Exact solution to equations generalizing the Witten-Veneziano relation⋆

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Introduction and motivation

$U_A(1)$ symmetry is broken by the nonabelian ("gluon") axial anomaly: even in the chiral limit (ChLim, where $m_q \to 0$),

$$\partial_\alpha \bar{\psi}(x)\gamma^\alpha\gamma_5 \frac{\lambda^0}{2}\psi(x) \propto F^a(x)\cdot\widetilde{F}^a(x) \equiv \epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x) \neq 0\,.$$

This breaks the $U_A(1)$ symmetry of QCD and precludes the 9th Goldstone pseudoscalar meson $\Rightarrow$ very massive η': even in ChLim, where $M_\pi, M_K, M_\eta \to 0$, still ('ChLim WVR')

$$0 \neq \Delta M^2_{\eta_0} = \Delta M^2_{\eta'} = \frac{(A = \text{qty.dim.mass})^4}{(``f_{\eta'}")^2} = \frac{6\,\chi_{\text{YM}}}{f_\pi^2} + O(\frac{1}{N_c})$$

$$\text{Out of ChLim}: \quad {M_{\eta'}}^2 + {M_\eta}^2 - 2\,{M_K}^2 = \frac{2N_f}{f_\pi^2}\,\chi_{\text{YM}} \quad \left(+O(\frac{1}{N_c})\right)$$

Anomalous part of η_0 mass: $\Delta M_{\eta_0}^2 = \chi_{\rm YM} \frac{2N_f}{f_\pi^2} + O(\frac{1}{N_c})$

QCD chiral behavior (reproduced by DS approach) of the non-anomalous parts of masses of light $q\bar{q}'$ pseudoscalars (i.e., all parts except ΔM_{η_0}) : $M_{q\bar{q}'}^2 = \mathrm{const}\,(m_q + m_{q'}), \;\; (q, q' = u, d, s)$.

$\Rightarrow$ non-anomalous parts of the masses in WVR cancel:
$M_{\eta'}{}^2 + M_\eta{}^2 - 2\,M_K{}^2 \approx \Delta M_{\eta_0}{}^2\,,$ approx. as in ChLim WVR

$$\chi = \int d^4x \, \langle 0 | Q(x) Q(0) | 0 \rangle \; , \qquad Q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

- $Q(x)$ = topological charge density operator
- In WV rel., χ is the pure-glue, YM one, $\chi_{\rm YM} \leftrightarrow \chi_{quench}$.

Lattice: good $\chi_{\rm YM}$, subtleties with χ of light-flavor QCD [Bernard et al., JHEP 1206 (2012) 051] where $\chi = - \frac{\langle \bar{q} q \rangle_0}{\sum_{q=u,d,s} \frac{1}{m_q}} + \mathcal{O}(\text{higher } m)$.

Good DS results; e.g., "non-anomalous":

- E.g., separable model parameters reproducing experimental data:
- $\widetilde{m}_{u,d} = 5.5$ MeV, $\Lambda_0 = 758$ MeV, $\Lambda_1 = 961$ MeV, $p_0 = 600$ MeV, $D_0\Lambda_0^2 = 219$, $D_1\Lambda_1^4 = 40$ – fixed by fitting calculated quantities M_π, f_π, M_ρ, $g_{\rho\pi^+\pi^-}$, $g_{\rho e^+e^-}$ $\Rightarrow$ pertinent predictions $a_{u,d} = 0.672$, $b_{u,d} = 660$ MeV, i.e., $m_{u,d}(p^2)$, $\langle q\bar{q}\rangle_0$
- $\widetilde{m}_s = 115$ MeV – fixed by fitting M_K $\Rightarrow$ predictions $a_s = 0.657$, $b_s = 998$ MeV, i.e., $m_s(p^2)$, $M_{s\bar{s}}$, f_K, $f_{s\bar{s}}$
- Summary of results (all in GeV) for $q = u, d, s$ and pseudoscalar mesons without the influence of gluon anomaly:

PS	M_{PS}	M_{PS}^{exp}	f_{PS}	f_{PS}^{exp}	$m_q(0)$	$-\langle q\bar{q}\rangle_0^{1/3}$
π	0.140	0.1396	0.092	0.0924 ± 0.0003	0.398	0.217
K	0.495	0.4937	0.110	0.1130 ± 0.0010		
$s\bar{s}$	0.685		0.119		0.672	

Anomaly and mixing in η-η' complex

- $q\bar{q}'$ bound-state approach yields mass^2 eigenvalues $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_K, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$
- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles, although in the isospin limit ($m_u = m_d$, adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi$. I is a good quantum number!
- $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

$$I = 1 \qquad |\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)\ ,$$

$$I = 0 \qquad |\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle)\ ,$$

$$I = 0 \qquad |\eta_0\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)\ .$$

NA-part of the $I_3 = 0$ mass matrix in the π^0-η_8-η_0 basis

- the “non-anomalous” (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η’s is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M^2_\pi & 0 & 0 \\ 0 & M^2_{88} & M^2_{80} \\ 0 & M^2_{08} & M^2_{00} \end{pmatrix}$$

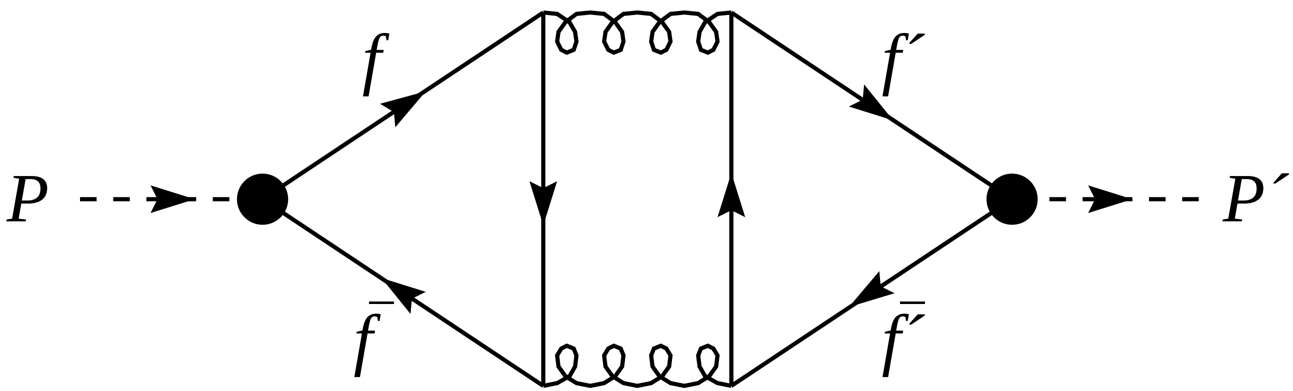
$$M^2_{88} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle = \frac{2}{3}(M^2_{s\bar{s}} + \frac{1}{2} M^2_\pi),$$

$$M^2_{80} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M^2_{08} = \frac{\sqrt{2}}{3}(M^2_\pi - M^2_{s\bar{s}})$$

$$M^2_{00} \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2} M^2_{s\bar{s}} + M^2_\pi),$$

- Not enough! In order to avoid the $\mathrm{U}_A(1)$ problem, one must break the $\mathrm{U}_A(1)$ symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.

Gluon anomaly is not accessible to ladder approximation!



- **Diamond graph**: just the simplest example of a transition $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ $(q, q' = u, d, s[...])$. Such transitions add off-diagonal terms to $\hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$, i.e., contribute to the anomalous masses in $\hat{M}^2_A$ of the η-η' complex. But, such transitions are not included in the interaction kernel in the ladder approximation.

Anomalous part of the mass matrix in the π^0-η_8-η_0 basis

- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\to$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_{A}$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \quad \text{does not vanish in the chiral limit.}$$

$3\beta = \Delta M^2_{\eta_0}$ = the anomalous mass^2 of η_0 [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.

SU(3)-flavor breaking in the A-part of the mass matrix

- we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$. $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.
- Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- $\to$ In the isospin limit, one can always restrict to 2×2 submatrix of etas $(I=0)$, as π^0 $(I=1)$ is decoupled then.

Anomaly and mixing in η-η' complex

- nonstrange (*NS*) – strange (*S*) basis

$$
\begin{aligned}
|\eta_{NS}\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \;, \\
|\eta_S\rangle &= |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \;.
\end{aligned}
$$

- the η–η' matrix in this basis is

$$
\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S \eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \stackrel{\phi}{\rightarrow} \begin{pmatrix} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{pmatrix}
$$

- *NS*–*S* mixing relations

$$
|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \;, \qquad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \;.
$$

$$
\theta = \phi - \arctan\sqrt{2}
$$

The empirical η-η' mass matrix

- Let lowercase m_M's denote the empirical mass of meson M. From our calculated, model mass matrix in *NS*–*S* basis, we form its empirical counterpart $\hat{m}^2_{\rm exp}$ by
- *i)* obvious substitutions $M_{u\bar{u}} \equiv M_\pi \to m_\pi$, $M_{s\bar{s}} \to m_{s\bar{s}}$
- *ii)* by noting that $m_{s\bar{s}}$, the "empirical" mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m^2_{s\bar{s}} \approx 2m^2_K - m^2_\pi$. Then,

$$\hat{m}^2_{\rm exp} = \left[\begin{array}{cc} m^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m^2_K - m^2_\pi + \beta X^2 \end{array} \right] \xrightarrow[\phi_{\rm exp}]{} \left[\begin{array}{cc} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{array} \right].$$

Finally, fix anomalous contribution to η-η':

- the trace of the empirical $\hat{m}^2_{\text{exp}}$ demands the 1^{st} equality in

$$\beta(2+X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2}\chi_{\text{YM}} \quad (2^{\text{nd}}\text{equality} = \text{WV relation})$$

- requiring that the experimental trace $(m_\eta^2 + m_{\eta'}^2)_{exp} \approx 1.22$ GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\text{fit}} = \frac{1}{2+X^2}[(m_\eta^2 + m_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)]$
- Or, get β from lattice χ_{YM} ! Then no free parameters!
- then, can calculate the NS-S mixing angle ϕ

$$\tan 2\phi = \frac{2\,M^2_{\eta_S\eta_{NS}}}{M^2_{\eta_S} - M^2_{\eta_{NS}}} = \frac{2\sqrt{2}\beta X}{M^2_{\eta_S} - M^2_{\eta_{NS}}} \qquad \text{and}$$

$$M^2_{\eta_{NS}} = M^2_{u\bar{u}} + 2\beta = M^2_\pi + 2\beta, \quad M^2_{\eta_S} = M^2_{s\bar{s}} + \beta X^2 = M^2_{s\bar{s}} + \beta\frac{f_\pi^2}{f_{s\bar{s}}^2}$$

Anomaly and mixing in η-η' complex

- The diagonalization of the NS-S mass matrix then finally gives us the *calculated* η and η' masses:

$$\begin{aligned} M_\eta^2 &= \cos^2\phi\, M_{\eta_{NS}}^2 - \sqrt{2}\beta X \sin 2\phi + \sin^2\phi\, M_{\eta_S}^2 \\ M_{\eta'}^2 &= \sin^2\phi\, M_{\eta_{NS}}^2 + \sqrt{2}\beta X \sin 2\phi + \cos^2\phi\, M_{\eta_S}^2 \end{aligned}$$

- Equivalently, from the secular determinant,

$$\begin{aligned} M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4M_{\eta_S\eta_{NS}}^4} \right] \\ &= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \\ M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4M_{\eta_S\eta_{NS}}^4} \right] \\ &= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \end{aligned}$$

E.g., separable model results on η and η' mesons (at $T=0$)

	$\beta_{\rm fit}$	$\beta_{\rm latt.}$	Exp.
θ	-12.22°	-13.92°	
M_η	548.9	543.1	547.75
$M_{\eta'}$	958.5	932.5	957.78
X	0.772	0.772	
3β	0.845	0.781	

- masses are in units of MeV, 3β in units of GeV^2 and the mixing angles are dimensionless.
- $\beta_{\rm latt.}$ was obtained from $\chi_{\rm YM}(T=0) = (175.7\ {\rm MeV})^4$
- $X = f_\pi/f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.

For three DS models: summary of $T = 0$ results from WV

from Ref.	J-M&WV	A^2 &WV	separab&WV	orig. Shore	Experiment
M_π	137.3	135.0	140.0		$(138.0)^{isospin}_{average}$
M_K	495.7	494.9	495.0		$(495.7)^{isospin}_{average}$
$M_{s\bar{s}}$	700.7	722.1	684.8		
f_π	93.1	92.9	92.0		92.4 ± 0.3
f_K	113.4	111.5	110.1		113.0 ± 1.0
$f_{s\bar{s}}$	135.0	132.9	119.1		
M_η	568.2	577.1	542.3		547.75 ± 0.12
$M_{\eta'}$	920.4	932.0	932.6		957.78 ± 0.14
ϕ	41.42°	39.56°	40.75°	38.24°	
θ	-13.32°	-15.18°	-13.98°	-16.5°	
θ_0	-2.86°	-5.12°	-6.80°	-12.3°	
θ_8	-22.59°	-24.14°	-20.58°	-20.1°	
f_0	108.8	107.9	101.8	106.6	
f_8	122.6	121.1	110.7	104.8	
f_η^0	5.4	9.6	12.1	22.8	
$f_{\eta'}^0$	108.7	107.5	101.1	104.2	
f_η^8	113.2	110.5	103.7	98.4	
$f_{\eta'}^8$	-47.1	-49.5	-38.9	-36.1	

Shore's generalization of WV valid to all orders in $1/N_c$

- WV rel. – lowest order in $1/N_c$ – improved like this:

$$(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_{\eta})^2 M^2_{\eta} = \frac{1}{3}\left(f^2_\pi M^2_\pi + 2 f^2_K M^2_K\right) + 6A \qquad (1)$$

$$f^0_{\eta'} f^8_{\eta'} M^2_{\eta'} + f^0_{\eta} f^8_{\eta} M^2_{\eta} = \frac{2\sqrt{2}}{3}\left(f^2_\pi M^2_\pi - f^2_K M^2_K\right) \qquad (2)$$

$$(f^8_{\eta'})^2 M^2_{\eta'} + (f^8_{\eta})^2 M^2_{\eta} = -\frac{1}{3}\left(f^2_\pi M^2_\pi - 4 f^2_K M^2_K\right) \qquad (3)$$

A is the full QCD topological charge,

$$A = \frac{\chi}{1 + \chi\left(\frac{1}{\langle \bar{u}u\rangle\, m_u} + \frac{1}{\langle \bar{d}d\rangle\, m_d} + \frac{1}{\langle \bar{s}s\rangle\, m_s}\right)} \qquad (4)$$

= hard to calculate on lattice ...

However, it is known that $A = \chi_{\mathrm{YM}} + \mathcal{O}(\frac{1}{N_c})$

Reduction to the standard WV relation (= large N_c result)

Replacement 3 different condensates $\to \langle\bar{q}q\rangle_0$ reduces the full QCD topological charge A (4) to the combination $\widetilde{\chi}$ on the RHS of Leutwyler-Smilga relation (lowest $\mathcal{O}(\frac{1}{N_c})$):

$$\chi_{\rm YM} = \frac{\chi}{1 + \frac{\chi}{\langle\bar{q}q\rangle_0} \sum\limits_{q=u,d,s} \frac{1}{m_q}} \longrightarrow \widetilde{\chi}(T,\mu) \approx \frac{\langle\bar{q}q(T,\mu)\rangle_0}{\sum\limits_{q=u,d,s} \frac{1}{m_q}} + corr's$$

Previously, we only **conjectured** $\chi_{\rm YM}(T) \to \widetilde{\chi}(T)$ [Benić et al, Phys. Rev. D84 (2011) 016006] , to explain increased η' multiplicity at RHIC noted by Csörgő et al.

Also note (1)+(3) $\Rightarrow$

$$(f_{\eta'}^0)^2 M_{\eta'}^2 + (f_\eta^0)^2 M_\eta^2 + (f_\eta^8)^2 M_\eta^2 + (f_{\eta'}^8)^2 M_{\eta'}^2 - 2 f_K^2 M_K^2 = 6A$$

- Then, large N_c limit and 'off-diagonal' $f_\eta^0, f_{\eta'}^8 \to 0$, as well as $f_{\eta'}^0, f_\eta^8, f_K \to f_\pi$, recovers the standard WV.

η' and η have 4 independent decay constants

$$f^0_{\eta'}, f^8_{\eta}, f^0_{\eta}, f^8_{\eta'} : \quad \langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \;\; a = 8, 0; \;\; P = \eta, \eta' .$$

- Equivalently, one has 4 related but different constants $f^{NS}_{\eta'}, f^{NS}_{\eta}, f^{S}_{\eta}, f^{S}_{\eta'}$ if instead of octet and singlet axial currents ($a = 8, 0$) one takes this matrix element of the nonstrange-strange axial currents ($a = NS, S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left(\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right) ,$$

$$A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}} \bar{s}(x)\gamma^\mu\gamma_5 s(x) ,$$

$$\begin{bmatrix} f^{NS}_{\eta} & f^{S}_{\eta} \\ f^{NS}_{\eta'} & f^{S}_{\eta'} \end{bmatrix} = \begin{bmatrix} f^{8}_{\eta} & f^{0}_{\eta} \\ f^{8}_{\eta'} & f^{0}_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} ,$$

$$a, P = NS, S : \qquad \langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{NS}(x)|\eta_{S}(p)\rangle = 0 ,$$

$$a, P = NS, S : \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0 ,$$

- Note: in a DS approach, $f_{NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$, $f_S = f_{s\bar{s}}$ are calculated quantities

Two Mixing Angles and FKS one-angle scheme

- Any 4 η-η' decay constants conveniently parametrized in terms of two decay constants and two angles:

$$f_\eta^8 = \cos\theta_8\, f_8\,, \qquad f_\eta^0 = -\sin\theta_0\, f_0\,, \qquad f_\eta^{NS} = \cos\phi_{NS}\, f_{NS}\,, \qquad f_\eta^S = -\sin\phi_S\, f_S\,,$$

$$f_{\eta'}^8 = \sin\theta_8\, f_8\,, \qquad f_{\eta'}^0 = \cos\theta_0\, f_0\,, \qquad f_{\eta'}^{NS} = \sin\phi_{NS} f_{NS}\,, \qquad f_{\eta'}^S = \cos\phi_S f_S$$

- Big practical difference between 0-8 and NS-S schemes:

- while θ_8 and θ_0 differ a lot from each other and from $\theta \approx (\theta_8 + \theta_0)/2$, FKS showed that $\phi_{NS} \approx \phi_S \approx \phi$.

$$\begin{bmatrix} f_\eta^{NS} & f_\eta^S \\ f_{\eta'}^{NS} & f_{\eta'}^S \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} .$$

For four decay constants, can use FKS one-angle scheme!

- ϕ relates $\{f_\eta^8, f_{\eta'}^8, f_\eta^0, f_{\eta'}^0\}$ with $\{f_{NS}, f_S\} = \{f_\pi, f_{s\bar{s}}\}$:

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

- Some other useful relations between quantities of NS-S (FKS) and 0-8 schemes:

$$f_8 = \sqrt{\frac{1}{3} f_{NS}^2 + \frac{2}{3} f_S^2}\,, \qquad \theta_8 = \phi - \arctan\left(\frac{\sqrt{2} f_S}{f_{NS}}\right)\,,$$

$$f_0 = \sqrt{\frac{2}{3} f_{NS}^2 + \frac{1}{3} f_S^2}\,, \qquad \theta_0 = \phi - \arctan\left(\frac{\sqrt{2} f_{NS}}{f_S}\right)\,.$$

Solve numerically Shore's Eqs. (1)-(3) for $M_{\eta'}$, M_η, and ϕ:

Inputs:	$M_\pi, M_K, f_\pi = f_{\rm NS}, f_{s\bar{s}} = f_{\rm S}$ and f_K, calculated in 3 different DS models					
$\chi_{\rm YM}$	191^4	175.7^4	191^4	175.7^4	191^4	175.7^4
M_η	499.8	485.7	496.7	482.8	526.2	507.0
$M_{\eta'}$	931.4	815.8	934.9	818.4	983.2	868.7
ϕ	52.01°	46.11°	51.85°	46.07°	47.23°	40.86°
θ	-2.72°	-8.62°	-2.89°	-8.67°	-7.51°	-13.87°
θ_0	7.74°	1.84°	7.17°	1.39°	-0.33°	-6.69°
θ_8	-12.00°	-17.90°	-11.85°	-17.6°	-14.12°	-20.47°
f_0	108.8	108.8	107.9	107.9	101.8	101.8
f_8	122.6	122.6	121.1	121.1	110.7	110.7
f_η^0	-14.7	-3.5	-13.5	-2.6	0.6	11.9
$f_{\eta'}^0$	107.9	108.8	107.1	107.9	101.8	101.1
f_η^8	119.9	116.7	118.5	115.4	107.4	103.7
$f_{\eta'}^8$	-25.5	-37.7	-2.49	-37.6	-27.0	-38.7

(in D. Horvatić et al., Eur. Phys. J. A **38** (2008) 257.) $M_{\eta,\eta'}$ and f's in MeV, $\chi_{\rm YM}$ is in MeV^4.

The same is now reproduced analytically:

- Eqs. (1)-(3) $\Rightarrow$ four closed-form solutions for M_η, $M_{\eta'}$ and $\tan\phi$ in terms of $f_\pi, f_{s\bar{s}}, M_\pi$, M_K and A.
  The set reproducing the previous numerical results is:

$$\tan\phi = \frac{-2Af_\pi^2 + 4Af_{s\bar{s}}^2 - 2f_K^2 f_\pi^2 M_K^2 + f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 + \Delta}{4\sqrt{2}\,A\,f_\pi\,f_{s\bar{s}}}$$

$$M_{\eta,\eta'}^2 = \frac{2Af_\pi^2 + 4Af_{s\bar{s}}^2 + 2f_K^2 f_\pi^2 M_K^2 - f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 \mp \Delta}{2f_\pi^2 f_{s\bar{s}}^2}$$

where $\quad \Delta^2 =$

$$\frac{64A^2 f_\pi^2 f_{s\bar{s}}^2 + 2\left[-2A(f_\pi^2 - 2f_{s\bar{s}}^2) + f_\pi^2(-2f_K^2 M_K^2 + (f_\pi^2 + f_{s\bar{s}}^2)M_\pi^2)\right]^2}{2}$$

Find matrix elem's in NS-S basis from these $M_\eta, M_{\eta'}, \phi$:

$$M^2_{\eta_{NS}} \equiv M^2_{\text{NS}} = \cos^2\phi\, M^2_\eta + \sin^2\phi\, M^2_{\eta'}$$

$$M^2_{\eta_S} \equiv M^2_{\text{S}} = \sin^2\phi\, M^2_\eta + \cos^2\phi\, M^2_{\eta'}$$

$$M^2_{\eta_{NS}\eta_S} \equiv M^2_{\text{NSS}} = \sin\phi\,\cos\phi\,(M^2_\eta - M^2_{\eta'})$$

to use $M^2_{\eta,\eta'} = \frac{1}{2}\left[M^2_{\text{NS}} + M^2_{\text{S}} \mp \sqrt{(M^2_{\text{NS}} - M^2_{\text{S}})^2 + 4M^4_{\text{NSS}}}\right]$

Mathematica leads to surprisingly simple results:

$$M^2_{\text{NS}} = M^2_\pi + \frac{4A}{f^2_\pi}, \qquad M^2_{\text{NSS}} = \frac{2\sqrt{2}A}{f_\pi f_{s\bar{s}}}$$

$$M^2_{\text{S}} = \frac{1}{f^2_{s\bar{s}}}\left[2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi\right] + \frac{2A}{f^2_{s\bar{s}}} = M^2_{s\bar{s}} + \frac{2A}{f^2_{s\bar{s}}}$$

since in the isospin limit, $f^2_\pi\, M^2_\pi = -(m_u\langle u\bar{u}\rangle + m_d\langle d\bar{d}\rangle)$ and

$f^2_K\, M^2_K = -(m_u\langle u\bar{u}\rangle + m_s\langle s\bar{s}\rangle) \Rightarrow 2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi = f^2_{s\bar{s}}\, M^2_{s\bar{s}}$

Compare M_{NS}, M_{NSS} and M_{S} with NS-S mass matrix:

$$\begin{bmatrix} M^2_{\text{NS}} & M^2_{\text{NSS}} \\ M^2_{\text{NSS}} & M^2_{\text{S}} \end{bmatrix} = \begin{bmatrix} M^2_{\pi} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{bmatrix}$$

$\Rightarrow$ Very similar formulas in WV case and "Shore case":

$$1.) \quad \beta_{\text{WV}} = \frac{6\chi_{YM}}{f^2_\pi (2 + X^2)}\,, \qquad \beta_{\text{Shore}} = \frac{2A}{f^2_\pi} \approx \frac{2\chi_{YM}}{f^2_\pi}$$

Explains why Shore’s scheme needs higher values of χ_{YM} than WV, to approach empirical masses.

$2.) \quad X = \dfrac{f_\pi}{f_{\text{ss}}}$ the SAME in the both WV and Shore cases ...

... but in the "Shore case", it follows from equations! Before, incl. WV, it was an input – estimate, educated guess.

Summary

- The results of the approach through Witten-Veneziano relation and Shore's approach were shown to be similar numerically.
- The results for Shore's approach (with FKS 1-angle scheme) are also available as analytic, closed-form expressions, and they explain both the similarities and diffrences in the results on the η-η' complex.
- The full QCD topological charge A (to which χ_{YM} appears only as a numerical approximation at $T = 0 = \mu$) is not a pure-gauge quantity, but a full QCD quantity, and Leutwyler-Smilga quantity $\widetilde{\chi}$ is its approximation $\langle u\bar{u}\rangle, \langle d\bar{d}\rangle, \langle s\bar{s}\rangle \to \langle q\bar{q}\rangle_0$.
  This fact gives support to our explanation of the data on η' enhanced multiplicity in RHIC experiments at $T > 0$, where we replace the T-dependence of $\chi_{\rm YM}$ by that of the chiral condensate as in Leutwyler-Smilga quantity $\widetilde{\chi}(T)$.
  It also motivates additionally our work on extending the same approach to $\mu > 0$ for NICA and GSI and ...
- $\Rightarrow$ Increased motivation for lattice to calculate A and χ of full QCD