Resurgence in Quantum Field Theory and Quantum Mechanics

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GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302
Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
  (i) IR renormalons $\Rightarrow$ perturbation theory ill-defined
  (ii) $\mathcal{I}\mathcal{I}$ interactions $\Rightarrow$ instanton-gas ill-defined

- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- “exact” asymptotics in QFT and string theory: relation to localization in QFT
Mathematical Motivation

Resurgence: ‘new’ idea in mathematics (Écalle, 1980; Stokes, 1850)

- goal: explore implications for physics

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally ⇒ divergent series
- series expansion → \textit{trans-series} expansion
- trans-series ‘well-defined under analytic continuation’
- perturbative and non-perturbative physics entwined
- philosophical shift:
  view semiclassical expansions as potentially exact
- applications: ODEs, PDEs, QM, Matrix Models, QFT, String Theory, ...
• trans-series expansion in QM and QFT applications:

\[
f(g^2) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \left( \frac{1}{g^{2N+1}} \exp \left[ -\frac{c}{g^2} \right] \right)^k c_{k,l,p} g^{2p} \left( \ln \left[ \pm \frac{1}{g^2} \right] \right)^l
\]

- known as “multi-instanton calculus” in QFT
- trans-monomial elements, \( e^{-\frac{1}{g^2}} \), \( \ln(g^2) \), \( g^2 \), are familiar
- many exact results in supersymmetric QFT
- **does the resurgence perspective add something new?**
- trans-series expansion coefficients highly correlated
- exponentially improved asymptotic expansions
resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities.

J. Écalle, 1980
recap: rough basics of Borel summation

(i) divergent, alternating:
\[ \sum_{n=0}^{\infty} (-1)^n \frac{n!}{g^{2n}} = \int_0^\infty dt \ e^{-t} \frac{1}{1+g^2t} \]

(ii) divergent, non-alternating:
\[ \sum_{n=0}^{\infty} \frac{n!}{g^{2n}} = \int_0^\infty dt \ e^{-t} \frac{1}{1+g^2t} \]

⇒ ambiguous imaginary non-pert. term:
\[ \pm i\pi \frac{e^{-1}}{g^2} \]

avoid singularities on \( \mathbb{R}^+ \): lateral Borel sums:
\[ C_+ + C_- \]

\[ \theta = 0 \]
\[ \pm \rightarrow \] non-perturbative ambiguity:
\[ \pm \text{Im}[S_0(f(g^2))] \]

challenge: use physical input to resolve ambiguity
recap: rough basics of Borel summation

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avoid singularities on \( \mathbb{R}^+ \): lateral Borel sums:

\[ \theta = 0^\pm \rightarrow \text{non-perturbative ambiguity: } \pm \text{Im}[S_0 f(g^2)] \]

challenge: use physical input to resolve ambiguity
direct quantitative correspondence between:

rate of growth ↔ Borel poles ↔ non-perturbative exponent

non-alternating factorial growth: \( c_n \sim b^n n! \)

positive Borel singularity: \( t_c = \frac{1}{b g^2} \)

non-perturbative exponent: \( \pm i \frac{\pi}{b g^2} \exp \left[ - \left( \frac{1}{b g^2} \right) \right] \)
Analogue of IR Renormalon Problem in QM

- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$
Analogue of IR Renormalon Problem in QM

- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: \( \Delta E \sim e^{-\frac{S}{g^2}} \)

surprise: pert. theory non-Borel summable: \( c_n \sim \frac{n!}{(2S)^n} \)
  - stable systems
  - ambiguous imaginary part
  - \( \pm ie^{-\frac{2S}{g^2}} \), a 2-instanton effect
“Bogomolny/Zinn-Justin mechanism” in QM

- degenerate vacua: double-well, Sine-Gordon, ...

1. perturbation theory non-Borel summable: ill-defined/incomplete

2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\mathcal{I}'$ attract

- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel!
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- regularize both by analytic continuation of coupling

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“resurgence” $\Rightarrow$ cancellation to all orders
Paradigm of Resurgence in QFT

- effective actions, partition functions, etc., have natural integral representations permitting resurgent asymptotic expansions

- resurgent asymptotic expansions: analytic continuation of external parameters: temperature, chemical potential, external fields (electromagnetic, gravitational, ...)

- e.g., magnetic $\leftrightarrow$ electric; de Sitter $\leftrightarrow$ anti de Sitter, ...
Euler-Heisenberg and Matrix Models, Large N, Strings, ...

- scalar QED EH in self-dual background ($F = \pm \tilde{F}$):
  \[
  S = \frac{F^2}{16\pi^2} \int_0^\infty \frac{dt}{t} e^{-t} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)
  \]

- Gaussian matrix model: $\lambda = g N$
  \[
  \mathcal{F} = -\frac{1}{4} \int_0^\infty \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)
  \]

- $c = 1$ String: $\lambda = g N$
  \[
  \mathcal{F} = \frac{1}{4} \int_0^\infty \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sin^2(t)} - \frac{1}{s^2} - \frac{1}{3} \right)
  \]

- Chern-Simons matrix model:
  \[
  \mathcal{F} = -\frac{1}{4} \sum_{m \in \mathbb{Z}} \int_0^\infty \frac{dt}{t} e^{-2(\lambda + 2\pi i m) t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)
  \]
one of many views of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

key problem: analytic continuation of functional integrals
QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: “renormalons”
- both positive and negative Borel poles
IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: \[ \pm ie^{-\frac{2S}{\beta_0 g^2}} \]

instantons on \( \mathbb{R}^2 \) or \( \mathbb{R}^4 \): \[ \pm ie^{-\frac{2S}{g^2}} \]

appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

\(^{\prime}t\) Hooft, 1980; David, 1981
resolution: there is another problem with the non-perturbative instanton gas analysis  
(Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{C}P^{N-1}$ model:

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UV renormalon poles

\[\text{instanton/anti-instanton poles}\]

IR renormalon poles

neutral bion poles

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Cancellation occurs!
Topological Molecules in Spatially Compactified Theories

\( \mathbb{CP}^{N-1} \): regulate scale modulus problem with (spatial) compactification

\( \mathbb{R}^2 \rightarrow S_L^1 \times \mathbb{R}^1 \)

Euclidean time

instantons fractionalize

Kraan/van Baal; Lee/Yi; Bruckmann; Brendel et al,
temporal compactification: information about deconfined phase

spatial compactification: semi-classical small $L$ regime
continuously connected to large $L$:

principle of continuity

"continuity"
\( \mathbb{CP}^{N-1} \) Model

\( \mathbb{CP}^{N-1} \) model: 2d sigma model analogue of 4d Yang-Mills

- asymptotically free: \( \beta_0 = N \) (independent of \( N_f \))
- instantons, theta vacua, fermion zero modes, ...
- divergent perturbation theory (non-Borel summable)
- renormalons (both UV and IR)
- large-\( N \) analysis
- non-perturbative mass gap: \( m_g = \Lambda = \mu e^{-4\pi/(g^2 N)} \)
- couple to fermions, SUSY, ...
- analogue of center symmetry (GD, Ünsal, 1210.2423)
Fractionalized Instantons in $\mathbb{CP}^{N-1}$ on $S^1 \times \mathbb{R}^1$

$\mathbb{Z}_N$ twisted instantons fractionalize Bruckmann, 2007; Brendel et al, 2009

- *spatial* compactification $\Rightarrow \mathbb{Z}_N$ twist:

$$v_{\text{twisted}} = \left( \left( \lambda_1 + \lambda_2 e^{-\frac{2\pi}{L} z} \right) e^{\frac{2\pi}{L} \mu_2 z} \right)$$

(twist in $x_2$) + (holomorphicity) $\Rightarrow$ fractionalization along $x_1$

$$\Rightarrow \quad S_{\text{inst}} \rightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$$
bions: topological molecules of instantons/anti-instantons

- characterized by (extended) Cartan matrix (as in YM)
- “orientation” dependence of $I\bar{I}$ interaction:
- charged bions: $\hat{A}_{ij} < 0$; repulsive bosonic interaction
  \[ \mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j] \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{i \theta (\alpha_i - \alpha_j)} \]
- neutral bions: $\hat{A}_{ii} > 0$; attractive bosonic interaction
  \[ \Re \mathcal{B}_{ii} = \Re [\mathcal{K}_i \bar{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)} \]
- kink-anti-kink amplitude is two-fold ambiguous:
  \[ [\mathcal{K}_i \bar{\mathcal{K}}_i]_\pm = \left( \ln \left( \frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i \pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \]
Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- Small radius limit $\rightarrow$ effective QM Hamiltonian
  \[
  H^{\text{zero}} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2 \theta + \frac{g^2}{2 \sin^2 \theta} P_\phi^2 , \quad \xi = \frac{2\pi}{N}
  \]

- Born-Oppenheimer approximation: drop high $\phi$-sector modes
  effective Mathieu equation:
  \[
  -\frac{1}{2} \psi'' + \frac{\xi^2}{2g^2} \sin^2 (g\theta) \psi = E \psi
  \]

- Stone-Reeve (Bender-Wu methods):
  \[
  \mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{n=0}^{\infty} a_n (g^2)^n , \quad a_n \sim -\frac{2}{\pi} \left( \frac{N}{8\pi} \right)^n n! \left( 1 - \frac{5}{2n} + \ldots \right)
  \]

- Non-Borel summable!
BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$

- perturbative sector: lateral Borel summation

\[
B_{\pm} \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_{\pm}} dt \, B\mathcal{E}(t) \, e^{-t/g^2} = \text{Re} \, B\mathcal{E}(g^2) \mp i\pi \frac{16}{g^2 N} \, e^{-\frac{8\pi}{g^2 N}}
\]
BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$ (GD, Ünsal, 1210.2423)

- perturbative sector: lateral Borel summation

$$B_\pm \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt \ B\mathcal{E}(t) \ e^{-t/g^2} = \text{Re} \ B\mathcal{E}(g^2) \mp i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

- non-perturbative sector: bion-bion amplitudes

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_\pm = \left( \ln \left( \frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

exact cancellation!

application of resurgence to nontrivial QFT
Q: should we expect resurgent behavior in QM and QFT?

• QM with degenerate vacua: trans-series arise naturally from uniform WKB, and perturbation theory generates everything: all multi-instanton effects are encoded in perturbation theory!

(QD, Ünsal, 1306.4405, 1401.5202)

Q: what is behind this resurgent structure?

• basic property of all-orders steepest descents integrals: could this extend to (path) functional integrals?

• resurgence ‘enforces’ proper analytic continuation properties
Uniform WKB and Resurgent Trans-Series for Eigenvalues

\[ -g^4 \frac{d^2}{dy^2} \psi(y) + V(y) \psi(y) = g^2 E \psi(y) \]

- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: \( g^2 \leftrightarrow \hbar \Rightarrow \exp \left( -\frac{c}{g^2} \right) \)
- approximately harmonic

\( \Rightarrow \) uniform WKB with parabolic cylinder functions
Uniform WKB and Resurgent Trans-Series for Eigenvalues

- uniform WKB ansatz ($\nu$ a parameter)

$$
\psi(y) = \frac{D_\nu \left( \frac{1}{g} u(y) \right)}{\sqrt{u'(y)}}
$$

- nonlinear equation for $u(y)$

- perturbative expansion $\rightarrow u(y)$ and energy:

$$
E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)
$$

- $\nu = N$: Rayleigh-Schrödinger perturbation theory:

$$
E \left( \nu = N, g^2 \right) \equiv E_{\text{pert. theory}}^{(N)}(g^2) \text{ not Borel summable!}
$$
Uniform WKB and Resurgent Trans-Series for Eigenvalues

- global analysis ⇒ boundary conditions:

\[ y - \frac{1}{y^2} - 1 \]

- midpoint ∼ \( \frac{1}{g} \); non-Borel summability ⇒ \( g^2 \rightarrow e^{\pm i\epsilon} g^2 \)

\[ D_\nu(z) \sim z^\nu e^{-z^2/4} (1 + \ldots) + e^{\pm i\pi \nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} e^{z^2/4} (1 + \ldots) \]

→ exact quantization condition

\[ \frac{1}{\Gamma(-\nu)} \left( \frac{e^{\pm i\pi} 2}{g^2} \right)^{-\nu} = e^{-S/g^2} \sqrt{\pi g^2} F(\nu, g^2) \]
• exact quantization condition

\[
\frac{1}{\Gamma(-\nu)} \left( \frac{e^{\pm i\pi/2}}{g^2} \right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi g^2}} \mathcal{F}(\nu, g^2)
\]

\[
\Rightarrow \ \nu \text{ is only exponentially close to } N \text{ (here } \xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi g^2}}): \]

\[
\nu = N + \left( \frac{2}{g^2} \right)^N \mathcal{F}(N, g^2) - \frac{\left( \frac{2}{g^2} \right)^{2N}}{(N!)^2} \left[ \mathcal{F} \frac{\partial \mathcal{F}}{\partial N} + \left( \ln \left( \frac{e^{\pm i\pi/2}}{g^2} \right) - \psi(N + 1) \right) \mathcal{F}^2 \right] \xi^2 + O(\xi^3)
\]
Uniform WKB and Resurgent Trans-Series for Eigenvalues

- exact quantization condition

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\frac{1}{\Gamma(-\nu)} \left( \frac{e^{\pm i\pi/2}}{g^2} \right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi g^2}} \mathcal{F}(\nu, g^2)
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\[
\nu = N + \frac{\left( \frac{2}{g^2} \right)^N}{N!} \xi
\]

\[
- \frac{\left( \frac{2}{g^2} \right)^{2N}}{(N!)^2} \left[ \mathcal{F} \frac{\partial \mathcal{F}}{\partial N} + \left( \ln \left( \frac{e^{\pm i\pi/2}}{g^2} \right) - \psi(N + 1) \right) \mathcal{F}^2 \right] \xi^2 + O(\xi^3)
\]

- insert: \( E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu) \Rightarrow \text{trans-series!} \)
for QM problems with degenerate harmonic vacua, the trans-series form of the exact expressions for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions.

generic and universal

Zinn-Justin/Jentschura: generate entire trans-series from

(i) perturbative expansion $B = B(E, g^2)$ ($B \equiv \nu + \frac{1}{2}$)
(ii) single-instanton fluctuation function $\mathcal{F}(E, g^2) \sim \exp[-\frac{1}{2}A(E, g^2)]$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)


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(ii) single-instanton fluctuation function

\[ \mathcal{F}(E, g^2) \sim \exp\left[ -\frac{1}{2} A(E, g^2) \right] \]

(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

something surprising happens when

\[ B(E, g^2) \rightarrow E(B, g^2) \]
Connecting Perturbative and Non-Perturbative Sector

- perturbative function \((g^2 \rightarrow g)\):

\[
E_{DW}(B, g) = B - g \left(3B^2 + \frac{1}{4}\right) - g^2 \left(17B^3 + \frac{19}{4}B\right) - \frac{g^3}{3} \left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - g^4 \left(\frac{10689}{4}B^5 + \frac{23405}{8}B^3 + \frac{22709}{64}B\right) - \ldots
\]

- non-perturbative function \((\mathcal{F} \sim (...) \exp[-A/2])\):

\[
A_{DW}(B, g) = \frac{1}{3g} + g \left(17B^2 + \frac{19}{12}\right) + g^2 \left(125B^3 + \frac{153B}{4}\right) + \frac{g^3}{12} \left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) + g^4 \left(\frac{87549}{4}B^5 + \frac{50715}{2}B^3 + \frac{217663}{64}B\right)
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\]

- simple relation:

\[
\frac{\partial E_{DW}}{\partial B} = -6Bg - 3g^2 \frac{\partial A_{DW}}{\partial g}
\]
Connecting Perturbative and Non-Perturbative Sector

- similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and $O(d)$ AHO, ...

- general expression: \((\text{GD, Ünsal, 1306.4405, 1401.5202})\)

\[
\frac{\partial E}{\partial B} = -\frac{g^2}{2S} \left(2B + g \frac{\partial A}{\partial g}\right)
\]

implication: perturbation theory generates everything!
all orders of multi-instanton trans-series encoded in perturbation theory of fluctuations about perturbative vacuum

\[
\mathcal{F}(\nu, g^2) = \exp \left[ S \int_0^{g^2} \frac{dg^2}{g^4} \left( \frac{\partial E}{\partial \nu} - 1 + \left(\nu + \frac{1}{2}\right) \frac{g^2}{S} \right) \right]
\]

why? turn to path integrals...
Connecting Perturbative and Non-Perturbative Sector

• similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and $O(d)$ AHO, ...

• general expression: (GD, Ünsal, 1306.4405, 1401.5202)

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$$\mathcal{F}(\nu, g^2) = \exp \left[ S \int_0^{g^2} \frac{dg^2}{g^4} \left( \frac{\partial E}{\partial \nu} - 1 + \frac{(\nu + \frac{1}{2}) g^2}{S} \right) \right]$$

why? turn to path integrals ....
All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals
  (Berry/Howls 1991: hyperasymptotics)

\[ I^{(n)}(k) = \int_{C_n} dz \ e^{-k f(z)} = \frac{1}{\sqrt{k}} e^{-k f_n} T^{(n)}(k) \]

- \( T^{(n)}(k) \): beyond the Gaussian approximation

- asymptotic expansion of fluctuations about the saddle \( n \):

\[ T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r} \]
All-Orders Steepest Descents: Darboux Theorem

• universal resurgent relation between different saddles:

\[ T^{(n)}(k) = \frac{1}{2\pi i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(kF_{nm})} T^{(m)} \left( \frac{v}{F_{nm}} \right) \]

• exact resurgent relation between fluctuations about \( n^{th} \) saddle and about neighboring saddles \( m \)
All-Orders Steepest Descents: Darboux Theorem

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• exact resurgent relation between fluctuations about \( n^{\text{th}} \) saddle and about neighboring saddles \( m \)

\[ T_r^{(n)} = \frac{(r - 1)!}{2\pi i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[ T_0^{(m)} + \frac{F_{nm}}{(r - 1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r - 1)(r - 2)} T_2^{(m)} + \ldots \right] \]

• universal factorial divergence of fluctuations (Darboux)

• fluctuations about different saddles explicitly related!
$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(k) = \int_0^\pi dz \, e^{-k \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$. 

![Diagram with vacuum, minima, and saddle points labeled $I\bar{I}$]
All-Orders Steepest Descents: Darboux Theorem

• large order behavior about saddle $z_0$:

$$T_r^{(0)} = \frac{\Gamma \left( r + \frac{1}{2} \right)^2}{\sqrt{\pi} \Gamma(r + 1)}$$

$$\sim \frac{(r - 1)!}{\sqrt{\pi}} \left( 1 - \frac{1}{4} \frac{1}{(r - 1)} + \frac{9}{32} \frac{1}{(r - 1)(r - 2)} - \frac{75}{128} \frac{1}{(r - 1)(r - 2)(r - 3)} \right)$$
All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle $z_0$:

$$T_r^{(0)} = \frac{\Gamma (r + \frac{1}{2})^2}{\sqrt{\pi} \Gamma (r + 1)}$$

$$\sim \frac{(r - 1)!}{\sqrt{\pi}} \left( 1 - \frac{1}{4} \cdot \frac{1}{r - 1} + \frac{9}{32} \cdot \frac{1}{(r - 1)(r - 2)} - \frac{75}{128} \cdot \frac{1}{(r - 1)(r - 2)(r - 3)} + \ldots \right)$$

- low order coefficients about saddle $z_1$:

$$T^{(1)}(k) \sim i \sqrt{\pi} \left( 1 - \frac{1}{4} \cdot \frac{1}{k} + \frac{9}{32} \cdot \frac{1}{k^2} - \frac{75}{128} \cdot \frac{1}{k^3} + \ldots \right)$$

- fluctuations about the two saddles are explicitly related
could something like this work for path integrals?

“functional Darboux theorem”?
Resurgence in Path Integrals: “Functional Darboux Theorem”

- periodic potential: \( V(x) = \frac{1}{g^2} \sin^2(gx) \)

- vacuum saddle point

\[ c_n \sim n! \left( 1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \cdots \right) \]

- instanton/anti-instanton saddle point:

\[ \text{Im} \ E \sim \pi e^{-2 \frac{1}{2g^2}} \left( 1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \cdots \right) \]
Resurgence in Path Integrals: “Functional Darboux Theorem”

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(gx)$

- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \cdots\right)$$

- instanton/anti-instanton saddle point:

$$\text{Im } E \sim \pi e^{-\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \cdots\right)$$

- double-well potential: $V(x) = x^2(1 - gx)^2$

- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \cdots\right)$$

- instanton/anti-instanton saddle point:

$$\text{Im } E \sim \pi e^{-\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \cdots\right)$$
Resurgence in Path Integrals: “Functional Darboux Theorem”

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle

“functional Darboux theorem”
Analytic Continuation of Path Integrals: Lefschetz Thimbles

functional version: path integral

\[ \int \mathcal{D}A \, e^{-\frac{1}{g^2} \left( S_{\text{real}}[A] + i S_{\text{imag}}[A] \right)} \sim \sum_{\text{thimbles } k} e^{-\frac{i}{g^2} S_{\text{imag}}[A]} \int_{\Gamma_k} \mathcal{D}A \, e^{-\frac{1}{g^2} S_{\text{real}}[A]} \]

thimble = “functional steepest descents contour”

remaining path integral has real measure: amenable to
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact results?

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

“functional Darboux” suggests possibilities ...
Ghost Instantons: Quantum Mechanical Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

periodic elliptic potential

\[ V(x) = \frac{1}{g^2} \text{sd}^2(gx|m) \]

\[
E(g^2|0) = 1 - \frac{g^2}{4} - \frac{g^4}{16} - \frac{3g^6}{64} - \frac{53g^8}{1024} - \frac{297g^{10}}{4096} - \ldots
\]

\[
E(g^2|1) = 1 + \frac{g^2}{4} - \frac{g^4}{16} + \frac{3g^6}{64} - \frac{53g^8}{1024} + \frac{297g^{10}}{4096} - \ldots
\]

\[
E\left(g^2\left|\frac{1}{4}\right.\right) = 1 - \frac{g^2}{8} - \frac{11g^4}{128} - \frac{3g^6}{128} - \frac{889g^8}{32768} - \frac{225g^{10}}{8192} - \ldots
\]

\[
E\left(g^2\left|\frac{3}{4}\right.\right) = 1 + \frac{g^2}{8} - \frac{11g^4}{128} + \frac{3g^6}{128} - \frac{889g^8}{32768} + \frac{225g^{10}}{8192} - \ldots
\]

\[
E\left(g^2\left|\frac{1}{2}\right.\right) = 1 + 0g^2 - \frac{3g^4}{32} + 0g^6 - \frac{39g^8}{2048} + 0g^{10} - \ldots
\]
Ghost Instantons: Quantum Mechanical Path Integrals

- Large order growth of perturbation theory:

\[ a_n(m) \sim -\frac{16}{\pi} n! \frac{1}{(S_{\bar{I}I}(m))^{n+1}} \]

fails miserably!
Ghost Instantons: Quantum Mechanical Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

\[ Z(g^2|m) = \int \mathcal{D}x \, e^{-S[x]} = \int \mathcal{D}x \, e^{-\int d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{g^2} \text{sd}^2(gx|m) \right)} \]

- doubly periodic potential: real & complex instantons

\[
\begin{align*}
S_{I}(m) & = \frac{2 \arcsin(\sqrt{m})}{g^2} = \frac{2 \arcsin(\sqrt{m})}{g^2 \sqrt{m(1-m)}} \\
S_{G}(m) & = -\frac{2 \arcsin(\sqrt{1-m})}{g^2} = -\frac{2 \arcsin(\sqrt{1-m})}{g^2 \sqrt{m(1-m)}}
\end{align*}
\]
Ghost Instantons: Quantum Mechanical Path Integrals

• large order growth of perturbation theory:

\[ a_n(m) \sim -\frac{16}{\pi} n! \left( \frac{1}{(S_{\Pi\Pi}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{G\bar{G}}(m)|^{n+1}} \right) \]

without ghost instantons

with ghost instantons

• complex instantons directly affect perturbation theory, even though they are not in original path integral measure!
Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, PCM, ... all have non-BPS solutions with finite action

- “unstable”: negative modes of fluctuation operator
- what do these mean?

**resurgence**: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

\[
\int \mathcal{D}A \, e^{-\frac{1}{g^2} S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2} S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})
\]
Non-perturbative Physics Without Instantons: Principal Chiral Model
(Cherman, Dorighini, GD, Ünsal, 1308.0127)

\[ S_b = \frac{N}{2\lambda} \int d^2 x \, \text{tr} \, \partial_\mu U \partial^\mu U^\dagger, \quad U \in SU(N), \]

- non-Borel-summable perturbation theory due to IR renomalons
- but, the theory has no instantons!

resolution: there exist non-BPS saddle point solutions to the second-order classical Euclidean equations of motion: “unitons” (Uhlenbeck)

\[ \partial_\mu \left( U^\dagger \partial_\mu U \right) = 0 \]

- have negative fluctuation modes: saddles, not minima
- fractionalize on cylinder \( \longrightarrow \) BZJ cancellation
Conclusions

- Resurgence systematically unifies perturbative and non-perturbative world
- Trans-series encode ‘all’ information
- There is extra ‘magic’ in perturbation theory
- Expansions about different saddles are related
- IR renormalon puzzle in asymptotically free QFT
- Multi-instanton physics from perturbation theory
- Basic property of steepest descents expansions
- Hints of analytic continuation for path integrals
- Moral: consider all saddles, not just minima
Open Problems

• natural path integral construction?
• analytic continuation of path integrals?
• relation to localization?
• relation to renormalization group flow?
• relating strong- & weak-coupling expansions: dualities?
• relation to operator product expansion (OPE)?
• relation to SUSY and extended SUSY?
• ODE/Integrable Model correspondence?
• …