(-E,-g) = \frac{1}{3g} + \sum_{k=1} g^k a_{k+1}(E).$$

The coefficients $b_k(E)$, $a_k(E)$ are odd or even polynomials in E of degree k. The three first orders, for example, are

$$
\begin{aligned}
B(E,g) &= E + g\left(3E^2 + \tfrac{1}{4}\right) + g^2\left(35E^3 + \tfrac{25}{4}E\right) + O\left(g^3\right),\\
A(E,g) &= \tfrac{1}{3}g^{-1} + g\left(17E^2 + \tfrac{19}{12}\right) + g^2\left(227E^3 + \tfrac{187}{4}E\right) + O\left(g^3\right).
\end{aligned}
$$

The function $B(E,g)$ can be inferred from the complex WKB perturbative expansion. The function $A(E,g)$ has initially been determined at this order by a combination of analytic and numerical calculations.

However, recently, it has been proved (Dunne and Unsal) for the double well and cosine potentials, using differential equation techniques, the intriguing relation

$$\frac{\partial E}{\partial B} = -6Bg - 3g^2 \frac{\partial A}{\partial g}\,,$$

which reduces the determination of both functions to the determination of the perturbative spectral function $B(E,g)$.

The n-instanton contributions at leading order

Replacing the functions A and B by their leading terms, one obtains

$$\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E}=-\frac{\epsilon i}{\Gamma(\frac{1}{2}-E)}\ \Leftrightarrow\ \frac{\cos\pi E}{\pi}=\epsilon i\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\left(-\frac{2}{g}\right)^{E}\frac{1}{\Gamma(\frac{1}{2}+E)}\,.$$

Expanding then the equation in powers of $\mathrm{e}^{-1/6g}$, one obtains terms that, from the point of view of the path integral representation, correspond to the successive multi-instanton contributions at leading order.

For example, the term proportional to $\mathrm{e}^{-1/6g}$, which can be identified with the one-instanton contribution at leading order, is

$$E_N^{(1)}(g)=-\frac{\epsilon}{N!}\left(\frac{2}{g}\right)^{N+1/2}\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}(1+O(g)).$$

The next term, (the two-instanton contribution), is ($\psi=(\ln\Gamma)'$)

$$E_N^{(2)}(g)=\frac{1}{(N!)^2}\left(\frac{2}{g}\right)^{2N+1}\frac{\mathrm{e}^{-1/3g}}{2\pi}\left[\ln(-2/g)-\psi(N+1)+O\left(g\ln g\right)\right].$$

More generally, the nth power, which can be identified with the n-instanton contribution at leading order, has the form

$$E_N^{(n)}(g) = (-1)^n \left(\frac{2}{g}\right)^{n(N+1/2)} \left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}\right)^n \left[P_n^{(N)}(\ln(-2/g)) + O\left(g\,(\ln g)^{n-1}\right)\right],$$

in which $P_n^N(\sigma)$ is a polynomial of degree $(n-1)$. For example, for $N=0$ one finds (γ is Euler's constant)

$$P_1^{(0)}(\sigma) = 1\,, \quad P_2^{(0)}(\sigma) = \sigma + \gamma\,, \quad P_3^{(0)}(\sigma) = \frac{3}{2}(\sigma+\gamma)^2 + \frac{\pi^2}{12}\,.$$

.

An application: Large order behaviour of perturbation series

After an analytic continuation from g negative to g positive, the Borel sums become complex with an imaginary part exponentially smaller by about a factor $\mathrm{e}^{-1/3g}$ than the real part.

Consistently, the function $\ln(-2/g)$ also becomes complex with an imaginary part $\pm i\pi$. Since the sum of all contributions is real, imaginary parts must cancel.

For example, the non-perturbative imaginary part of the Borel sum of the perturbation series cancels the perturbative imaginary part of the two-instanton contribution. For the ground state,

$$\operatorname{Im} E^{(0)}(g) \underset{g\to 0}{\sim} \frac{1}{\pi g}\,\mathrm{e}^{-1/3g}\,\operatorname{Im}\left[P_2^{(0)}\big(\ln(-2/g)\big)\right] = -\frac{1}{g}\,\mathrm{e}^{-1/3g}\,.$$

The coefficients of the perturbative expansion

$$E^{(0)}(g) = \sum_k E_k^{(0)} g^k$$

of the ground state energy, are related to the imaginary part by a Cauchy integral ($k > 1$):

$$E_k^{(0)} = \frac{1}{\pi} \int_0^\infty \mathrm{Im} \left[E^{(0)}(g) \right] \frac{\mathrm{d}g}{g^{k+1}} .$$

For $k \to \infty$, the integral is dominated by small g values. Thus,

$$E_k^{(0)} \underset{k\to\infty}{\sim} -\frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-1/3g}}{g^{k+2}} \mathrm{d}g = -\frac{1}{\pi} 3^{k+1} k! \, .$$

Similarly, since $\operatorname{Im} E^{(1)}(g)$ and $\operatorname{Im} E^{(3)}(g)$ cancel at leading order,

$$\operatorname{Im} E^{(1)}(g) \sim 3\pi \left(\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^{3} \left[\ln(2/g) + \gamma + O(g \ln(g))\right].$$

The coefficients of the expansion

$$E^{(1)}(g) = -\frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/6g} \left(1 + \sum^{\infty} E_k^{(1)} g^k\right)$$

are given by the dispersion integral

$$E_k^{(1)} = -\frac{1}{\pi}\int_0^{\infty} \left\{\operatorname{Im}\left[E^{(1)}(g)\right] \sqrt{\pi g}\,\mathrm{e}^{1/6g}\right\} \frac{\mathrm{d}g}{g^{k+1}}.$$

Combining both equations, one finds

$$E_k^{(1)} \sim -\frac{3}{\pi}\int_0^{\infty} \left(\ln\frac{2}{g} + \gamma\right) \mathrm{e}^{-1/3g}\,\frac{\mathrm{d}g}{g^{k+2}} \sim -\frac{3^{k+2}}{\pi} k!\,(\ln 6k + \gamma)\,.$$

Both results have been confirmed numerically by calculating many terms of the corresponding series.

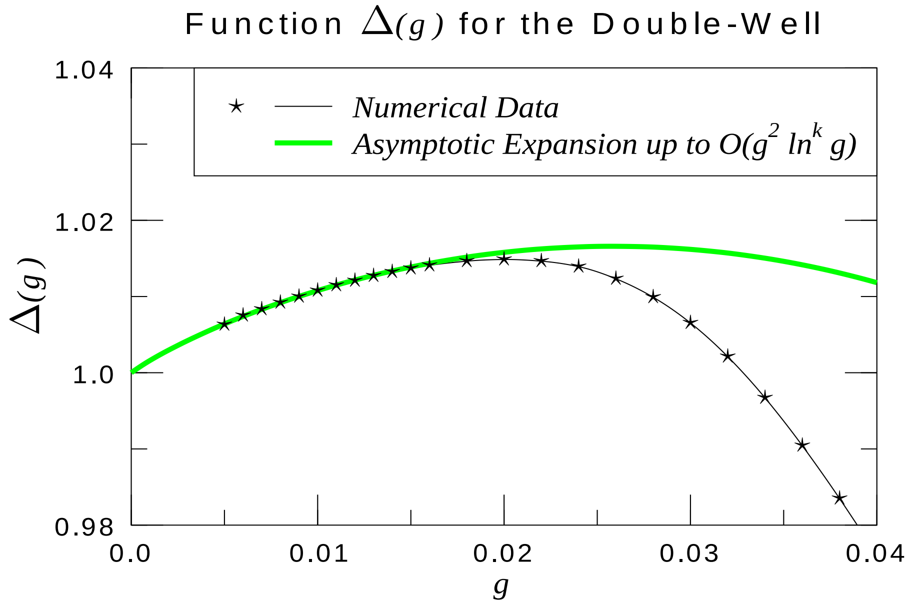

Fig. 1 Comparison of numerical evaluation of $\Delta(g)$ with the asymptotic expansion for g small.

The real part of the two-instanton contribution

Another test has been provided by the evaluation of the ratio dominated for $g \ll 1$ by the two-instanton contribution, (see Jentschura's talk)

$$\Delta(g) = 4\frac{\left\{\frac{1}{2}\left(E_{0,+} + E_{0,-}\right) - \mathrm{Re}\left[\text{Borel sum } E^{(0)}(g)\right]\right\}}{\left(E_{0,+} - E_{0,-}\right)^2 \left(\ln 2g^{-1} + \gamma\right)} = 1 + 3g + \cdots .$$

Asymmetric wells

For a potential with two asymmetric wells,

$$V(q)=\tfrac{1}{2}\omega_1^2 q^2+O(q^3),\quad V(q)=\tfrac{1}{2}\omega_2^2(q-q_0)^2+O\left((q-q_0)^3\right),$$

the spectral equation takes the form

$$\frac{1}{\Gamma(\frac{1}{2}-B_1)\Gamma(\frac{1}{2}-B_2)}+\frac{1}{2\pi}\left(-\frac{2C_1}{g}\right)^{B_1(E,g)}\left(-\frac{2C_2}{g}\right)^{B_2(E,g)}\mathrm{e}^{-A(g,E)}=0\,,$$

where $B_1(E,g)$ and $B_2(E,g)$ are determined by the perturbative expansions around the two minima of the potential

$$B_1(E,g)=E/\omega_1+O(g),\quad B_1(E,g)=E/\omega_2+O(g),$$

and the constants C_1 and C_2 are adjusted in such a way that

$$A(E,g)-a/g=O(g),\quad a=2\int_0^{q_0}\mathrm{d}q\,\sqrt{2V(q)}\,.$$

From the poles of Γ-functions for $g \to 0$, one sees that the spectral equation yields two sets of energy eigenvalues,

$$E_N = \left(N + \tfrac{1}{2}\right)\omega_1 + O(g)\,, \quad E_N = \left(N + \tfrac{1}{2}\right)\omega_2 + O(g).$$

The same expression contains the instanton contributions to the two different sets of eigenvalues.

One verifies that multi-instanton contributions are singular for $\omega = 1$ but the spectral equation is regular in the symmetric limit.

One-instanton contribution and large order behaviour. The spectral equation can again be used to infer the large order behaviour of perturbation theory from the imaginary part of the leading instanton contribution by writing a dispersion integral. Setting $\omega_1 = 1$, $\omega_2 = \omega$, for the energy $E_N(g) = N + \frac{1}{2} + O(g)$ one infers that the coefficients E_{Nk} of the perturbative expansion of $E_N(g)$ behave, for order $k \to \infty$, like

$$E_{Nk} \underset{k\to\infty}{=} K_N \frac{\Gamma(k + (N + 1/2)(1 + 1/\omega))}{a^{k+(N+1/2)(1+1/\omega)}} \left(1 + O\left(k^{-1}\right)\right).$$

Other investigated potentials: the example of the periodic cosine potential

The cosine potential is still an entire function but no longer a polynomial. On the other hand the periodicity of the potential simplifies the analysis, because eigenfunctions can be classified according to their behaviour under a translation of one period T,

$$\psi_\varphi(q+T) = \mathrm{e}^{i\varphi}\, \psi_\varphi(q).$$

For the potential $\frac{1}{16}(1-\cos 4q)$ (and thus $T=\pi/2$), the conjecture then takes the form

$$\left(\frac{2}{g}\right)^{-B} \frac{\mathrm{e}^{A(E,g)/2}}{\Gamma(\frac{1}{2}-B)} + \left(\frac{-2}{g}\right)^{B} \frac{\mathrm{e}^{-A(g,E)/2}}{\Gamma(\frac{1}{2}+B)} = \frac{2\cos\varphi}{\sqrt{2\pi}}.$$

Instantons

The conjectures were initially motivated by a summation of leading order multi-instanton contributions. The method may be worth recalling since it could still be useful for other, less understood, problems.

Partition function and resolvent

The path integral formalism allows calculating directly the quantum partition function, which for a Hamiltonian H with discrete spectrum has the expansion

$$\mathcal{Z}(\beta) \equiv \operatorname{tr} \mathrm{e}^{-\beta H} = \sum_{N \geq 0} \mathrm{e}^{-\beta E_N} .$$

From the partition function, on infers the trace $G(E)$ of the resolvent of H (after analytic continuation and possible subtraction),

$$G(E) = \operatorname{tr} \frac{1}{H - E} = \int_0^{\infty} \mathrm{d}\beta \; \mathrm{e}^{\beta E} \, \mathcal{Z}(\beta) \, .$$

The poles of $G(E)$ yield the spectrum of the Hamiltonian H. The Fredholm determinant $\mathcal{D}(E) = \det(H - E)$, which vanishes on the spectrum, is then given by

$$\frac{\partial}{\partial E} \ln \mathcal{D}(E) = -G(E).$$

For a symmetric double-well potential, one can separate eigenvalues according to the parity of eigenfunctions by considering the two partition functions

$$\mathcal{Z}_{\pm}(\beta) = \operatorname{tr}\left[\tfrac{1}{2}(1 \pm P)\, \mathrm{e}^{-\beta H}\right] = \sum_{N=0} \mathrm{e}^{-\beta E_{\pm,N}},$$

where P is the reflection operator. The eigenvalues are then poles of ($\epsilon = \pm 1$)

$$G_{\epsilon}(E) = \int_0^{\infty} \mathrm{d}\beta \; \mathrm{e}^{\beta E}\, \mathcal{Z}_{\epsilon}(\beta)\,.$$

For the periodic cosine potential, one uses a generalized partition function with twisted boundary conditions depending on a rotation angle.

Path integrals and spectra of Hamiltonians

In the path integral formulation of quantum mechanics, the partition function is given by a summation over closed paths,

$$\mathcal{Z}(\beta) \propto \int_{q(-\beta/2)=q(\beta/2)} [\mathrm{d}q(t)] \exp\left[-\frac{1}{g}\int_{-\beta/2}^{\beta/2} \left[\tfrac{1}{2}\dot{q}^2(t) + V\big(q(t)\big)\right] \mathrm{d}t\right],$$

In the case of a symmetric potential with two degenerate minima, it is convenient to also consider (P is the reflection operator)

$$\mathcal{Z}_\mathrm{a}(\beta) \equiv \operatorname{tr}\left(P\,\mathrm{e}^{-\beta H}\right),$$

which is obtained by a path integral with the boundary conditions $q(-\beta/2) = P(q(\beta/2))$.

Then, eigenvalues corresponding to symmetric and antisymmetric eigenfunctions can be inferred from the combinations

$$\mathcal{Z}_\pm(\beta) = \operatorname{tr}\left[\tfrac{1}{2}(1 \pm P)\,\mathrm{e}^{-\beta H}\right] = \tfrac{1}{2}\big(\mathcal{Z}(\beta) \pm \mathcal{Z}_\mathrm{a}(\beta)\big).$$

Perturbative expansion

For $g \to 0$, the path integral can be evaluated by the steepest descent method. Saddle points are solutions $q_c(t)$ to the Euclidean equations of motion. When the potential has a unique minimum, for example, located at $q = 0$, the leading saddle point is $q_c(t) \equiv 0$. A systematic expansion around the saddle point then yields the perturbative expansion of the eigenvalues of the Hamiltonian.

In the case of potentials with degenerate minima, one must sum over several saddle points: to each saddle point corresponds an eigenvalue and thus several eigenvalues are degenerate at leading order. For the symmetric double-well potential, all eigenvalues are twice degenerate to all orders in perturbation theory:

$$E_{\pm,N}(g) = E_N^{(0)}(g) \equiv \sum_{n=0}^{\infty} E_{N,n}^{(0)} g^n .$$

Instantons

Eigenvalues can be extracted from the large β expansion. For $\beta \to \infty$, leading contributions to the path integral come from finite action solutions of the Euclidean equations of motion. In the case of the path integral representation of $\mathcal{Z}_{\mathrm{a}}(\beta)$, constant solutions do not satisfy the boundary conditions. Finite action solutions (instantons) necessarily correspond to paths that connect the two minima of the potential (see Fig. 2).

In the example of the quartic double-well potential, such solutions are

$$q_{\mathrm{c}}(t) = \left(1 + \mathrm{e}^{\pm(t-t_0)}\right)^{-1} \ \Rightarrow \ \mathcal{S}(q_{\mathrm{c}}) = 1/6\,.$$

Since the two solutions depend on an integration constant t_0 (the instanton position), one finds two one-parameter families of degenerated saddle points.

The corresponding contribution to the path integral is proportional, at leading order for $g \to 0$ and for $\beta \to \infty$, to $\mathrm{e}^{-1/(6g)}$ and thus is non-perturbative.

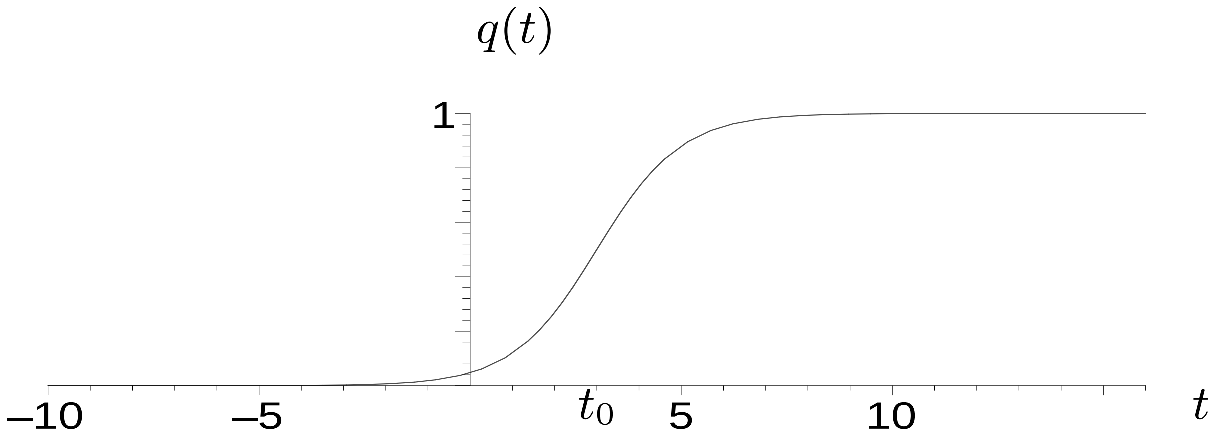

Fig. 2 The instanton configuration.

The complete calculation involves integrating exactly over the time t_0 (the collective coordinate), which for β finite varies in $[0,\beta]$, and over the remaining fluctuations in the Gaussian approximation. The two lowest eigenvalues are given by ($\epsilon=\pm 1$)

$$E_{\epsilon,0}(g)=\lim_{\beta\to\infty}-\frac{1}{\beta}\ln\mathcal{Z}_{\epsilon}(\beta)\underset{g\to 0,\beta\to\infty}{=}E_0^{(0)}(g)-\epsilon E_0^{(1)}(g),$$

$$E_0^{(1)}(g)=\frac{1}{\sqrt{\pi g}}\,\mathrm{e}^{-1/6g}\big(1+O(g)\big).$$

Multi-instantons

For β finite, one finds subleading saddle points, which correspond to oscillations in the well of the potential $-V(q)$. For $\beta \to \infty$, the action of the solutions with n oscillations goes to $n \times 1/6$.

However, the Gaussian integral at the saddle point diverges for $\beta \to \infty$. Indeed, the classical solutions decompose into a succession of largely separated instantons and fluctuations that change the distances between instantons induce only infinitesimal variations of the action.

Therefore, one has to sum over all configurations of largely separated instantons, connected in a smooth way, which become solutions of the equation of motion only asymptotically, for infinite separation. They depend on n collective coordinates, the distance between instantons. The action then has a dependence on the collective coordinates, called instanton interaction.

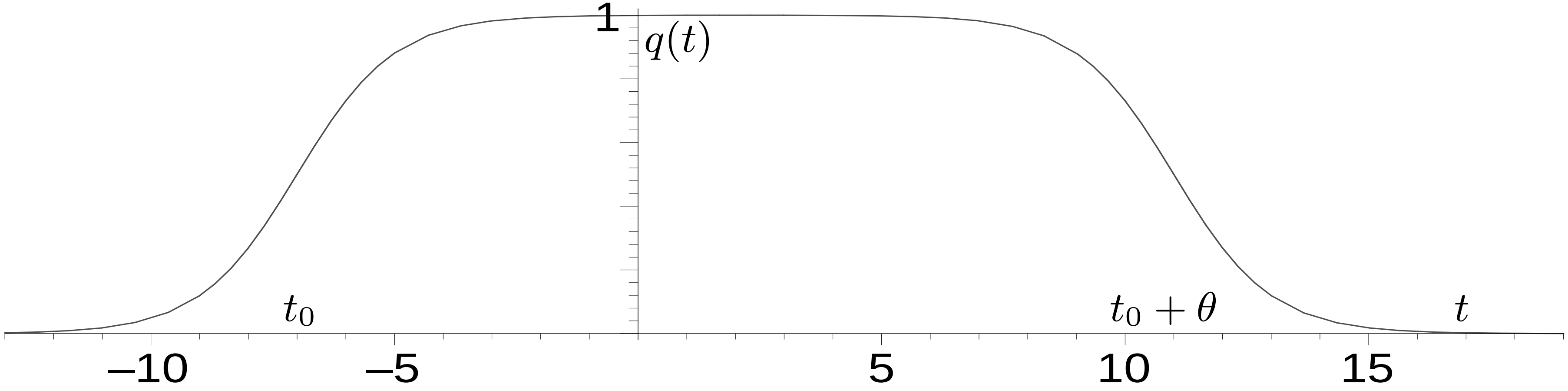

Fig. 3 A two-instanton configuration.

Example: the two-instanton configurations

In the infinite β limit, the one-instanton configuration can be written as

$$q_{\pm}(t) = f\left(\mp(t - t_0)\right), \;\; f(t) \equiv 1/\left(1 + \mathrm{e}^t\right) = 1 - f(-t),$$

where the constant t_0 characterizes the instanton position.

One verifies that a configuration $q_{\mathrm{c}}(t)$ that is the sum of instantons separated by a distance θ, up to an additive constant adjusted in such a way as to satisfy the boundary conditions (Fig. 3),

$$q_{\mathrm{c}}(t) = f(t-\theta/2) + f(-t-\theta/2) - 1 = f(t-\theta/2) - f(t+\theta/2),$$

has the required properties: it is differentiable and for θ large, but fixed, it minimizes the variation of the action. The corresponding action is

$$\mathcal{S}(q_c) = \tfrac{1}{3} - 2\,\mathrm{e}^{-\theta} + O\left(\mathrm{e}^{-2\theta}\right).$$

We show later that contributions to the classical action of order $\mathrm{e}^{-2\theta}$ give only a correction of order g.

For β large, but finite, symmetry between θ and $(\beta-\theta)$ implies

$$\mathcal{S}(q_c) = \tfrac{1}{3} - 2\,\mathrm{e}^{-\theta} - 2\,\mathrm{e}^{-(\beta-\theta)} + \text{ negligible contributions.}$$

The n-instanton action

For a succession of n instantons (more precisely, alternatively instantons and anti-instantons) separated by times θ_i with

$$\sum_{i=1}^{n} \theta_i = \beta \,,$$

the classical action $\mathcal{S}_c(\theta_i)$ is then

$$\mathcal{S}_c(\theta_i) = \frac{n}{6} - 2 \sum_{i=1}^{n} e^{-\theta_i} + O\left(e^{-(\theta_i+\theta_j)}\right).$$

At leading order, for $\theta_i \gg 1$, it is the sum of nearest-neighbour interactions. For n even, the n-instanton configurations contribute to $\operatorname{tr} e^{-\beta H}$, while for n odd they contribute to $\operatorname{tr}\left(P\, e^{-\beta H}\right)$ (P is the reflection operator).

The n-instanton contribution

The evaluation, at leading order, of the contribution to the path integral of the neighbourhood of the n-instanton configuration is simple but slightly technical. One finds that the n-instanton contribution to the combination

$$\mathcal{Z}_\epsilon(\beta) = \tfrac{1}{2}\,\mathrm{tr}\left[(1+\epsilon P)\,\mathrm{e}^{-\beta H}\right],$$

$(\epsilon = \pm 1)$, can be written as

$$\mathcal{Z}_\epsilon^{(n)}(\beta) = \mathrm{e}^{-\beta/2}\,\frac{\beta}{n}\left(\epsilon\frac{\mathrm{e}^{-1/6g}}{\sqrt{\pi g}}\right)^n \int_{\substack{\theta_i \ge 0 \\ \sum \theta_i = \beta}} \prod_i \mathrm{d}\theta_i \exp\left(\frac{2}{g}\sum_{i=1}^n \mathrm{e}^{-\theta_i}\right).$$

Neglecting the instanton interaction and summing over n one recovers the one-instanton approximation to the energy eigenvalues.

Beyond the one-instanton approximation: a problem. If one examines the classical action for multi-instantons, one discovers that the interaction between instantons is attractive. Therefore, for g small, the dominant contributions to the integral come from configurations in which the instantons are close. For such configurations, the concept of instanton is no longer meaningful, since the configurations cannot be distinguished from fluctuations around the constant or the one-instanton solution.

Such a difficulty could have been expected. In the case of potentials with degenerate minima the perturbative expansion is not Borel summable and the series determines eigenvalues only up to exponentially decreasing terms that are of the order of two-instanton contributions. But if the perturbative expansion is ambiguous at the two-instanton order, n-instanton contributions with $n \geq 2$ are not defined. To proceed any further, it is necessary to first give a meaning to the sum of the perturbative expansion.

In the example of the quartic double-well potential, one can show that the perturbation series is Borel summable for g negative. Therefore, we define the sum of the perturbation series as the analytic continuation of this Borel sum from g negative to $g = |g| \pm i0$. This corresponds in the Borel transformation to eventually integrate above or below the real positive axis. Simultaneously, for g negative, the interaction between instantons becomes repulsive and the multi-instanton contributions become meaningful.

Therefore, we first calculate, for g small and negative, both the sum of the perturbation series and the multi-instanton instanton contributions, and then perform an analytic continuation to g positive of all quantities consistently.

The sum of leading order instanton contributions

We assume that initially g is negative and calculate the sum of leading n-instanton contributions to the trace of the resolvents,

$$\mathcal{G}_\epsilon(E)=\sum_{n=1}\int_0^\infty \mathrm{d}\beta\ \mathrm{e}^{\beta E}\ \mathcal{Z}_\epsilon^{(n)}(\beta),$$

where

$$\mathcal{Z}_\epsilon^{(n)}(\beta)\sim\frac{\beta}{n}\,\mathrm{e}^{-\beta/2}\left(\frac{\epsilon}{\sqrt{2\pi}}\right)^n \mathrm{e}^{-n/6g}\int_{\substack{\theta_i\geq 0\\ \sum\theta_i=\beta}}\prod_{i=1}^n \mathrm{d}\theta_i \exp\left[\frac{2}{g}\sum_{i=1}^n \mathrm{e}^{-\theta_i}\right].$$

The integration over β is immediate, the integrals over the θ_i then factorize. Evaluating the unique integral for $g\to 0_-$, summing over n and adding the resolvent of the harmonic oscillator, one finds for the resolvent $G_\epsilon(E)$ a result consistent with the conjectures:

$$G_\epsilon(E)=-\frac{\partial}{\partial E}\ln\mathcal{D}_\epsilon(E)\ \Rightarrow\ \mathcal{D}_\epsilon(E)=\frac{1}{\Gamma(\frac{1}{2}-E)}+\epsilon i\left(-\frac{2}{g}\right)^E\frac{\mathrm{e}^{-1/6g}}{\sqrt{2\pi}}.$$

Perturbative and WKB expansions from Schrödinger equations

These conjectures, motivated by semi-classical evaluation of path integrals (instanton calculus), have been confirmed by considerations based on the Schrödinger equation,

$$[H\psi](q) \equiv -\frac{g}{2}\psi''(q) + \frac{1}{g}V(q)\psi(q) = E\psi(q),$$

where the potential V is an entire function. This allows extending the Schrödinger equation and its solutions to the q-complex plane.

A Riccati equation is obtained by setting

$$S(q) = -g\psi'(q)/\psi(q).$$

One obtains

$$gS'(q) - S^2(q) + 2V(q) - 2gE = 0\,.$$

One decomposes

$$S(q) = S_+(q) + S_-(q) \text{ where, formally, } S_\pm(q; g, E) = \pm S_\pm(q; -g, -E)\,.$$

Then,

$$g S'_- - S_+^2 - S_-^2 + 2V(q) - 2gE = 0\,, \;\; g S'_+ - 2S_+ S_- = 0\,.$$

The quantization condition (or spectral equation) can then be written as

$$-\frac{1}{2i\pi g}\oint_C \mathrm{d}z\, S_+(z, E) = N + \tfrac{1}{2}\,,$$

where N is also the number of real zeros of the eigenfunction, and C is a contour that encloses them. This elegant formulation, restricted, however, to one dimension and analytic potentials, bypasses the difficulties generally associated with turning points.

It allows a smooth transition between WKB expansion ($g \to 0$, gE fixed), in our normalization, and perturbative expansion ($g \to 0$, E fixed), which can be derived by expanding the WKB expansion at E fixed.

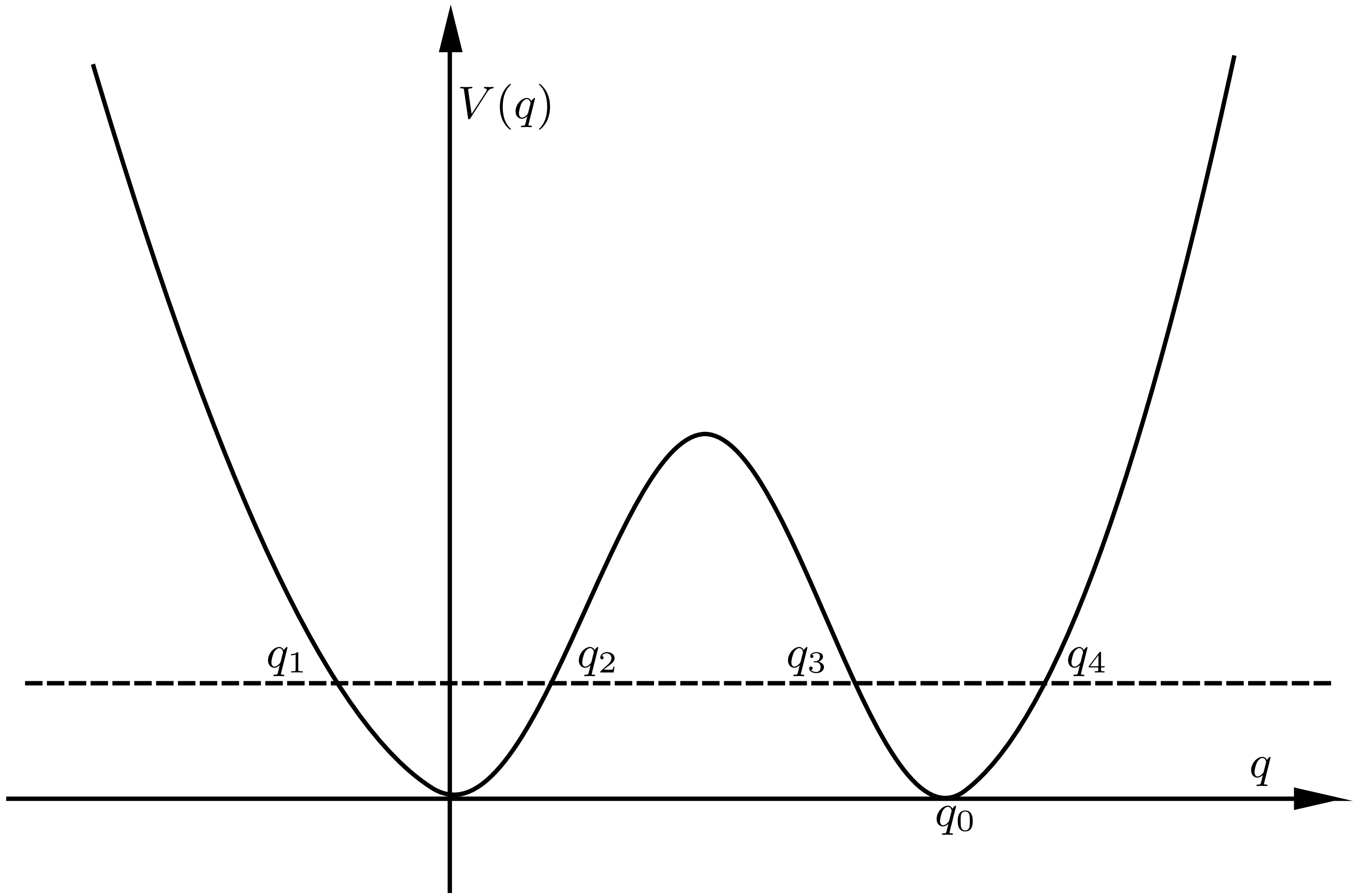

Fig. 4 Degenerate minima: The four turning points.

WKB expansion

At leading order in the WKB limit, the function S_+ reduces to

$$S_+(q) = S(q) = S_0(q), \quad S_0(q) = \sqrt{2V(q) - 2gE}$$

and the quantization condition becomes

$$N + \tfrac{1}{2} = B(E,g) = -\frac{1}{2i\pi g} \oint_C \mathrm{d}z \, S_0(z,E),$$

where the contour C encloses the cut of $S_0(q)$ which joins the turning points.

If the potential has two degenerate, non necessarily symmetric, minima, for E small enough, the function $S_0(q)$ has four branch points $q_1 < q_2 < q_3 < q_4$ on the real axis (Fig. 4).

One can define two functions $B_1(E,g)$ and $B_2(E,g)$ which, at leading order, correspond to contours enclosing the cuts $[q_1, q_2]$ and $[q_3, q_4]$.

Moreover, comparing with the conjecture, one infers the decomposition

$$\frac{1}{g}\oint_{C[q_2,q_3]} \mathrm{d}z\, S_+(z) = A(E,g) + \ln(2\pi) - \sum_{i=1}^{2} \ln\Gamma\left(\tfrac{1}{2} - B_i(E,g)\right) + B_i(E,g)\ln(-g/2C_i),$$

where, at leading order in the WKB expansion, the contour now encloses a cut $[q_2, q_3]$ and the constants C_i are chosen such that $A(E,g)$ has no term of order g^0.

For details see Jentschura's talk.