Self-similar solutions for the two dimensional non-Newtonian Navier-Stokes equation

Imre Ferenc Barna

Wigner Research Center of the Hungarian Academy of Sciences

2 Dec. 2013 Zimányi School'13

Outline

Solutions of PDEs *self-similar, traveling wave*

Equations of various dissipative flows &

my2/3D Ansatz and geometry *my solutions, replay from last years*

non-newtonian Navier-Stokes equation *with the same Ansatz, some part of the solutions*

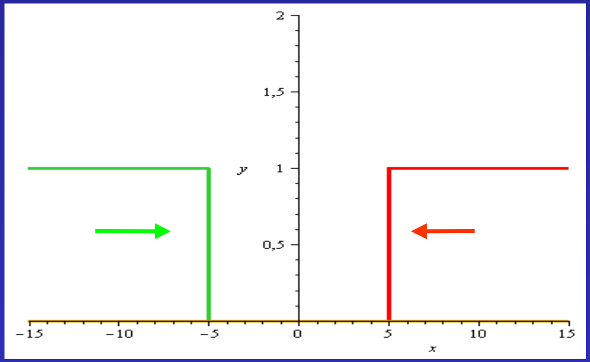
Summary & Outlook

Physically important solutions of PDEs

- *Travelling waves:*

  arbitrary wave fronts

  u(x,t) ~ g(x-ct), g(x+ct)



- *Self-similar*

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

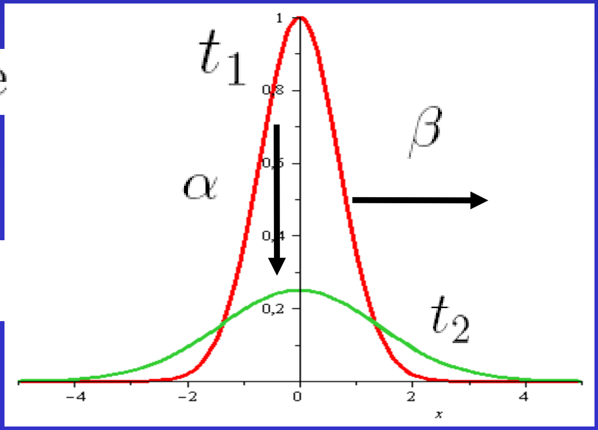
Sedov, Barenblatt, Zeldovich

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$ *in Fourier heat-conduction*



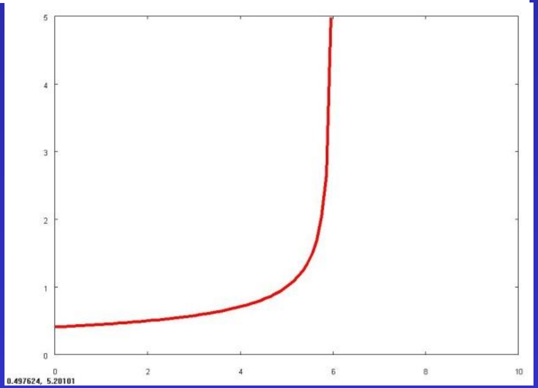
Physically important solutions of PDEs II

- generalised self-similar:

$$u(x,t) = l(t) f(x/g(t))$$

- one case: blow-up solution:
  goes to infinity in finite time
  other forms are available too

$$u(x,t) = \frac{1}{\sqrt{T-t}} f\left(\frac{x}{\sqrt{T-t}}\right)$$



- additional:
  additive separable
  multiplicative separable
  generalised separable eg.

$$u(x,t) = \varphi(t) + \phi(x)$$

$$u(x,t) = \varphi(t)\phi(x)$$

$$u(x,t) = \frac{x + C_1}{at + C_2} + \frac{2ab}{(at + C_1)^2}$$

Various dissipative fluid equations

Non-compressible and Newtonian (I) $$\nabla \mathbf{v} = 0, \quad \mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nu \triangle \mathbf{v} - \frac{\nabla p}{\rho} + a$$	Compressible and Newtonian (II) EOS $p = \kappa \rho^n$, $$\rho_t + div[\rho \mathbf{v}] = 0, \quad \rho[\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v}] = \nu_1 \triangle \mathbf{v} + \frac{\nu_2}{3} grad\, div\, \mathbf{v} - \nabla p + a$$
Non-compressible and Non-Newtonian (III) $$\nabla \mathbf{v} = 0, \quad \mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nabla[(\mu_0 + \mu_1 \lvert \underline{\underline{E}} \rvert^r) \cdot \underline{\underline{E}}] - \frac{\nabla p}{\rho} + a, \quad \underline{\underline{E}} = E_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i}\right)$$	Compressible and Non-Newtonian tooo complicated, we will see…

Euler description; Cartesian coordinate; v velocity field, p pressure, a external field ν viscosity, ρ density, κ kompressibility, E stress tensor

My 3 dimensional Ansatz

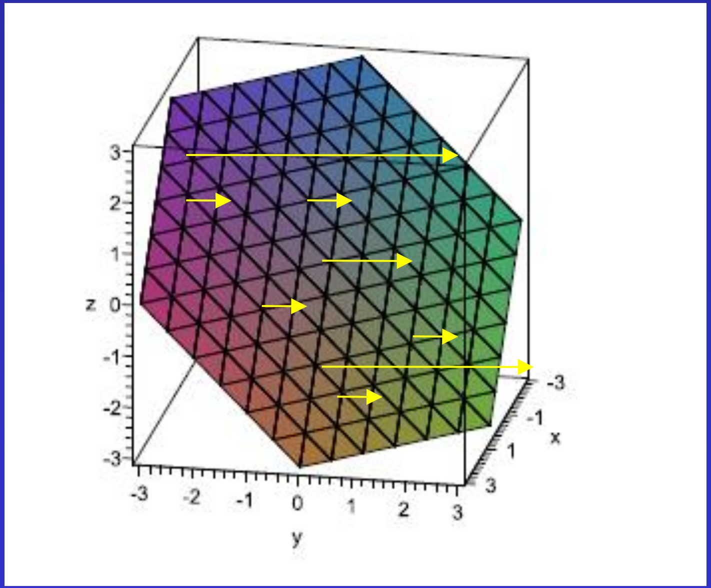
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{F(x,y,z)}{t^{\beta}}\right) := t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) := t^{-\alpha} f(\omega)$$

A more general function does not work for N-S

$$F(x,y,z) = x+y+z = 0$$

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{\sqrt{x^2+y^2+z^2} - a}{t^{\beta}}\right)$$



The graph of the $x+y+z=0$ plane.

Geometrical meaning:
all v components with
coordinate constrain x+y+z=0
lie in a plane = equivalent

The final applied forms:

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right)$$

$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad \begin{matrix} p(x,y,z,t) \\ \rho(x,y,z,t) \end{matrix} = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right)$$

The non-compressible Newtonian Navier-Stokes equation(I)

Considering the most general Cartesian case:

$\mathbf{v}(x,y,z,t) = \quad u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$

$p(x,y,z,t)$

just to write out all the coordinates:

$$u_x + v_y + w_z = 0$$

$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$

$$v_t + uv_x + vv_y + wv_z = \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}$$

$$w_t + uw_x + vw_y + ww_z = \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a.$$

as constraints the exponents: $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta = 1/2, \qquad \eta = 1$

universality relations, are all fixed no free paramater(s)

$$u(x,y,z,t) = t^{-1/2} f\left(\frac{x+y+z}{t^{1/2}}\right) = t^{-1/2} f(\omega), \quad v(x,y,z,t) = t^{-1/2} g(\omega),$$

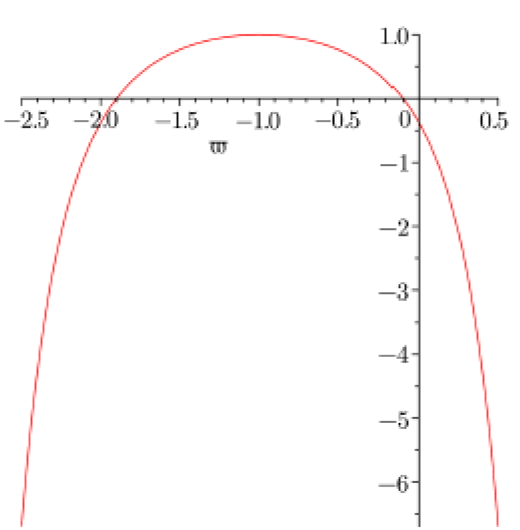
$$w(x,y,z,t) = t^{-1/2} h(\omega), \quad p(x,y,z,t) = t^{-1} l(\omega),$$

Solutions of the ODE

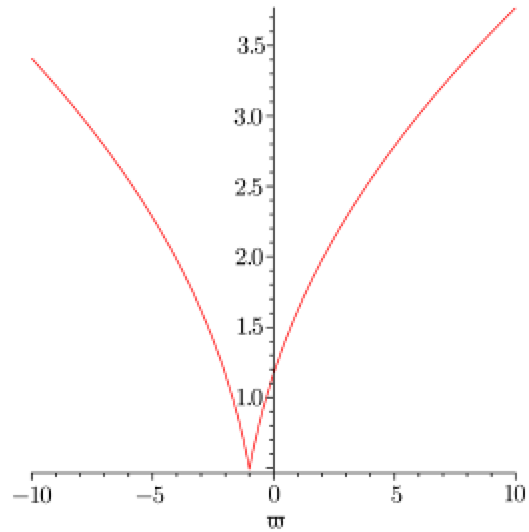
a single Eq. remains

$$9\nu f''(\omega) - 3(\omega + c) f'(\omega) + \frac{3}{2} f(\omega) - \frac{c}{2} + a = 0.$$

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}$$



The KummerM$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.



The KummerU$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

Kummer is spec. func.

$(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$$

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!},$$

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right]$$

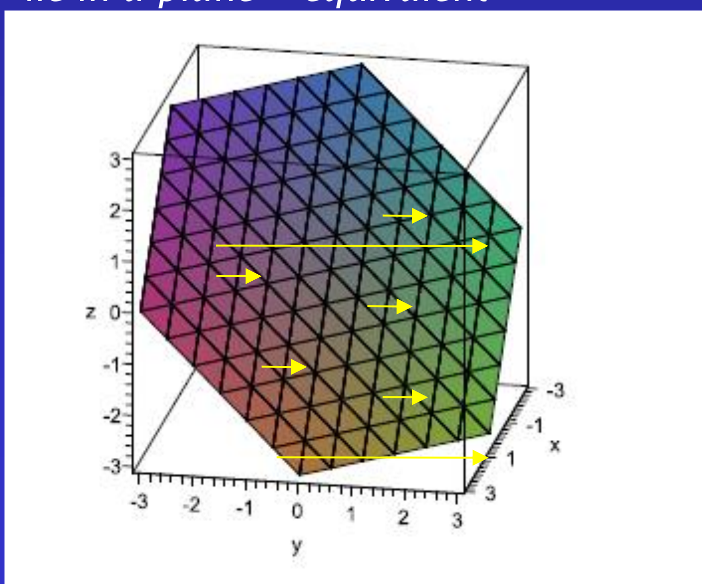
Solutions of N-S

$$u(x,y,z,t) = t^{-1/2} f(\omega) = t^{-1/2}\Big[c_1 \cdot \mathrm{KummerU}\Big(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big)\Big]$$
$$+ t^{-1/2}\Big[c_2 \cdot \mathrm{KummerM}\Big(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big) + \frac{c}{3} - \frac{2a}{3}\Big]$$

analytic only for one velocity component ☹

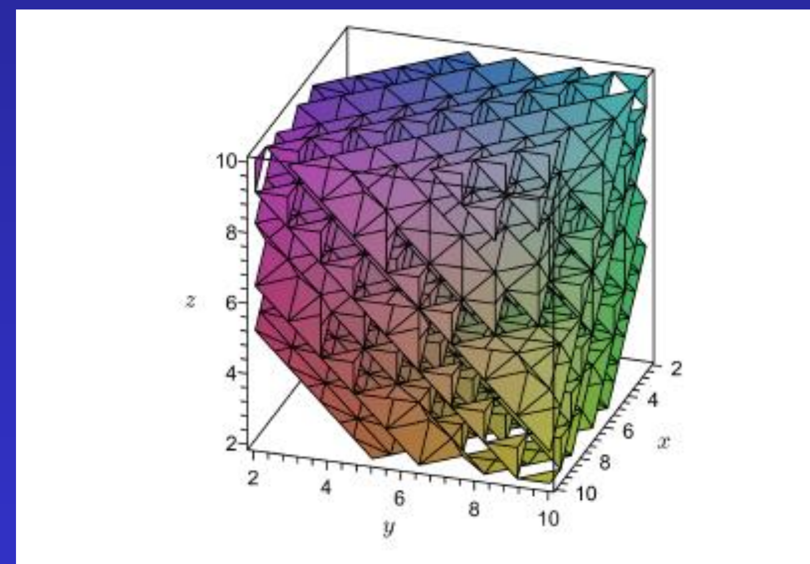
Geometrical explanation:

all v components with coordinate constrain x+y+z=0 lie in a plane = equivalent



Naver-Stokes makes a dynamics of this plane

getting a multi-valued surface



The implicit plot of the self-similar solution . Only the KummerU function is presented for $t = 1$, $c_1 = 1$, $c_2 = 0$, $a = 0$, $c = 1$, and $\nu = 0.1$.

I.F. Barna http://arxiv.org/abs/1102.5504
Commun. Theor. Phys. 56 (2011) 745-750

for fixed space it decays in time
t^-1/2 KummerT or U(1/t) ☺

Other analytic solutions

Without completeness, usually from Lie algebra studies

all are for non-compressible N-S

W. I. Fushchich, W. M. Shtelen and S. L. Slavutsky J. Phys. A: Math. Gen. 24 (1990) 971.

$\omega = z/\sqrt{t}$

Presented 19 various solutions one of them is:

$$u(z,t) = \frac{f(\omega)}{\sqrt{t}}, \quad v(y,z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \quad w(z,t) = \frac{h(\omega)}{\sqrt{t}}, \quad p(t,z) = \frac{l(\omega)}{\sqrt{t}}$$

V. Grassi, R.A. Leo, G. Soliani and P. Tempesta, Physica **286** (2000) 79

Ansatz:

$$U_1 = Y(y)T(z)\Phi(t).$$

Solutions are Kummer functions as well

JOURNAL OF MATHEMATICAL PHYSICS **50**, 083101 (2009)

Analytical solutions to the Navier–Stokes equations with density-dependent viscosity and with pressure

Ling Hei Yeung[1,a)] and Yuen Manwai[2,b)]

"Only" Radial solution for 2 or 3 D

Ansatz:

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

Nonlinear Instability of the Solutions of the Navier–Stokes Equations: Formulas for Constructing Exact Solutions

A. D. Polyanin

Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia

e-mail: polyanin@ipmnet.ru

Received May 5, 2009

Ansatz:

$$V_n = f_n(z,t)x + g_n(z,t)y, \quad n = 1, 2; \quad V_3 = F(z,t).$$

Ukrainian Mathematical Journal, Vol. 49, No. 9, 1997

ON NAVIER–STOKES FIELDS WITH LINEAR VORTICITY

G. V. Popovich and R. O. Popovich

$$\text{rot}\,\vec{u} = H(t)\vec{x} + \vec{k}(t).$$

$$\vec{u} = \nabla\varphi + \frac{1}{3}(H\vec{x})\times\vec{x} + \frac{1}{2}\vec{k}\times\vec{x},$$

Sedov, stationary N-S, only the angular part

$$v_r = \frac{\nu}{r}f(\theta), \quad v_\theta = \frac{\nu}{r}\varphi(\theta), \quad v_\lambda = \frac{\nu}{r}\psi(\theta), \quad U - \frac{p}{\rho} = \frac{\nu^2}{r^2}F(\theta).$$

The compressible Newtonian Navier-Stokes eq. (II)

$$\rho_t + div(\rho \mathbf{v}) = 0$$
$$\rho[\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v}] = \nu_1 \triangle \mathbf{v} + \frac{\nu_2}{3} grad\, div\, \mathbf{v} - \nabla p + a; \qquad EOS\ p = \kappa \rho^n$$

polytropic EOS

most general case: $\mathbf{v}(x,y,z,t) = u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ $\rho(x,y,z,t)$

as constraints the exponents are not fixed:

universality relations are depend on the EOS exponent

$$\alpha = \beta = \frac{2}{n+1} \quad \delta = \epsilon = \omega = 2 - \frac{4}{n+1}$$

There are different regimes for different ns

n > 1 all exponents are positive decaying, spreading solutions for speed and density

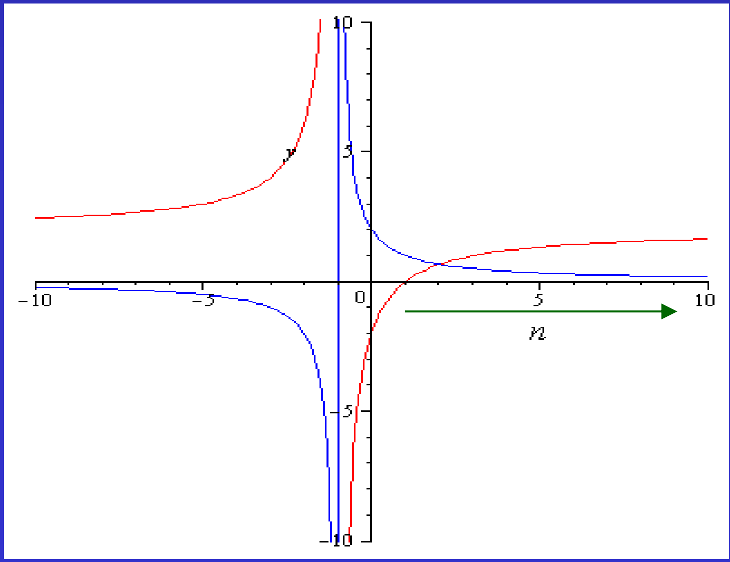
n = 1 see above

-1 ≤ n ≤ +1 decaying and spreading density & enhancing velocity in time

n ≠ -1

n ≤ -1 sharpening and enhancing density & decaying and sharpening velocity

Relevant physics is for n >1



The compressible Newtonian Navier-Stokes eq. (II)

there is an ODE for the density with analytic solutions for any n BUT more important the ODE for the velocity field:

$$-3\kappa n f^{n-1} f' + \left(\frac{2n-2}{n+1}\right)\left(\frac{2}{n+1}\right)\eta f = 0.$$

$$f(\eta) = 3^{\frac{-1}{n-1}}\left(\frac{2\eta^2[n-1]}{\kappa n[n+1]} + 3c_1\right)^{\frac{1}{n-1}}$$

$$-3\nu g'' + \left(2 - \frac{4}{n+1}\right) g f - \kappa n f^{n-1} f' = 0.$$

$$g = \frac{\tilde{c}_1}{\sqrt{\eta}} \mathrm{M}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}}\left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}}\right) + \frac{\tilde{c}_2}{\sqrt{\eta}} \mathrm{W}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}}\left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}}\right) + \frac{2}{3}\eta,$$

For n =1/2 and n= 3/2 HeunT functions tooo elaborate BUT for n=2 we have the Whitakker functions

The Whitakker funtions can be expressed via the Kummer's confluent hypergeometric functions M and U in general

$$M_{\lambda,\mu}(z) = e^{-z/2} z^{\mu+1/2} M(\mu - \lambda + 1/2, 1 + 2\mu; z)$$
$$W_{\lambda,\mu}(z) = e^{-z/2} z^{\mu+1/2} U(\mu - \lambda + 1/2, 1 + 2\mu; z)$$

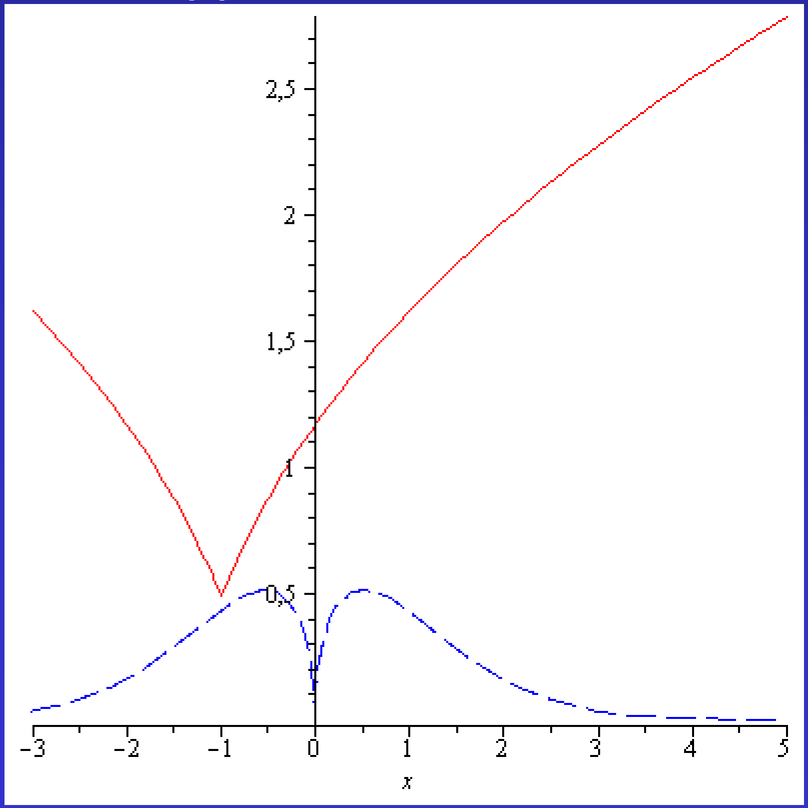
The solution for the non-compressible case

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}$$

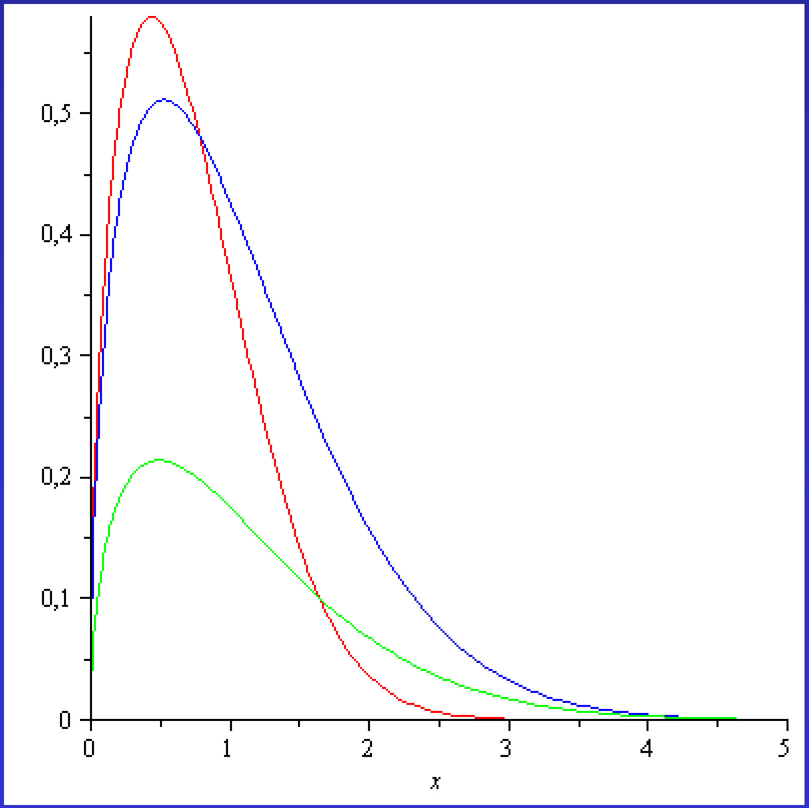
There is no kappa → 0 limit to compare the results

The compressible Newtonian Navier-Stokes eq. (II)

Solutions for the same parameters red is the non/compressible and blue is the compresible solution nu = 0.1, kappa =1, c1 = 1



the compresible solutions for kappa = 0.1,1,2 values red, blue green



I.F. Barna http://arxiv.org/abs/1309.0703 & under publication

The non-compressible non-newtonian Navier-Stokes eq. for 2D (III)

$$\nabla \mathbf{v} = 0,$$

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nabla[(\mu_0 + \mu_1 |\underline{\underline{\mathbf{E}}}|^{\mathrm{r}}) \cdot \underline{\underline{\mathbf{E}}}] - \frac{\nabla \mathbf{p}}{\rho} + \mathbf{a}$$

$$\underline{\underline{E}} = E_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i}\right)$$

Abs value of the tensor in 2D, in 3D much longer

$$|\underline{\underline{E}}| = [u_x^2 + v_y^2 + 1/2(u_y + v_x)^2]^{1/2}.$$

Ladyzhenskaya model for non-Newtonian fluids

Newtonian	for $\mu_0 > 0,\ \mu_1 = 0,$
Rabinowitsch	for $\mu_0, \mu_1 > 0,$ and $r = 2,$
Ellis	for $\mu_0, \mu_1 > 0,$ and $r > 0,$
Ostwald-de Waele	for $\mu_0 = 0,\ \mu_1 > 0,$ and $r > -1,$
Bingham	for $\mu_0, \mu_1 > 0,$ and $r = -1.$

$\mu_0 = 0$, if $r < 0$ then it is a pseudo-plastic fluid

if $r > 0$ dilattant fuid

paper pulp
$\mu_0 = 0, \mu_1 = 0.418, r = -0.425$

There are existence theorems for weak solutions for some rs
Our Ansatz:

$$u = t^{-\alpha} f(\eta), \qquad v = t^{-\delta} g(\eta), \qquad p = t^{-\epsilon} h(\eta), \qquad \eta = \frac{x+y}{t^{\beta}}.$$

The non-compressible non-newtonian Navier-Stokes eq. for 2D (III)

Thre most general case:
$\mu_0 = 0, \mu_1 \neq 0, \alpha = \delta = (r-1)/2, \beta = (3-r)/2, \epsilon = 2$

$$\mu_1(1+2r)f''[2f'^2]^r + \left(\frac{3-r}{2}\right)\eta f' + \left(\frac{r-1}{2}\right) f = 0$$

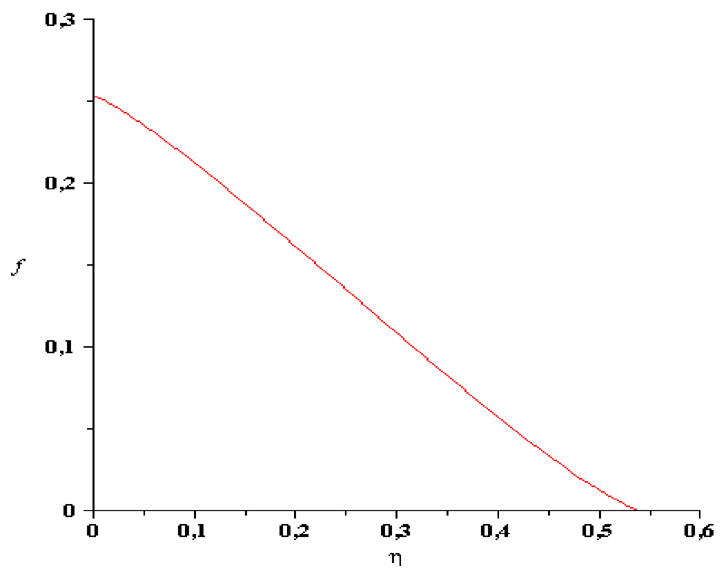
under heavy investigation

for $r = 2$ can be integrated

$$4\mu_1 f'^5 + f\eta/2 + c_1 = 0$$

For C1 =0, mu_1 = 0.1

$$f(\eta) = \frac{1}{243}\left(27\_C1 - 27\,40^{1/5}\,\eta^{6/5}\right)^{5/4}$$



Large number of solutions (20), some of them are explicit with compact support

The non-compressible non-newtonian Navier-Stokes eq. for 2D (III)

$$\mu_1(1+2r)f''[2f'^2]^r + \left(\frac{3-r}{2}\right)\eta f' + \left(\frac{r-1}{2}\right) f = 0$$

for r = 0 we get back the non-newtonian ODE with the Kummer M/W solutions but for 2D

$$f(x) = \_C1\ \mathrm{KummerM}\left(-\frac{1}{6}, \frac{1}{2}, \frac{3}{4}\ \frac{(x+c)^2}{\mu}\right) + \_C2\ \mathrm{KummerU}\left(-\frac{1}{6}, \frac{1}{2}, \frac{3}{4}\ \frac{(x+c)^2}{\mu}\right)$$

An important message:
the non-Newtonian contribution gives „more localised" velocity distributions then tha Newtonian

Summary & Outlook

- The self-similar Ansatz is presented as a tool for non-linear PDA
- The non-compressible N-S eq. & compressible N-S eq. & non-Newtonian non-compressible N-S are investigated
- The non-compressible N-S is the „simplest case”
- The other two systems have more „localized” or even compact support solutions which are important
- Clear out the dark points, include heat conduction

hank you for

our attention!

Questions, Remarks, Comments?...