Resurgence in Quantum Field Theory and Quantum Mechanics

Gerald Dunne

Helmholtz-Institut Jena and Friedrich-Schiller Universität, Jena
& University of Connecticut

CERN Theory Institute: *Resurgence and Transseries in Quantum, Gauge and String Theories*, June 2014

GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302

Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
  (i) IR renormalons $\Rightarrow$ perturbation theory ill-defined
  (ii) $\mathcal{I}\bar{\mathcal{I}}$ interactions $\Rightarrow$ instanton-gas ill-defined
- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- "exact" asymptotics in QFT and string theory: relation to localization in QFT

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

- goal: explore implications for physics

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- philosophical shift:
  view semiclassical expansions as potentially exact
- applications: ODEs, PDEs, QM, Matrix Models, QFT, String Theory, ...

Resurgent Trans-Series in Physics (QFT and QM)

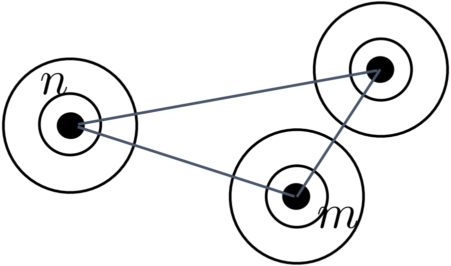
- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}} \exp\left[-\frac{c}{g^2} \right] \right)^k}_{\text{k}-\text{instantons}} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\ln\left[\pm \frac{1}{g^2} \right] \right)^l}_{\text{quasi-zero-modes}}$$

- known as “multi-instanton calculus” in QFT
- trans-monomial elements, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, g^2, are familiar
- many exact results in supersymmetric QFT
- does the resurgence perspective add something new?
- trans-series expansion coefficients highly correlated
- exponentially improved asymptotic expansions

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



recap: rough basics of Borel summation

(i) divergent, alternating:

$$\sum_{n=0}^{\infty}(-1)^n \, n! \, g^{2n} = \int_0^{\infty} dt \, e^{-t} \, \frac{1}{1+g^2 \, t}$$

recap: rough basics of Borel summation

(i) divergent, alternating:

$$\sum_{n=0}^{\infty}(-1)^n \, n! \, g^{2n} = \int_0^\infty dt \, e^{-t} \, \frac{1}{1+g^2 \, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n! \, g^{2n} = \int_0^\infty dt \, e^{-t} \, \frac{1}{1-g^2 \, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm \frac{i\,\pi}{g^2} \, e^{-1/g^2}$

recap: rough basics of Borel summation

(i) divergent, alternating:

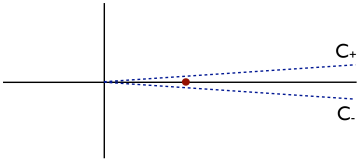
$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^\infty dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm\frac{i\,\pi}{g^2}\, e^{-1/g^2}$

avoid singularities on $\mathbb{R}^+$: lateral Borel sums:



$\theta = 0^{\pm} \longrightarrow$ non-perturbative ambiguity: $\pm\mathrm{Im}[\mathcal{S}_0 f(g^2)]$

challenge: use physical input to resolve ambiguity

Borel summation in practice (physical applications)

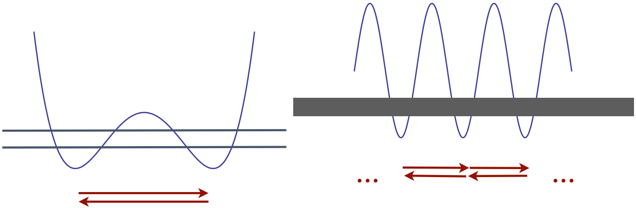
direct quantitative correspondence between:

rate of growth $\leftrightarrow$ Borel poles $\leftrightarrow$ non-perturbative exponent

non-alternating factorial growth: $c_n \sim b^n\, n!$

positive Borel singularity: $t_c = \frac{1}{b\, g^2}$

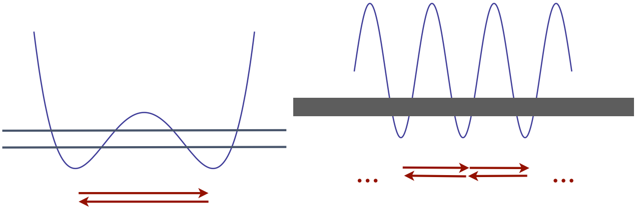
non-perturbative exponent: $\pm i\, \frac{\pi}{b\, g^2}\, \exp\left[-\left(\frac{1}{b\, g^2}\right)\right]$



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

Analogue of IR Renormalon Problem in QM



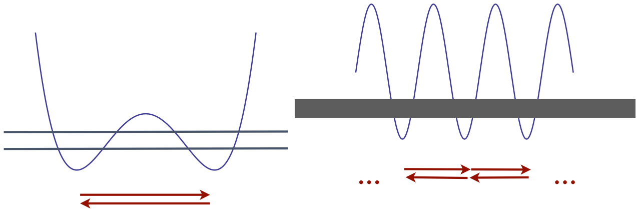
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

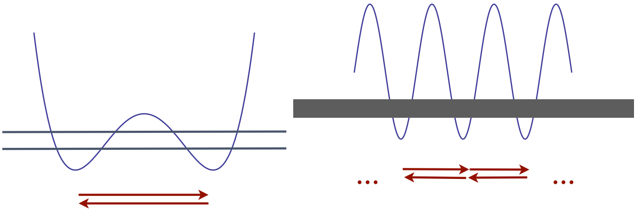
"Bogomolny/Zinn-Justin mechanism" in QM



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable:
     ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

“Bogomolny/Zinn-Justin mechanism” in QM



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel summable:
     ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

“resurgence” $\Rightarrow$ cancellation to all orders

Paradigm of Resurgence in QFT

• effective actions, partition functions, etc.., have natural integral representations permitting resurgent asymptotic expansions

• resurgent asymptotic expansions: analytic continuation of external parameters: temperature, chemical potential, external fields (electromagnetic, gravitational, ...)

• e.g., magnetic $\leftrightarrow$ electric; de Sitter $\leftrightarrow$ anti de Sitter, $\ldots$

Euler-Heisenberg and Matrix Models, Large N, Strings, ...

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\, N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\, N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{s^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

Resurgence and Analytic Continuation

one of many views of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

key problem: analytic continuation of **functional integrals**

QFT: Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

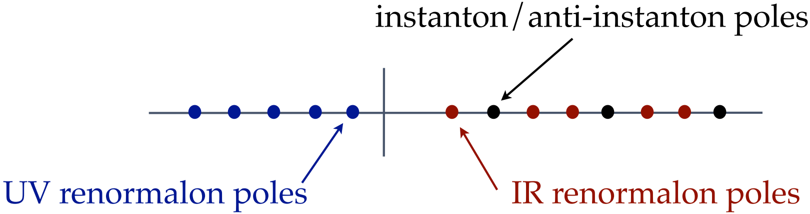
QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"
- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{g^2}}$



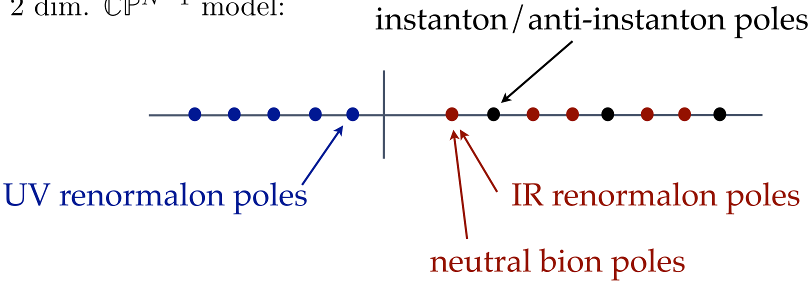
appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

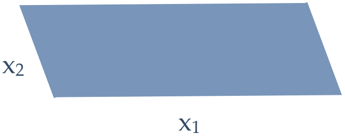
- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:

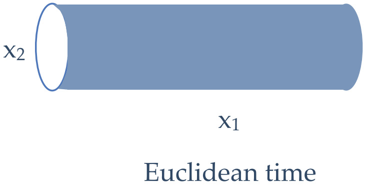


cancellation occurs !

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification

$$\mathbb{R}^2 \to S_L^1 \times \mathbb{R}^1$$





instantons fractionalize Kraan/van Baal; Lee/Yi; Bruckmann; Brendel et al, ...

Topological Molecules in Spatially Compactified Theories

temporal compactification: information about deconfined phase

spatial compactification: semi-classical small L regime continuously connected to large L:

principle of continuity

$\mathbb{CP}^{N-1}$ Model

$\mathbb{CP}^{N-1}$ model: 2d sigma model analogue of 4d Yang-Mills

- asymptotically free: $\beta_0 = N$ (independent of N_f)
- instantons, theta vacua, fermion zero modes, ...
- divergent perturbation theory (non-Borel summable)
- renormalons (both UV and IR)
- large-N analysis
- non-perturbative mass gap: $m_g = \Lambda = \mu\, e^{-4\pi/(g^2 N)}$
- couple to fermions, SUSY, ...
- analogue of center symmetry (GD, Ünsal, 1210.2423)

Fractionalized Instantons in $\mathbb{CP}^{N-1}$ on $S^1 \times \mathbb{R}^1$

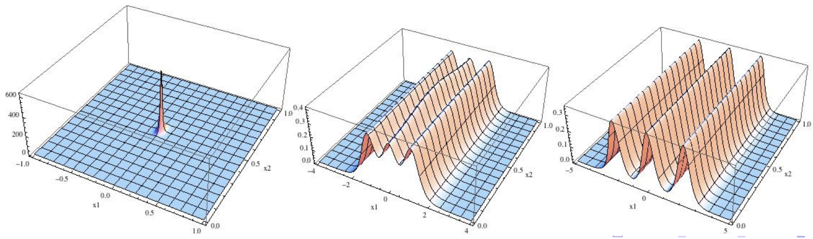
$\mathbb{Z}_N$ twisted instantons fractionalize Bruckmann, 2007; Brendel et al, 2009

- *spatial* compactification $\Rightarrow$ $\mathbb{Z}_N$ twist:

$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2\, e^{-\frac{2\pi}{L} z}\right)\, e^{\frac{2\pi}{L}\,\mu_2\, z} \end{pmatrix}$$

(twist in x_2)+(holomorphicity)$\Rightarrow$ fractionalization along x_1

$$\Rightarrow \qquad S_{\text{inst}} \longrightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$$



Fractionalized Bions in $\mathbb{CP}^{N-1}$ GD/Ünsal, 1210.2423

bions: topological molecules of instantons/anti-instantons

- characterized by (extended) Cartan matrix (as in YM)
- "orientation" dependence of $\mathcal{I}\bar{\mathcal{I}}$ interaction:
- charged bions: $\hat{A}_{ij} < 0$; repulsive bosonic interaction

$$\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j] \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{i\theta(\alpha_i - \alpha_j)}$$

- neutral bions: $\hat{A}_{ii} > 0$; attractive bosonic interaction

$$\Re \mathcal{B}_{ii} = \Re [\mathcal{K}_i \bar{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)}$$

- kink-anti-kink amplitude is two-fold ambiguous:

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_{\pm} = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right) \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- small radius limit $\longrightarrow$ effective QM Hamiltonian

$$H^{\rm zero} = \frac{g^2}{2}P_\theta^2 + \frac{\xi^2}{2g^2}\sin^2\theta + \frac{g^2}{2\sin^2\theta}P_\phi^2 \quad , \qquad \xi = \frac{2\pi}{N}$$

- Born-Oppenheimer approximation: drop high ϕ-sector modes

effective Mathieu equation:

$$-\frac{1}{2}\psi'' + \frac{\xi^2}{2g^2}\sin^2(g\theta)\psi = E\,\psi$$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0\xi^{-1} = \sum_{n=0}^{\infty} a_n (g^2)^n, \quad a_n \sim -\frac{2}{\pi}\left(\frac{N}{8\pi}\right)^n n!\left(1 - \frac{5}{2n} + \dots\right)$$

- non-Borel summable!

BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$ (GD, Ünsal, 1210.2423)

- perturbative sector: lateral Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi \, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$ (GD, Ünsal, 1210.2423)

- perturbative sector: lateral Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_{\pm}} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- non-perturbative sector: bion-bion amplitudes

$$\left[\mathcal{K}_i\bar{\mathcal{K}}_i\right]_{\pm} = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact cancellation !

application of resurgence to nontrivial QFT

The Bigger Picture

Q: should we expect resurgent behavior in QM and QFT?

• QM with degenerate vacua: trans-series arise naturally from uniform WKB, and perturbation theory generates everything: all multi-instanton effects are encoded in perturbation theory!

(GD, Ünsal, 1306.4405, 1401.5202)

Q: what is behind this resurgent structure?

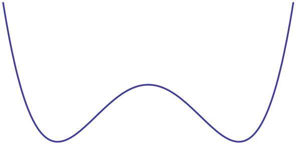
• basic property of all-orders steepest descents integrals: could this extend to (path) functional integrals ?

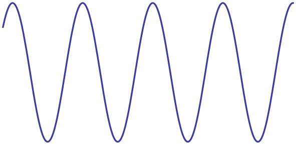
• resurgence 'enforces' proper analytic continuation properties

Uniform WKB and Resurgent Trans-Series for Eigenvalues

(GD, Ünsal, 1306.4405, 1401.5202)

$$-g^4 \frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2 \, E \, \psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \qquad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

Uniform WKB and Resurgent Trans-Series for Eigenvalues

- uniform WKB ansatz (ν a parameter)

$$\psi(y) = \frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$$

- nonlinear equation for $u(y)$
- perturbative expansion $\to u(y)$ and energy:

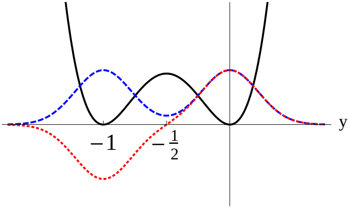
$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

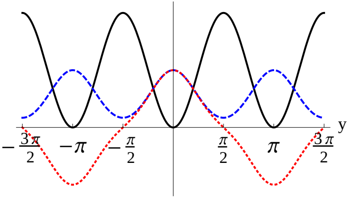
- $\nu = N$: Rayleigh-Schrödinger perturbation theory:

$$E\left(\nu = N, g^2\right) \equiv E^{(N)}_{\text{pert. theory}}(g^2) \quad \text{not Borel summable !}$$

Uniform WKB and Resurgent Trans-Series for Eigenvalues

- global analysis $\Rightarrow$ boundary conditions:





- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow$ $\quad g^2 \to e^{\pm i\,\epsilon}\, g^2$

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}\,(1+\dots) + e^{\pm i\pi\nu}\,\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, z^{-1-\nu}\, e^{z^2/4}\,(1+\dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\, 2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}\, \mathcal{F}(\nu, g^2)$$

- exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}\,\mathcal{F}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}$):

$$\begin{aligned}\nu &= N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{F}(N, g^2)}{N!}\,\xi \\ &- \frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{F}\frac{\partial \mathcal{F}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{F}^2\right]\xi^2 + O(\xi^3)\end{aligned}$$

- exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}\,\mathcal{F}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}$):

$$\begin{aligned}\nu &= N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{F}(N, g^2)}{N!}\,\xi \\ &- \frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{F}\frac{\partial \mathcal{F}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{F}^2\right]\xi^2 + O(\xi^3)\end{aligned}$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series!

Uniform WKB and Resurgent Trans-Series for Eigenvalues

for QM problems with degenerate harmonic vacua, the trans-series form of the exact expressions for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura: generate entire trans-series from

(i) perturbative expansion $B = B(E, g^2) \quad (B \equiv \nu + \frac{1}{2})$
(ii) single-instanton fluctuation function
$\mathcal{F}(E, g^2) \sim \exp[-\frac{1}{2}A(E, g^2)]$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

Uniform WKB and Resurgent Trans-Series for Eigenvalues

for QM problems with degenerate harmonic vacua, the trans-series form of the exact expressions for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura: generate entire trans-series from

(i) perturbative expansion $B = B(E, g^2) \quad (B \equiv \nu + \frac{1}{2})$
(ii) single-instanton fluctuation function
$\mathcal{F}(E, g^2) \sim \exp[-\frac{1}{2} A(E, g^2)]$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

something surprising happens when $B(E, g^2) \to E(B, g^2)$

Connecting Perturbative and Non-Perturbative Sector

- perturbative function ($g^2 \to g$):

$$E_{\mathrm{DW}}(B,g) \;=\; B - g\left(3B^2 + \frac{1}{4}\right) - g^2\left(17B^3 + \frac{19}{4}B\right) - g^3\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - g^4\left(\frac{10689}{4}B^5 + \frac{23405}{8}B^3 + \frac{22709}{64}B\right) - \ldots$$

- non-perturbative function ($\mathcal{F} \sim (\ldots)\exp[-A/2]$):

$$A_{\mathrm{DW}}(B,g) = \frac{1}{3g} + g\left(17B^2 + \frac{19}{12}\right) + g^2\left(125B^3 + \frac{153B}{4}\right) + g^3\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) + g^4\left(\frac{87549}{4}B^5 + \frac{50715}{2}B^3 + \frac{217663}{64}B\right)$$

Connecting Perturbative and Non-Perturbative Sector

- perturbative function ($g^2 \to g$):

$$E_{\mathrm{DW}}(B,g) = B - g\left(3B^2 + \frac{1}{4}\right) - g^2\left(17B^3 + \frac{19}{4}B\right) - g^3\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - g^4\left(\frac{10689}{4}B^5 + \frac{23405}{8}B^3 + \frac{22709}{64}B\right) - \dots$$

- non-perturbative function ($\mathcal{F} \sim (\dots)\exp[-A/2]$):

$$A_{\mathrm{DW}}(B,g) = \frac{1}{3g} + g\left(17B^2 + \frac{19}{12}\right) + g^2\left(125B^3 + \frac{153B}{4}\right) + g^3\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) + g^4\left(\frac{87549}{4}B^5 + \frac{50715}{2}B^3 + \frac{217663}{64}B\right)$$

- simple relation:

$$\frac{\partial E_{\mathrm{DW}}}{\partial B} = -6\,B\,g - 3g^2\frac{\partial A_{\mathrm{DW}}}{\partial g}$$

Connecting Perturbative and Non-Perturbative Sector

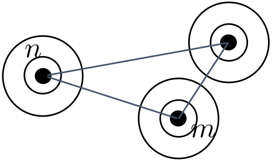
- similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and $O(d)$ AHO, ...

- general expression: (GD, Ünsal, 1306.4405, 1401.5202)

$$\frac{\partial E}{\partial B} = -\frac{g}{2S}\left(2B + g\frac{\partial A}{\partial g}\right)$$

implication: perturbation theory generates everything !
all orders of multi-instanton trans-series encoded in perturbation theory of fluctuations about perturbative vacuum

$$\mathcal{F}(\nu, g^2) = \exp\left[S\int_0^{g^2}\frac{dg^2}{g^4}\left(\frac{\partial E}{\partial \nu} - 1 + \frac{\left(\nu+\frac{1}{2}\right)\, g^2}{S}\right)\right]$$



Connecting Perturbative and Non-Perturbative Sector

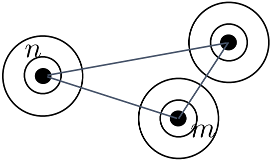
- similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and $O(d)$ AHO, ...

- general expression: (GD, Ünsal, 1306.4405, 1401.5202)

$$\frac{\partial E}{\partial B} = -\frac{g}{2S}\left(2B + g\frac{\partial A}{\partial g}\right)$$

implication: perturbation theory generates everything !
all orders of multi-instanton trans-series encoded in perturbation theory of fluctuations about perturbative vacuum

$$\mathcal{F}(\nu, g^2) = \exp\left[S\int_0^{g^2}\frac{dg^2}{g^4}\left(\frac{\partial E}{\partial \nu} - 1 + \frac{\left(\nu+\frac{1}{2}\right)\, g^2}{S}\right)\right]$$



why ? turn to path integrals ....

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals (Berry/Howls 1991: *hyperasymptotics*)

$$I^{(n)}(k) = \int_{C_n} dz\, e^{-k\, f(z)} = \frac{1}{\sqrt{k}}\, e^{-k\, f_n}\, T^{(n)}(k)$$

- $T^{(n)}(k)$: beyond the Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(k) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(k\, F_{nm})} T^{(m)} \left(\frac{v}{F_{nm}} \right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(k) = \frac{1}{2\pi\, i}\sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(k\, F_{nm})} T^{(m)}\left(\frac{v}{F_{nm}}\right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i}\sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r}\left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots\right.$$

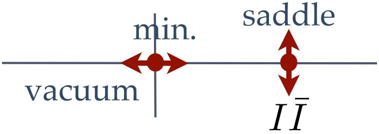
- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(k) = \int_0^\pi dz\, e^{-k\, \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.$$

All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{\frac{1}{4}}{(r-1)}+\frac{\frac{9}{32}}{(r-1)(r-2)}-\frac{\frac{75}{128}}{(r-1)(r-2)(r-3)}+\right.$$

- low order coefficients about saddle z_1:

$$T^{(1)}(k) \sim i\sqrt{\pi}\left(1-\frac{1}{4}\cdot\frac{1}{k}+\frac{9}{32}\cdot\frac{1}{k^2}-\frac{75}{128}\cdot\frac{1}{k^3}+\dots\right)$$

- fluctuations about the two saddles are explicitly related could something like this work for path integrals?

"functional Darboux theorem" ?

Resurgence in Path Integrals: "Functional Darboux Theorem"

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

Resurgence in Path Integrals: "Functional Darboux Theorem"

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

Resurgence in Path Integrals: “Functional Darboux Theorem”

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle

“functional Darboux theorem”

Analytic Continuation of Path Integrals: Lefschetz Thimbles

functional version: path integral

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}\left(S_{\rm real}[A]+i\, S_{\rm imag}[A]\right)} \sim \sum_{\text{thimbles}\, k} e^{-\frac{i}{g^2}\, S_{\rm imag}[A]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2} S_{\rm real}[A]}$$

thimble = "functional steepest descents contour"

remaining path integral has real measure: amenable to
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact results?

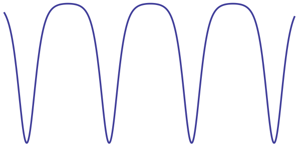
resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

"functional Darboux" suggests possibilities ...

Ghost Instantons: Quantum Mechanical Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)



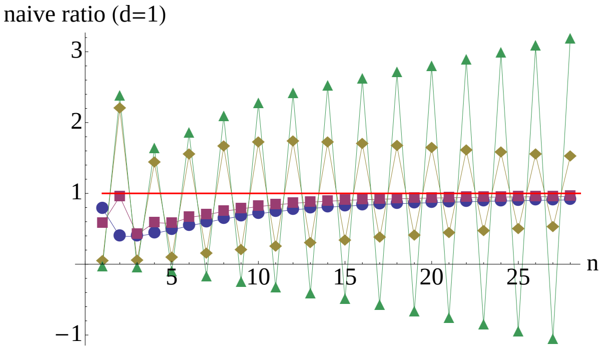
periodic elliptic potential

$$V(x) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)$$

$$
\begin{aligned}
E\left(g^2|0\right) &= 1 - \frac{g^2}{4} - \frac{g^4}{16} - \frac{3g^6}{64} - \frac{53g^8}{1024} - \frac{297g^{10}}{4096} - \dots \\
E\left(g^2|1\right) &= 1 + \frac{g^2}{4} - \frac{g^4}{16} + \frac{3g^6}{64} - \frac{53g^8}{1024} - \frac{297g^{10}}{4096} - \dots \\
E\left(g^2\middle|\frac{1}{4}\right) &= 1 - \frac{g^2}{8} - \frac{11g^4}{128} - \frac{3g^6}{128} - \frac{889g^8}{32768} - \frac{225g^{10}}{8192} - \dots \\
E\left(g^2\middle|\frac{3}{4}\right) &= 1 + \frac{g^2}{8} - \frac{11g^4}{128} + \frac{3g^6}{128} - \frac{889g^8}{32768} + \frac{225g^{10}}{8192} - \dots \\
E\left(g^2\middle|\frac{1}{2}\right) &= 1 + 0g^2 - \frac{3g^4}{32} + 0g^6 - \frac{39g^8}{2048} + 0g^{10} - \dots
\end{aligned}
$$

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \, \frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}}$$



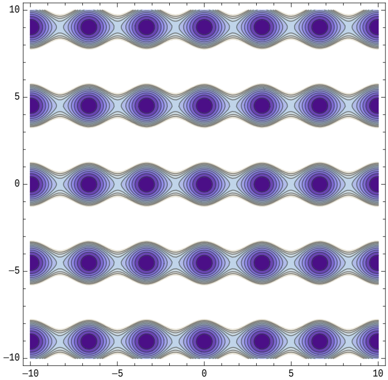
fails miserably !

Ghost Instantons: Quantum Mechanical Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\, \mathrm{sd}^2(g\, x|m)\right)}$$

- doubly periodic potential: *real* & complex instantons



instanton actions:

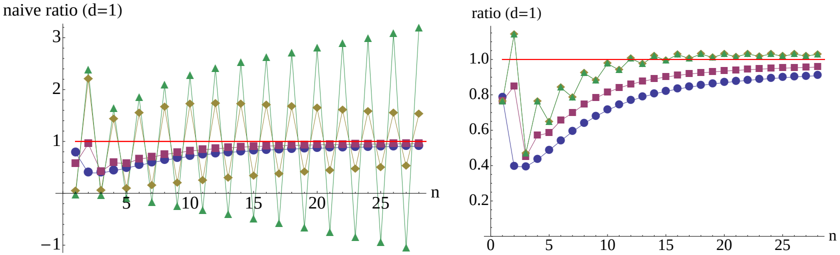
$$\frac{S_{\mathcal{I}}(m)}{g^2} = \frac{2\arcsin(\sqrt{m})}{g^2\sqrt{m(1-m)}}$$

$$\frac{S_{\mathcal{G}}(m)}{g^2} = \frac{-2\arcsin(\sqrt{1-m})}{g^2\sqrt{m(1-m)}}$$

Ghost Instantons: Quantum Mechanical Path Integrals

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons *with ghost instantons*

- complex instantons directly affect perturbation theory, even though they are not in original path integral measure !

Non-perturbative Physics Without Instantons

Dabrowski, GD, arXiv:1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127

Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, PCM, ... all have non-BPS solutions with finite action

- "unstable": negative modes of fluctuation operator
- what do these mean ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Non-perturbative Physics Without Instantons: Principal Chiral Model (Cherman, Dorigoni, GD, Ünsal, 1308.0127)

$$S_b = \frac{N}{2\lambda} \int d^2x \, \mathrm{tr} \, \partial_\mu U \partial^\mu U^\dagger, \quad U \in SU(N),$$

- non-Borel-summable perturbation theory due to IR renomalons
- but, the theory has no instantons !

resolution: there exist non-BPS saddle point solutions to the second-order classical Euclidean equations of motion: "unitons" (Uhlenbeck)

$$\partial_\mu \left(U^\dagger \partial_\mu U \right) = 0$$

- have negative fluctuation modes: saddles, not minima
- fractionalize on cylinder $\longrightarrow$ BZJ cancellation

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative world
- trans-series encode 'all' information
- there is extra 'magic' in perturbation theory
- expansions about different saddles are related
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- basic property of steepest descents expansions
- hints of analytic continuation for path integrals
- moral: consider all saddles, not just minima

Open Problems

- natural path integral construction ?
- analytic continuation of path integrals ?
- relation to localization ?
- relation to renormalization group flow ?
- relating strong- & weak-coupling expansions: dualities ?
- relation to operator product expansion (OPE) ?
- relation to SUSY and extended SUSY ?
- ODE/Integrable Model correspondence ?
- ...