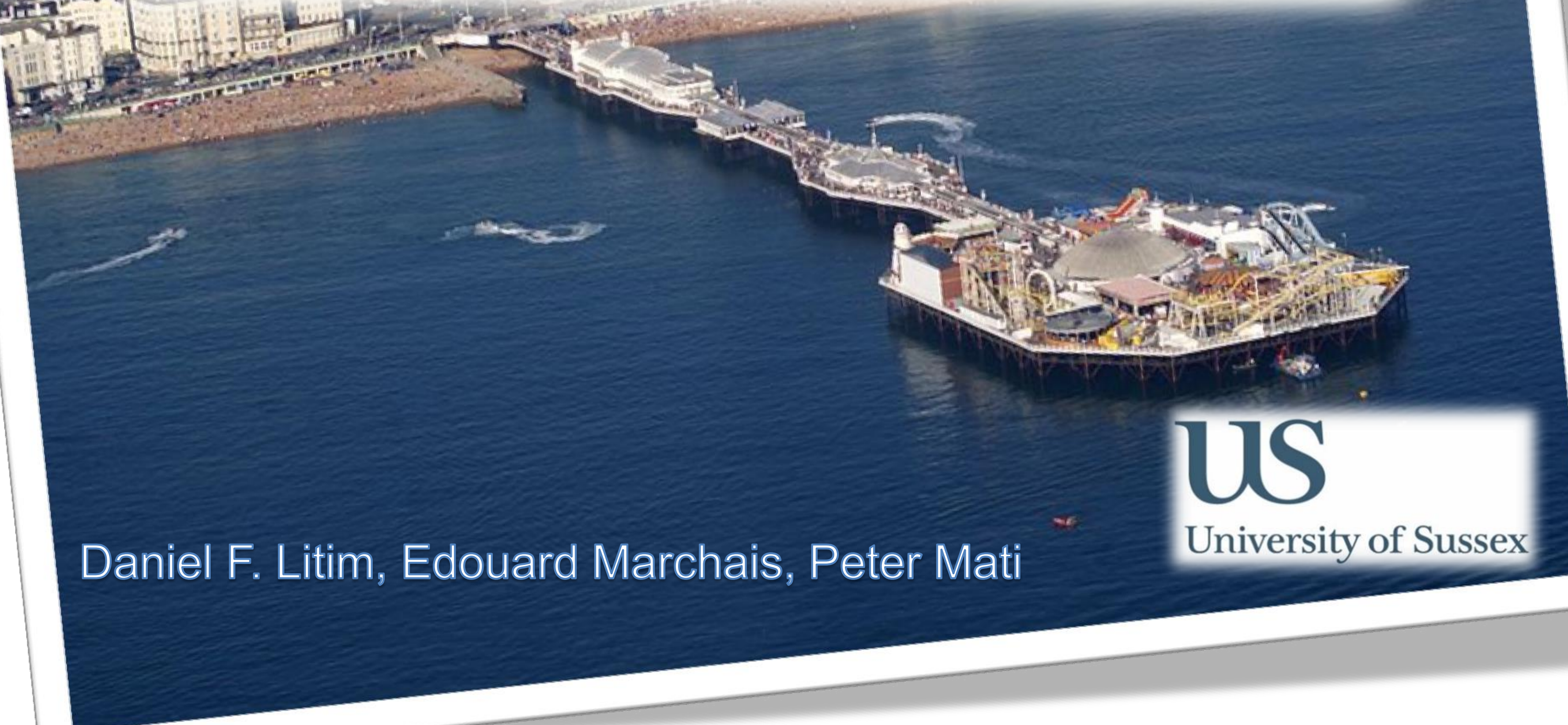
The fixed point structure of the 3d O(N) model in the large N limit



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Contents

Introduction

Euclidean QFT *n*-point function

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \dots \varphi(x_n) \, \mathrm{e}^{-S[\varphi]}$$

… can be produced from the generating functional

$$Z[J] \equiv \mathrm{e}^{W[J]} = \int \mathcal{D}\varphi \, \mathrm{e}^{-S[\varphi] + \int J\varphi} \longrightarrow \langle \varphi(x_1) \dots \varphi(x_n) \rangle = \frac{1}{Z[0]} \left(\frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right)_{J=0}$$

The effective action (by Legendre trf.)

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right)$$

$$\phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J$$

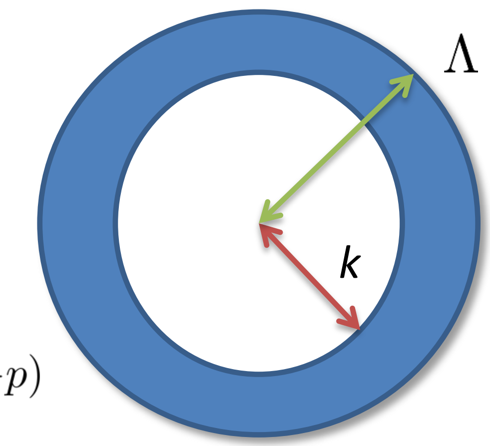
$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x)$$

Q-EOM:
Describes the dynamics of the VEV + quantum fluc. included

Introduction

Wilsonian idea: instead PT, we integrate out *momentum shell by momentum shell*

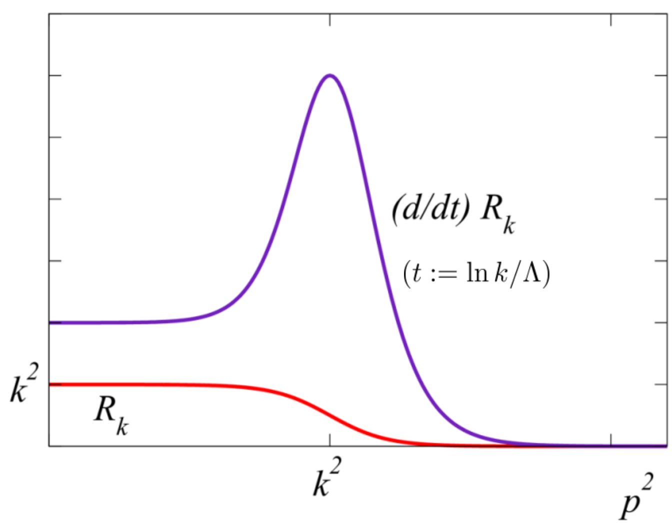
Effective average action: $\Gamma_{k\to\Lambda} \simeq S_{\text{bare}}, \quad \Gamma_{k\to 0} = \Gamma$



How?

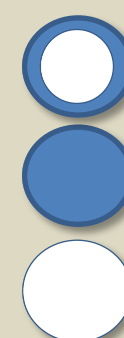
$$Z_k[J] := \int \mathcal{D}\varphi \; e^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi}$$

Momentum-dependent mass term $\Delta S_k[\varphi] := \frac{1}{2}\int_p \varphi(p) R_k(p) \varphi(-p)$



R_k: regulator function

- $\lim_{p^2/k^2\to 0} R_k(p) > 0$ IR regulator
- $\lim_{k^2/p^2\to 0} R_k(p) = 0$ original theory
- $\lim_{k^2\to\Lambda\to\infty} R_k(p) = \infty$ classical theory



Introduction

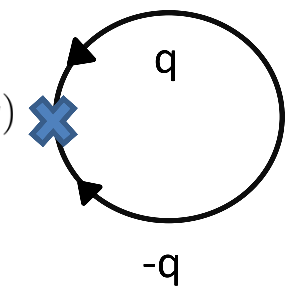
We apply the same routine for our new theory

- The "average" effective action $\quad \Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi]$
- The VEV $\quad \phi(x) = \langle \varphi(x) \rangle_J = \dfrac{\delta W_k[J]}{\delta J(x)}$
- Q-EOM $\quad \dfrac{\delta \Gamma[\phi]}{\delta \phi(x)} + (R_k \phi)(x) = J(x)$

The scale dependence of the avarage effective action: the flow eq.

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \left(\frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(q)\delta\phi(-q)} + R_k(q) \right)^{-1} \partial_t R_k(q)$$

$$\partial_t \Gamma_k = \frac{1}{2} \partial_t R_k(q)$$



$\left(\dfrac{\delta^2 \Gamma_k[\phi]}{\delta\phi(q)\delta\phi(-q)} + R_k(q) \right)^{-1} \equiv G_k \quad$ "full propagator"

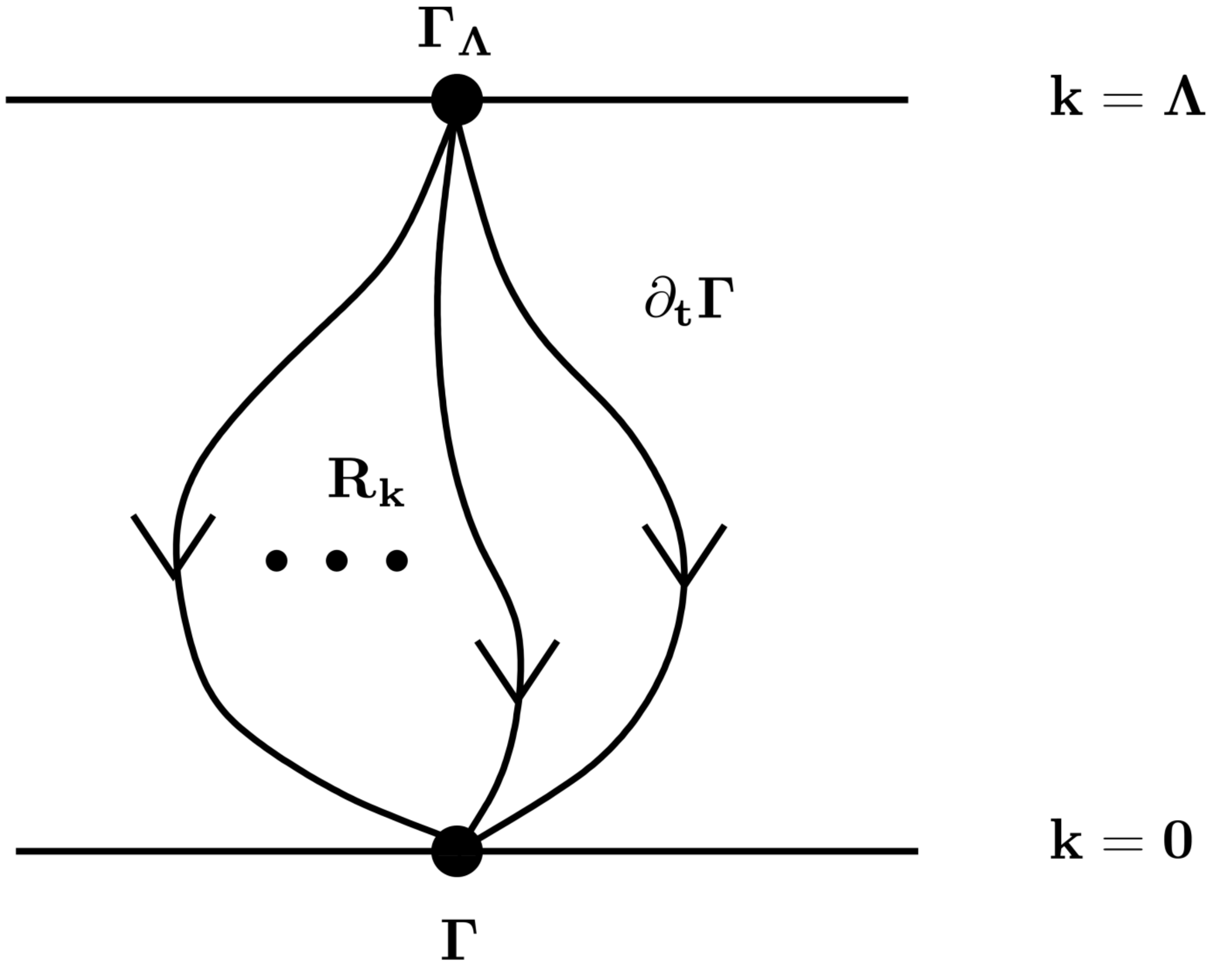
- No func. int.
- IR & UV regulator
- We can choose regulator

- One-loop structure
- PT expansion can be recovered

$(t := \ln k/\Lambda)$

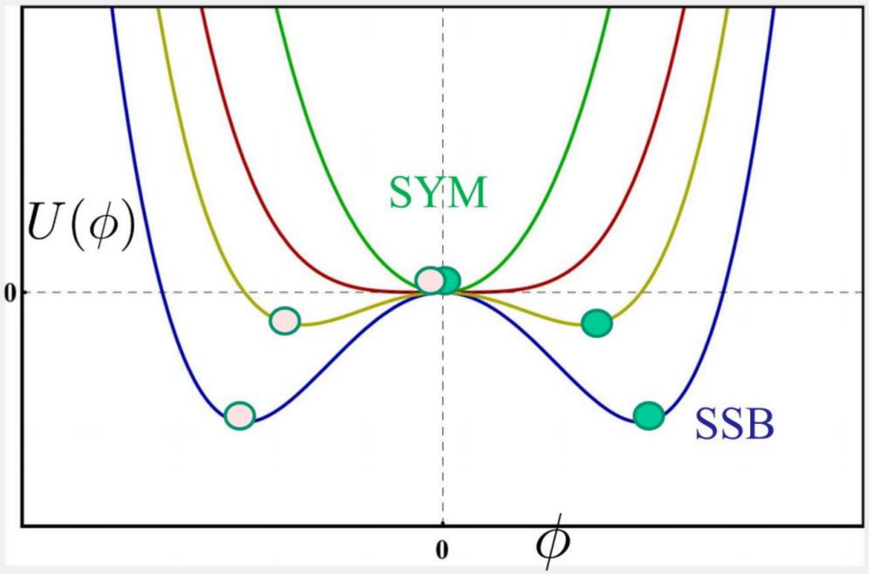
Introduction

FRG



Introduction

Second order (continous)

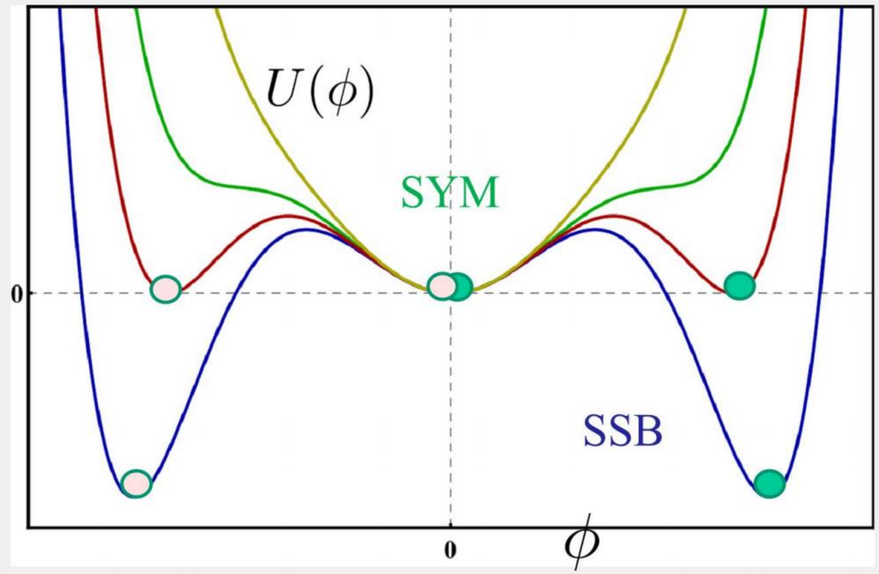


$$U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}(\phi^2)^2$$

Field VEV $\to 0$ continously

Universality

First order (discontinous)



$$U(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}(\phi^2)^2 + \frac{\tau}{6}(\phi^2)^3$$

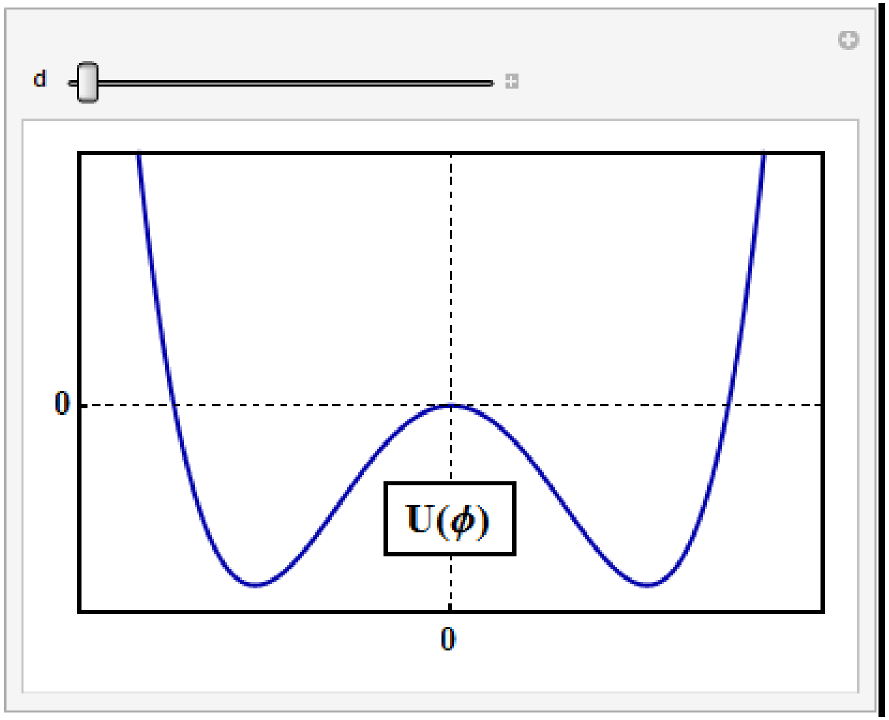
Field VEV $\to 0$ discontinously

Non-universality

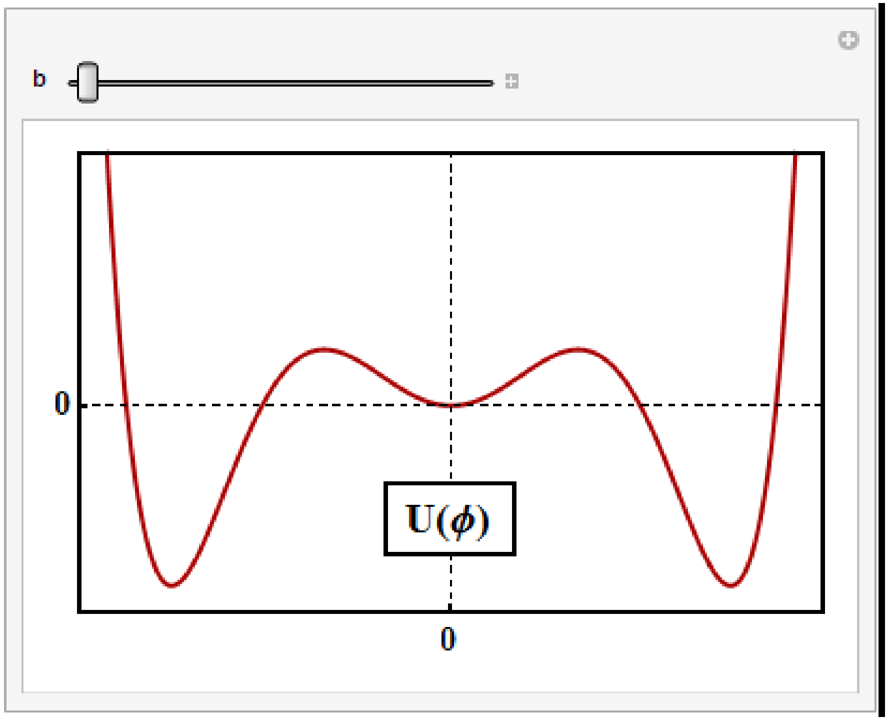
Introduction

PHASE TRANSITIONS

Second order phase transition



First order phase transition



Introduction

To solve the RG flow: an ansatz for the effective action is needed —"Slowly varying fields"→ Derivative Expansion/ Local Potential Approximation (LPA)

The *O(N)* symm. effective action $\Gamma_k = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + U_k(\phi^a\phi_a)\right]$

$Z \equiv 1$
$\bar{\rho} := \phi_a\phi^a/2$
$D = 3$

Plugging into the flow

Flow of effective potential $\partial_t U_k = \frac{1}{2}(2\pi)^{-3}\int_q \partial_t R_k \left(\frac{N-1}{M_0} + \frac{1}{M_1}\right)$

$M_0 := q^2 + R_k + U_k'$
$M_1 := q^2 + R_k + U_k' + 2\rho U_k''$
$(.)' := \frac{\delta}{\delta\rho}$

- Using the optimized regulator: $R_k = (k^2 - q^2)\theta(k^2 - q^2)$ the loop integral is analytic
- Taking the large *N*-limit (the universality class of the ideal Bose gas)

The flow for the dimensionless effective potential

$$\partial_t u' = -2u' + \rho u'' - \frac{u''}{(1+u')^2}$$

Dimensionless quantities

$u' \equiv U'/k^2$
$u'' \equiv U''/k$
$\rho \equiv \bar{\rho}/k$

LOCAL FLOW

Solving the flow eq.

Expanding the potential in terms of polynomial couplings

$$u = \sum_{n=1}^{n_{trunc}} \frac{a_n}{n!}(\rho - \rho_0)^n$$

$$u'(\rho_0) = 0 \quad \lambda \equiv a_2 \quad \tau \equiv a_3$$

The β–functions:

		Running	Fixed Points
(decoupled) $\partial_t \rho_0 = 1 - \rho_0$	⇒	$\rho(t) = 1 + c_\rho e^{-t}$	$\rho_{0*} = \rho_{0kr} = 1$
$\partial_t \lambda = -\lambda(1 - 2\lambda)$	⇒	$\lambda(t) = \dfrac{1}{2 + c_\lambda e^t}$	$\lambda_* = 0, \; \lambda_{**} = \frac{1}{2}$
(exactly marginal) $\partial_t \tau = -6\lambda(\lambda^2 - \tau)$	⇒	$\tau(t) = \dfrac{2 + c_\tau e^{3t}}{(2 + c_\lambda e^t)^3}$	$\tau_* = \tau, \; \tau_{**} = \frac{1}{4}$

$(\partial_t a_n = f(\lambda, \tau), n > 3)$

Tricritical (UV attractive)

Wilson-Fisher (IR attractive)

Constants from initial value $\quad c_\rho = \rho_{0,\Lambda} - 1, \;\; c_\lambda = 1/\lambda_\Lambda - 2, \;\; c_\tau = \tau_\Lambda/\lambda_\Lambda^3 - 2$

Solving the flow eq.

The flow equation can be solved analytically in the large N *(by method of characteristics)*

$$\partial_t u' = -2u' + \rho u'' - \frac{u''}{(1+u')^2}$$

$$\frac{\rho - 1}{\sqrt{u'}} - \frac{1}{2}\frac{\sqrt{u'}}{1+u'} - \frac{3}{2}\arctan\sqrt{u'} = G(u' e^{2t})$$

$u' \geq 0$

analytic continuation *

$u' \leq 0$

$$\frac{\rho - 1}{\sqrt{-u'}} + \frac{1}{2}\frac{\sqrt{-u'}}{1+u'} - \frac{3}{4}\ln\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}} = \bar{G}(u' e^{2t})$$

* $$\frac{1}{i}\arctan ix = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Solving the flow eq.

$$\partial_t u' = -2u' + \rho u'' - \frac{u''}{(1+u')^2}$$

$$\frac{\rho-1}{\sqrt{u'}} - \frac{1}{2}\frac{\sqrt{u'}}{1+u'} - \frac{3}{2}\arctan\sqrt{u'} = G(u'e^{2t}) \qquad u' \geq 0$$

$$\frac{\rho-1}{\sqrt{-u'}} + \frac{1}{2}\frac{\sqrt{-u'}}{1+u'} - \frac{3}{4}\ln\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}} = \bar{G}(u'e^{2t}) \qquad u' \leq 0$$

Fixed points are described by scaling solutions

$$G(u'e^{2t}) \to c$$

$$\bar{G}(u'e^{2t}) \to \bar{c}$$

$\partial_t u' \to 0$

$$\rho(u') = 1 + c\sqrt{u'} + H(u') \qquad u' \geq 0$$

$$\rho(u') = 1 + \bar{c}\sqrt{-u'} + \bar{H}(u') \qquad u' \leq 0$$

...it turns out: $c = \bar{c}$

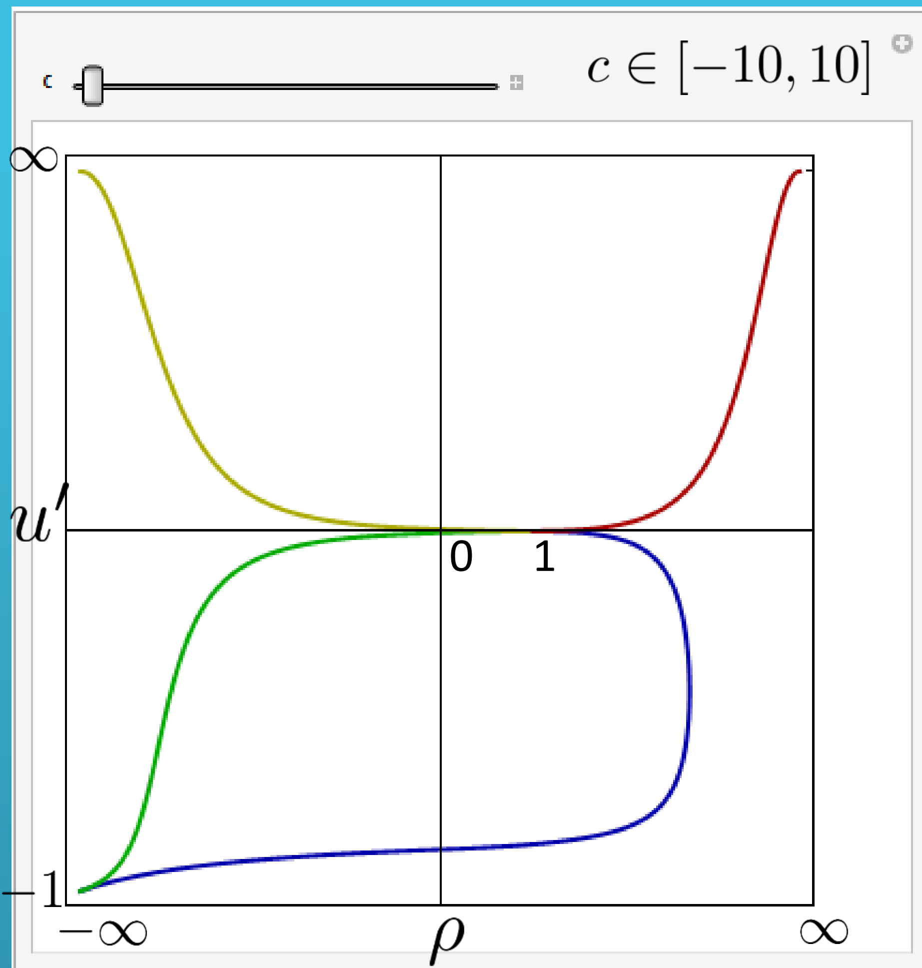
$$H(u') := \frac{1}{2}\frac{u'}{1+u'} + \frac{3}{2}\sqrt{u'}\arctan\sqrt{u'}$$

$$\bar{H}(u') := \frac{1}{2}\frac{u'}{1+u'} + \frac{3}{4}\sqrt{-u'}\ln\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}$$

Solving the flow eq.

GLOBAL FLOW

Four branches describe the solution epending on the sign of c and u'



Issues

1. Branch continuation

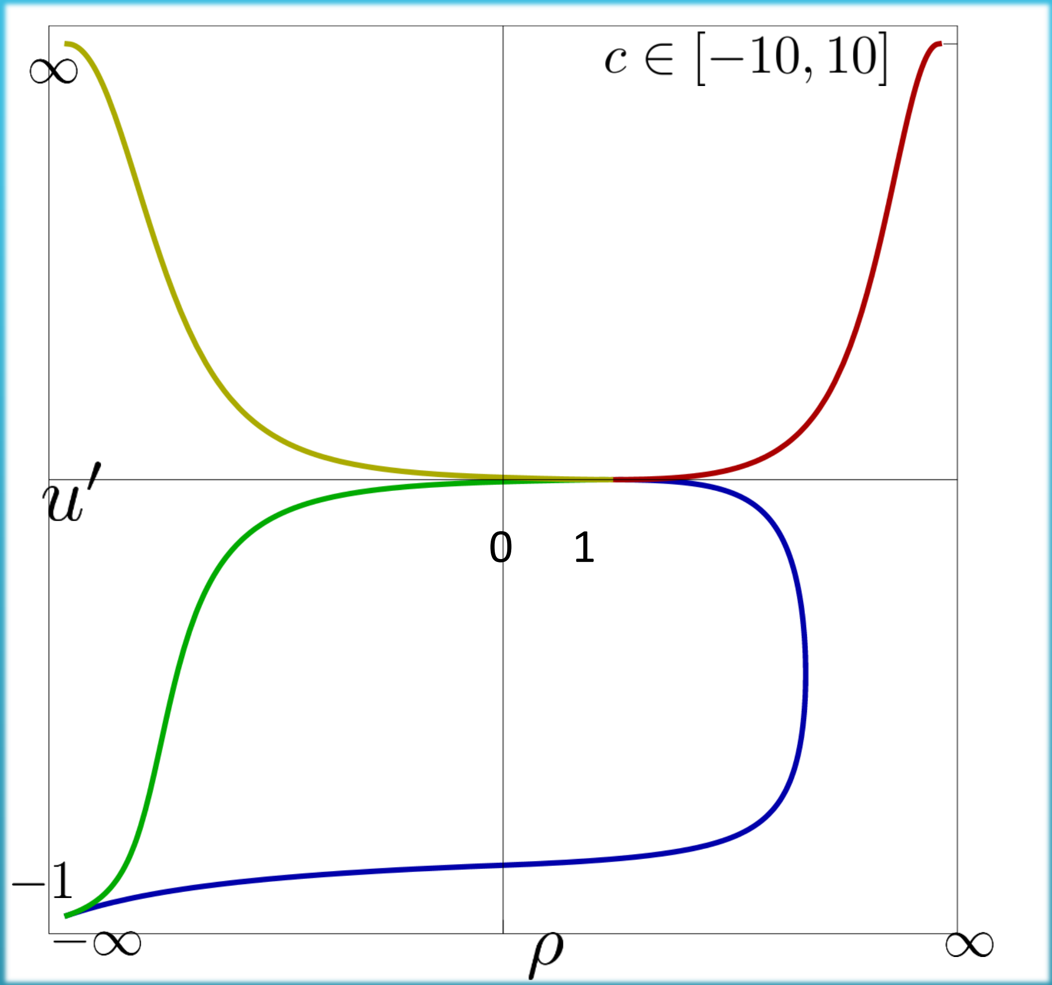
2. Turning points

Solving the flow eq.

1. Branch continuation

$$u'_+(1) = u'_-(1) = 0,$$
$$u''_+(1) = u''_-(1) = 0,$$
$$u'''_+(1) = \frac{2}{c^2} \neq u'''_-(1) = -\frac{2}{c^2}$$

Continuation from positive to negative branch is not smooth enough

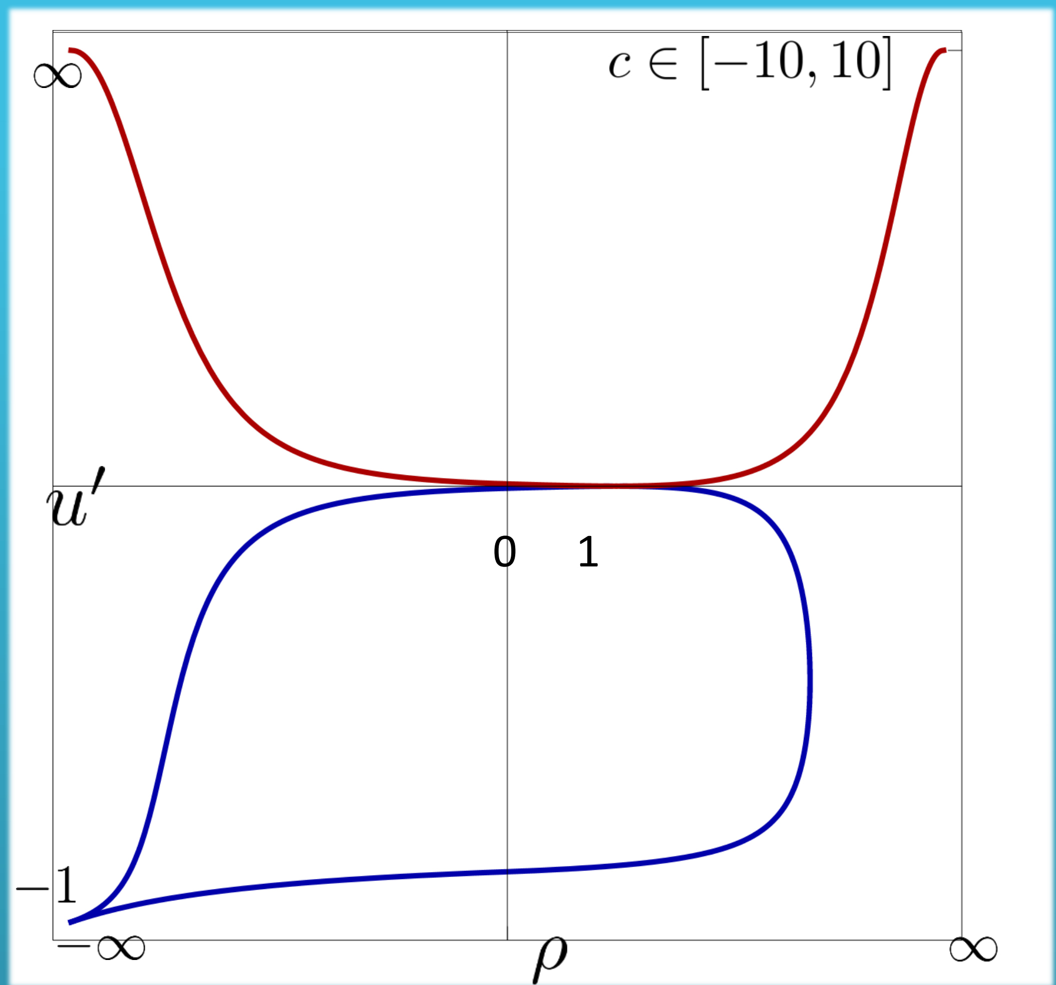


Solving the flow eq.

1. Branch continuation

$$
\begin{aligned}
u'_+(1) &= u'_-(1) = 0, \\
u''_+(1) &= u''_-(1) = 0, \\
u'''_+(1) = \frac{2}{c^2} &\neq u'''_-(1) = -\frac{2}{c^2}
\end{aligned}
$$

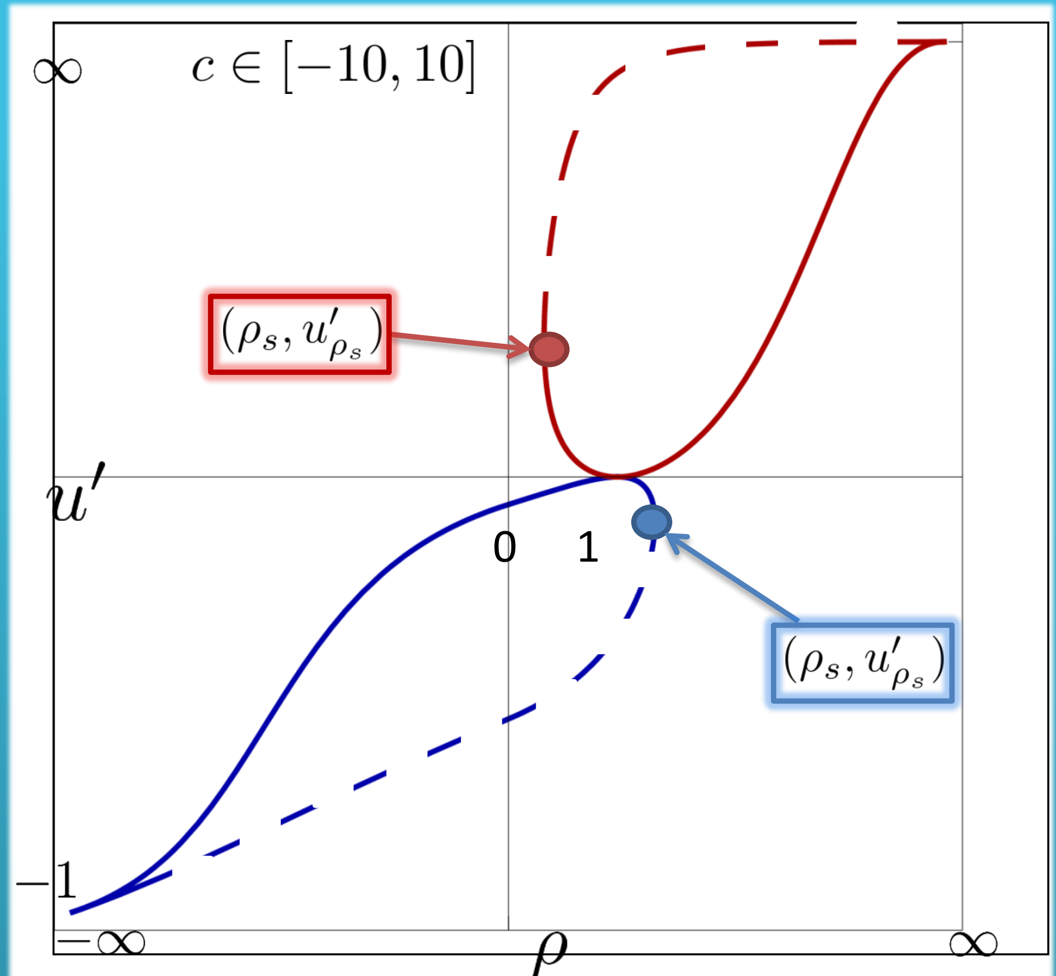
Continuation from positive to negative branch is not smooth enough



Solving the flow eq.

2. Turning points

In this regime we have to choose either the dashed or the full line: *"strong coupling"*



The fixed point structure

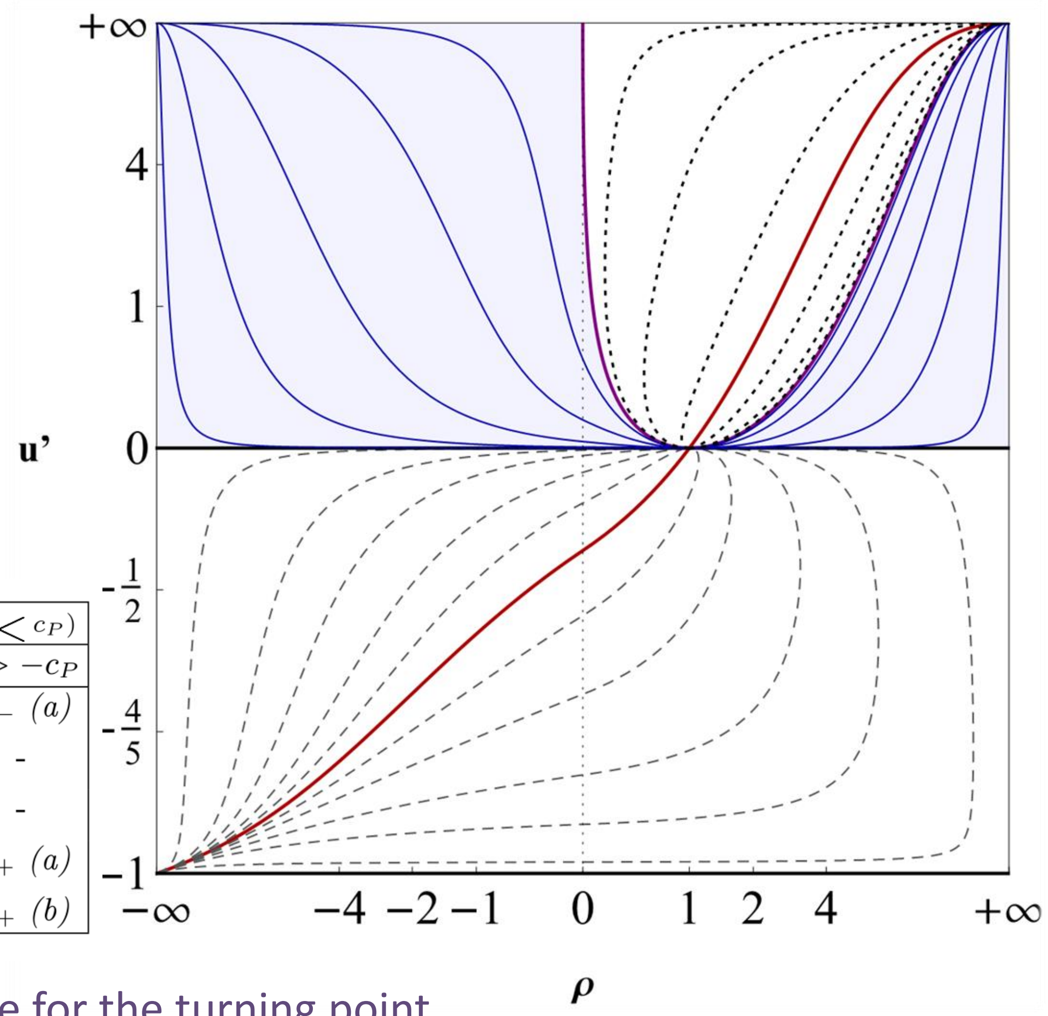
The proper definition of the function:

We consider only $u' \geq 0$

$$B_+(u') \equiv 1 + c\sqrt{u'} + H(u')$$

$$B_-(u') \equiv 1 - c\sqrt{u'} + H(u')$$

	Weak ($\lvert c\rvert > c_P$)		Critical ($\lvert c\rvert = c_P$)		Strong ($\lvert c\rvert < c_P$)	
Domain	$c > c_P$	$c < -c_P$	$c = c_P$	$c = -c_P$	$c < c_P$	$c > -c_P$
$\rho \geq 1$	B_+	B_-	B_+	B_-	B_+ (a)	B_- (a)
$\rho \leq 1$	B_-	B_+	-	-	-	-
$0 < \rho \leq 1$	-	-	B_-	B_+	-	-
$\rho_s \geq \rho \leq 1$	-	-	-	-	B_- (a)	B_+ (a)
$\rho \geq \rho_s$	-	-	-	-	B_- (b)	B_+ (b)



$c_P = 3\pi/4$ The treshold value for the turning point

The fixed point structure

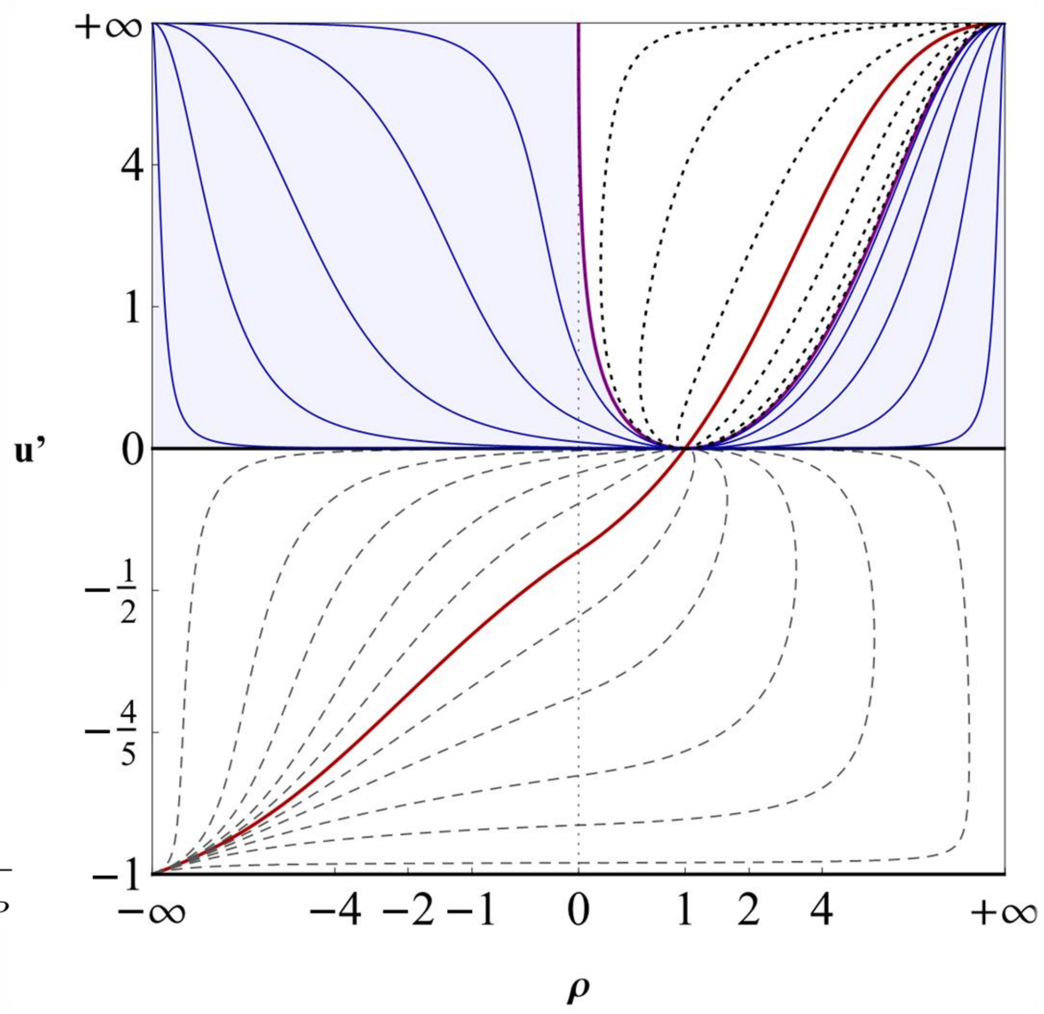
Using polynomial expansion we can identify the couplings as

$$u = \sum_{n=1}^{n_{trunc}} \frac{a_n}{n!}(\rho - \rho_0)^n$$

$$\lambda \equiv u''(\rho_0) \qquad \tau \equiv u'''(\rho_0)$$

Thus if we tune the VEV to its critical value we can distinguish different type of fixed point solutions

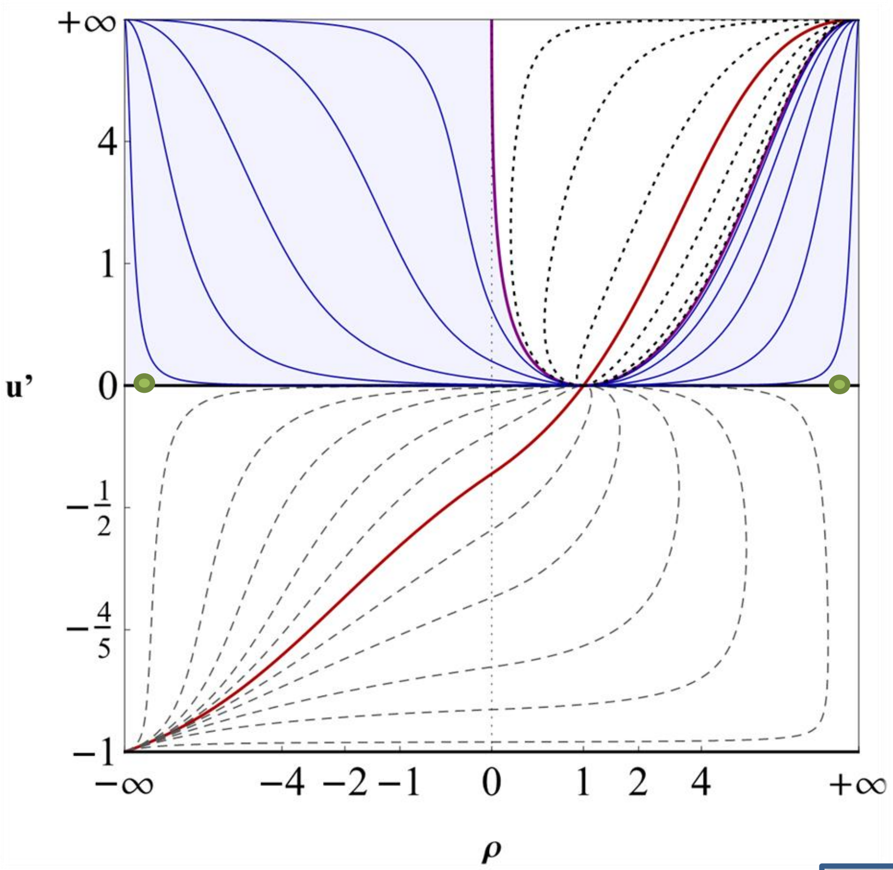
Gauss	Wilson-Fisher	Tricritical	BMB
$\rho_0 * = 1$	$\rho_0 * = 1$	$\rho_0 * = 1$	$\rho_0 * = 1$
$\lambda_* = 0$	$\lambda_* = \frac{1}{2}$	$\lambda_* = 0$	$\lambda_* = 0$
$\tau_* = 0$	$\tau_* = \frac{1}{4}$	$\tau_* = \frac{2}{c^2}$	$\tau_* = \frac{2}{c_P^2}$



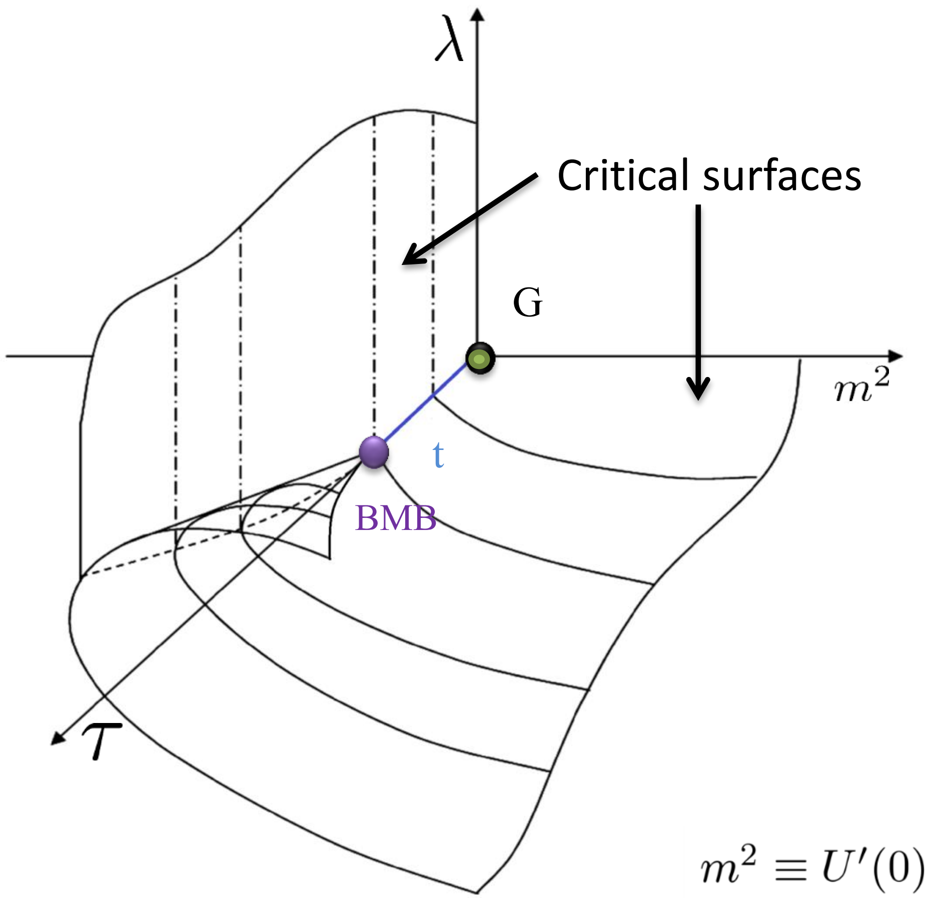
The fixed point structure

PARAMETERS

Fixed point solutions



Topology of the parameter space

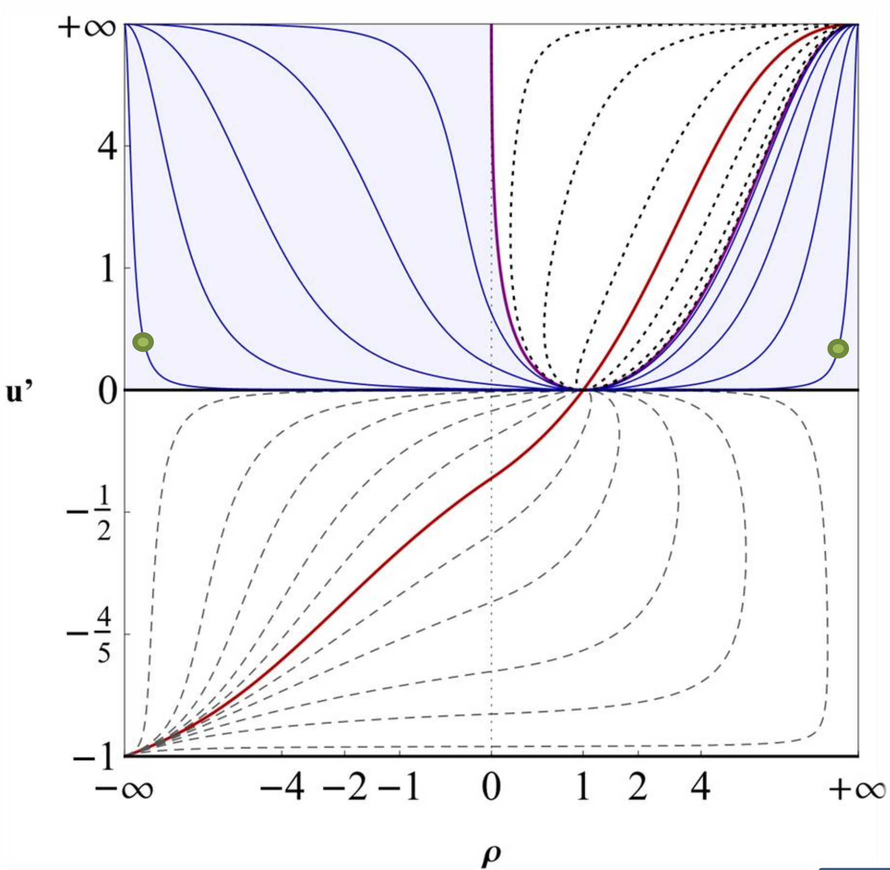


$$\tau = \frac{2}{c^2}$$

The fixed point structure

PARAMETERS

Fixed point solutions



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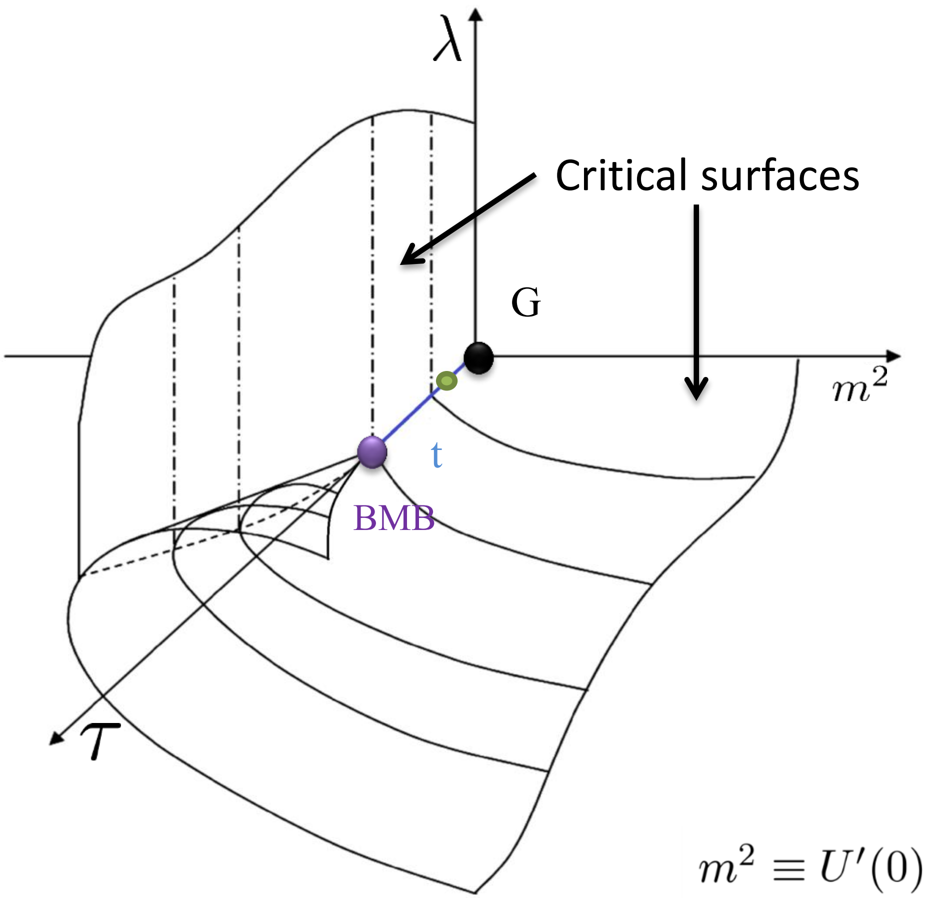
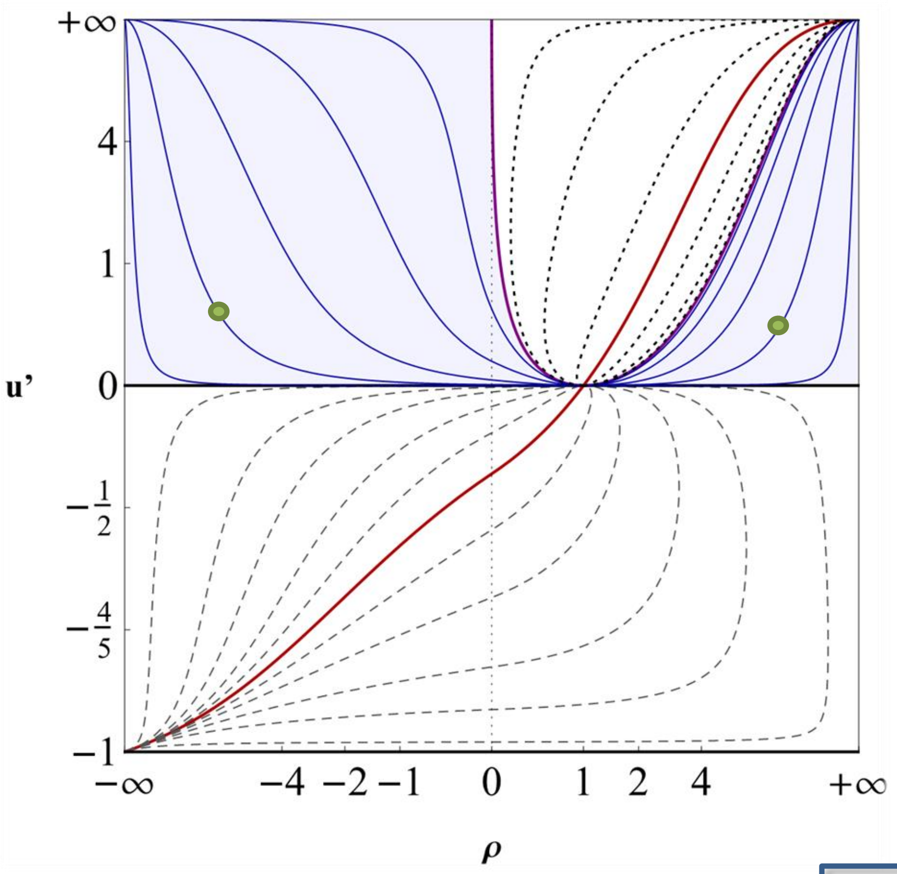


$$\tau = \frac{2}{c^2}$$

The fixed point structure

PARAMETERS

Fixed point solutions



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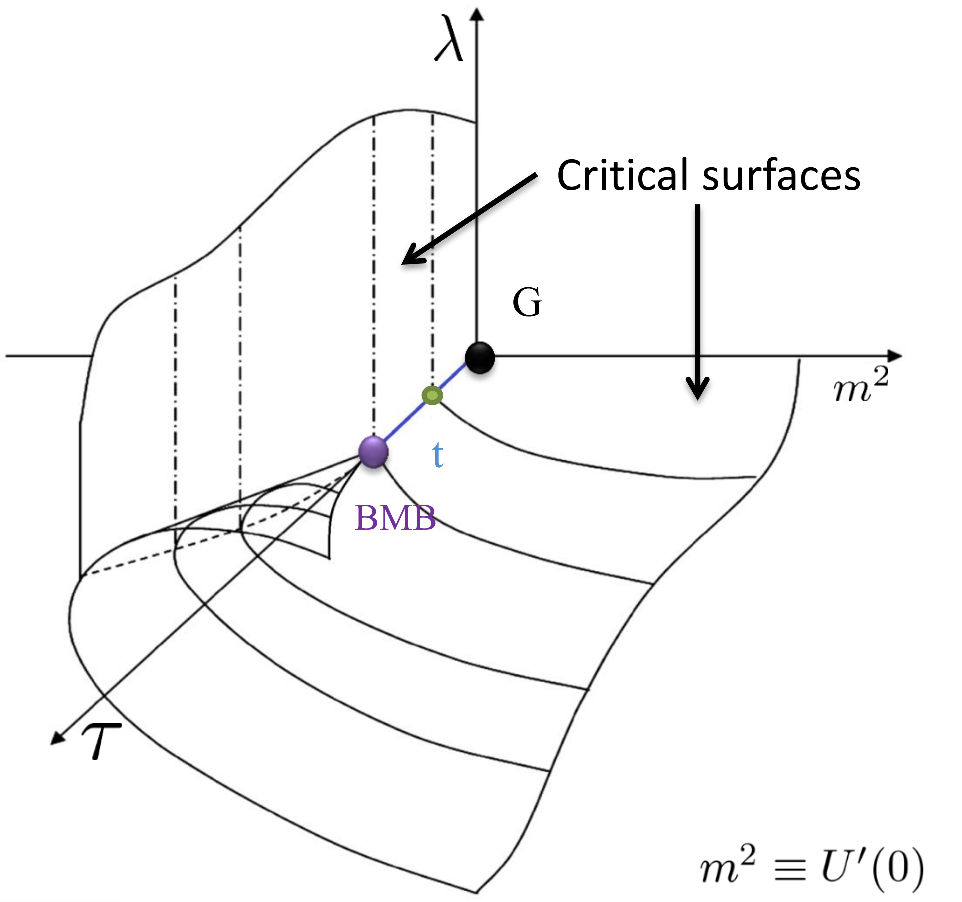
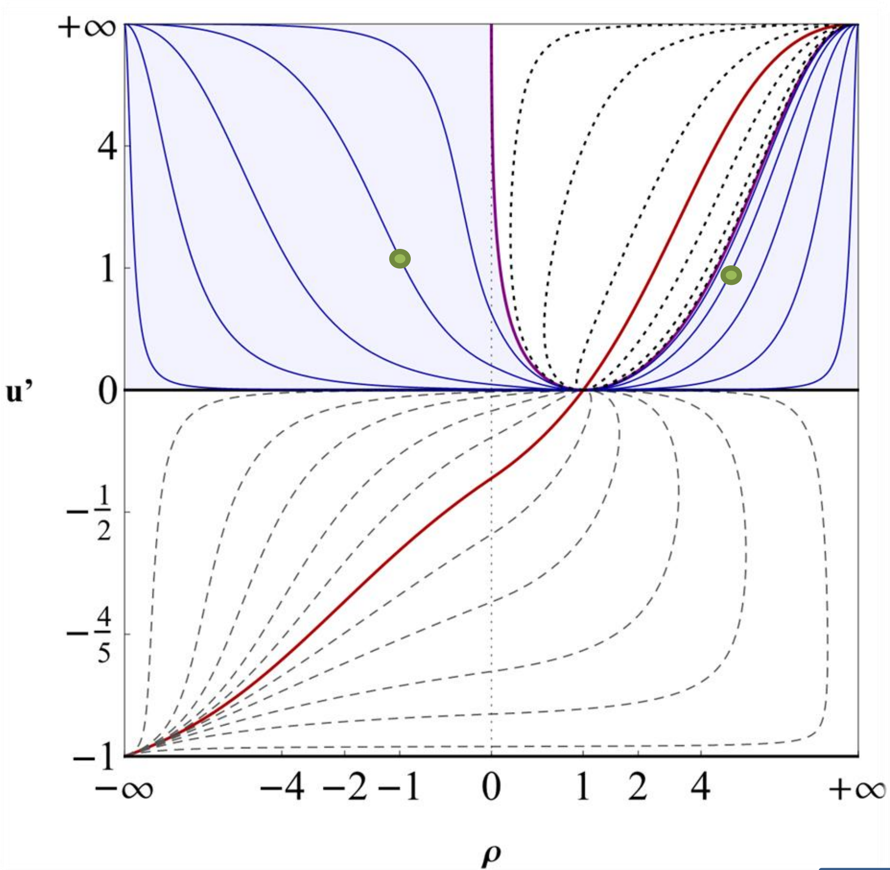


$$\tau = \frac{2}{c^2}$$

The fixed point structure

PARAMETERS

Fixed point solutions



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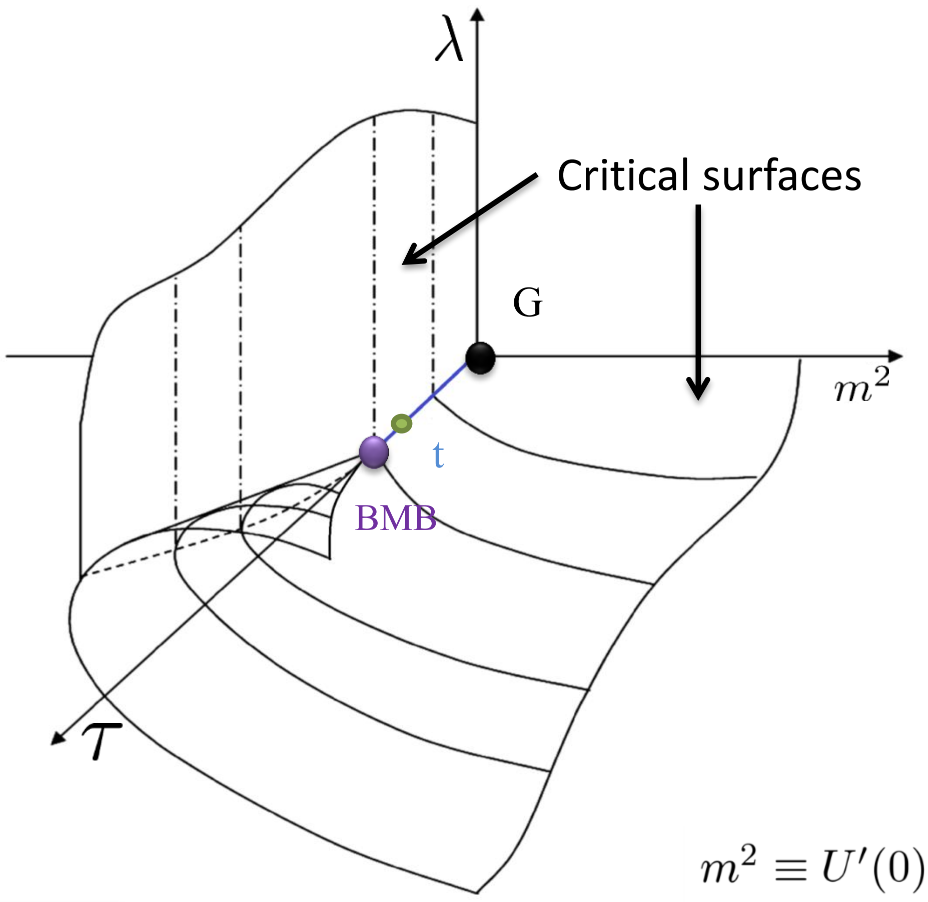
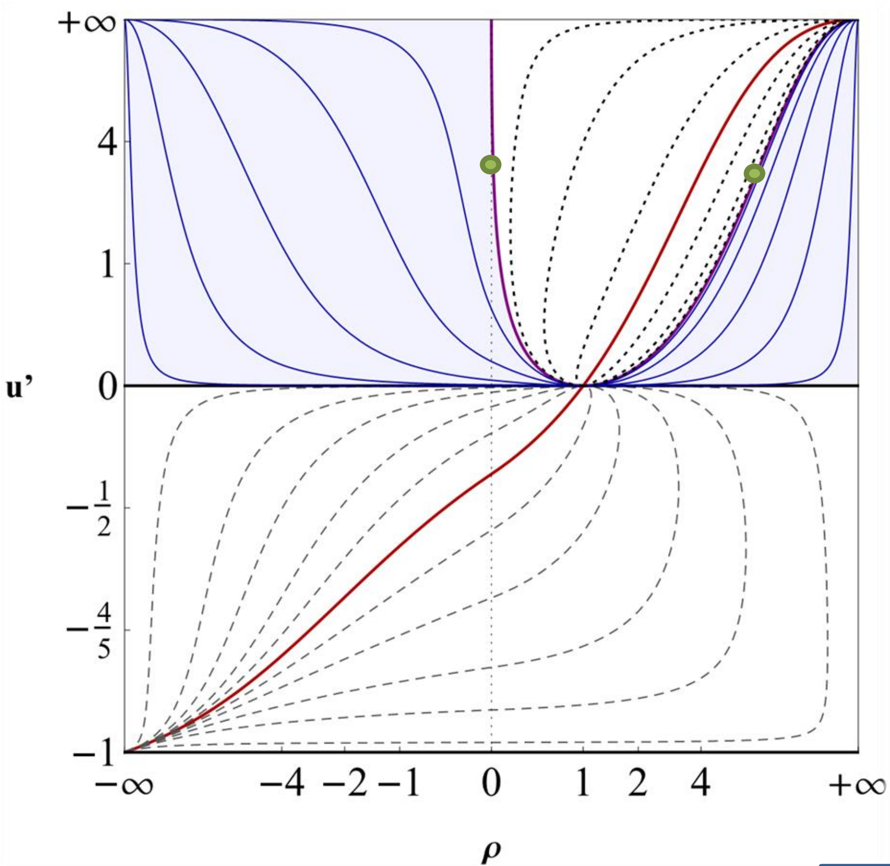


$$\tau = \frac{2}{c^2}$$

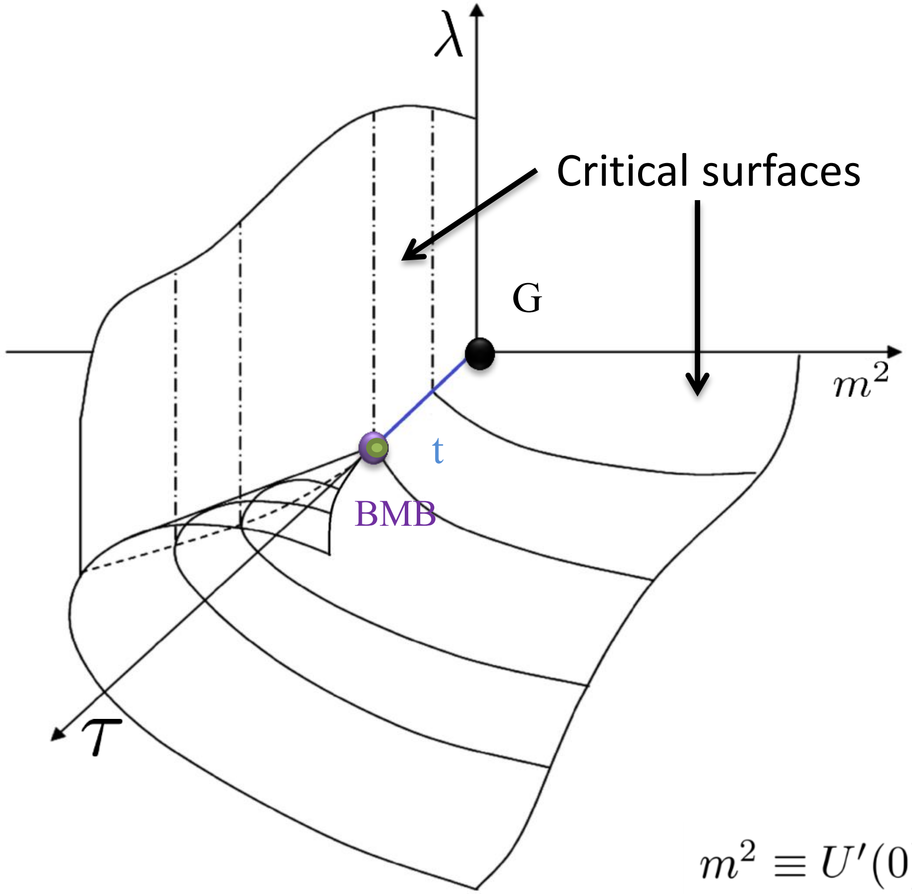
The fixed point structure

PARAMETERS

Fixed point solutions



Topology of the parameter space



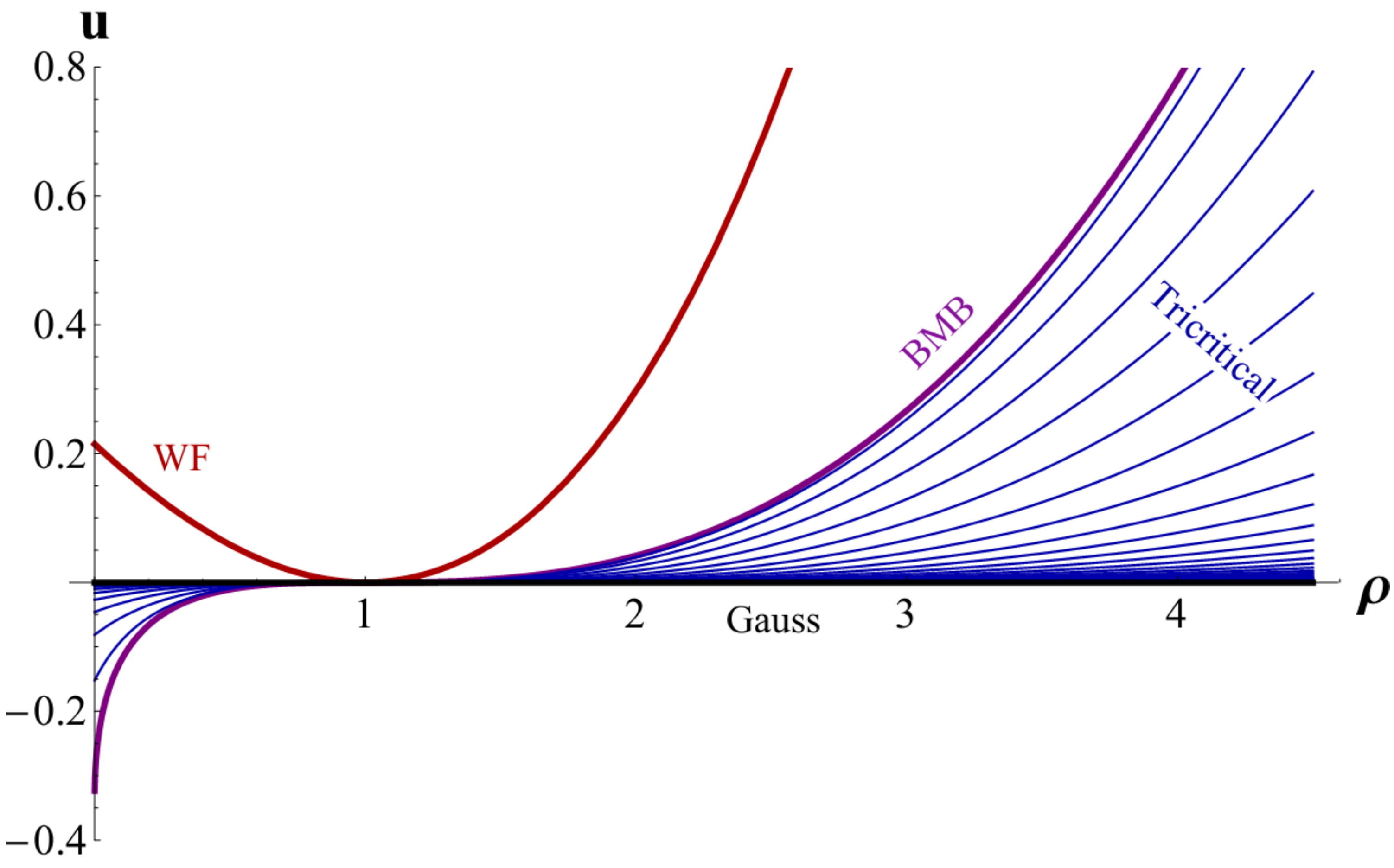
$$m^2 \equiv U'(0)$$

$$\tau = \frac{2}{c^2}$$

The fixed point structure

POTENTIAL

Integrating u' respect to ρ



The fixed point structure

The idea: perturbing around the scaling solution $u'(t,\rho,\epsilon) = u'_*(\rho) + \delta u'(t,\rho,\epsilon)$

Inserting it into the flow equation we obtain the fluctuation equation

$$\partial_t \, \delta u' = 2 \frac{u'}{u''} \left(\partial_\rho + \frac{(u'u'')'}{u'u''} \right) \delta u'$$

Solving it by separation of variables gives: $\delta u'(t,\rho,\epsilon) = \epsilon e^{\theta t} (u'_*)^{\frac{1}{2}(1+\theta)} u''_*$

The eigenperturbation equation reads: $\partial_t \delta u' = \theta \delta u'$

ANALITICITY CONDITION: the perturbation must be analytic

$\Downarrow$

Restriction on θ

Remark: $\xi^{-1} = m \propto |\bar{\rho}_0|^\nu$

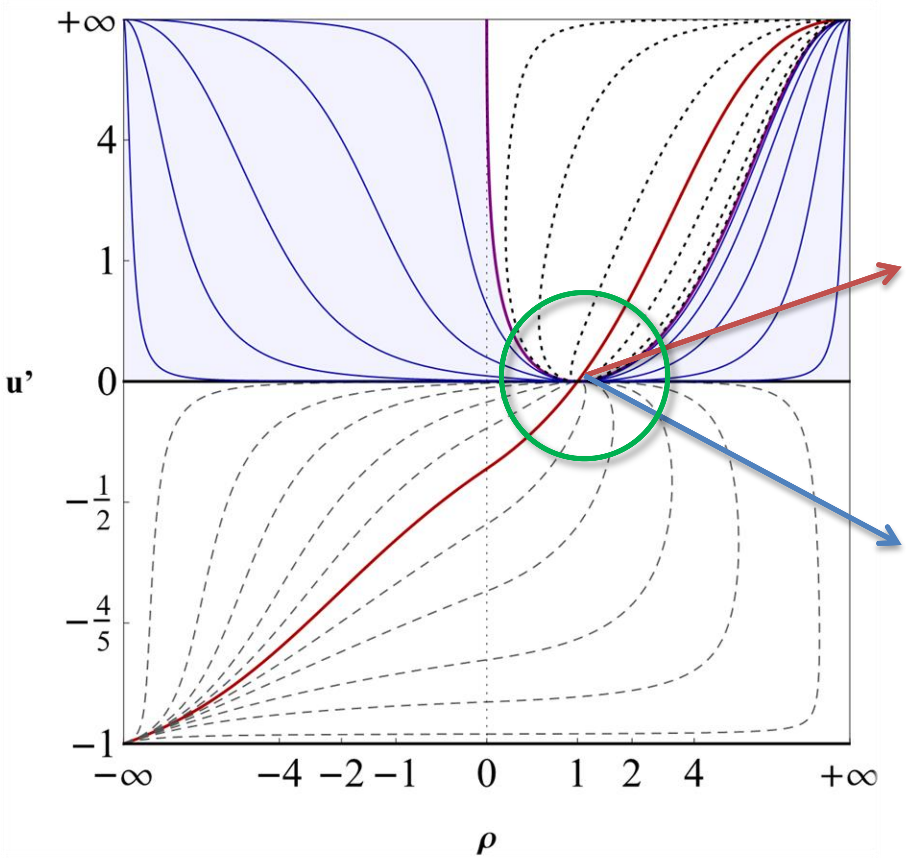
$\nu = -1/\theta$

CRITICAL EXPONENTS

The fixed point structure

$$\delta u'(t,\rho,\epsilon) = \epsilon e^{\theta t} (u'_*)^{\frac{1}{2}(1+\theta)} u''_*$$

$$\nu = -1/\theta$$



Wilson-Fisher

$$u'_* \propto \frac{1}{2}(\rho - 1) \quad \text{for} \quad \rho \to 1^{\pm}$$

$$u''_* = \text{const.}$$

$$\theta \in \{-1, 1, 3, ...\}$$

$$\delta u' \propto \epsilon e^{\theta t} (\rho - 1)^{\frac{1}{2}(1+\theta)}$$

$$\nu = 1$$

Tricritical

$$u'_* \propto \frac{1}{c^2}(\rho - 1)^2 \quad \text{for } \rho \to 1^{\pm}$$

$$u''_* \propto \rho$$

$$\theta \in \{-2, -1, 0, 1, 2, 3, ...\}$$

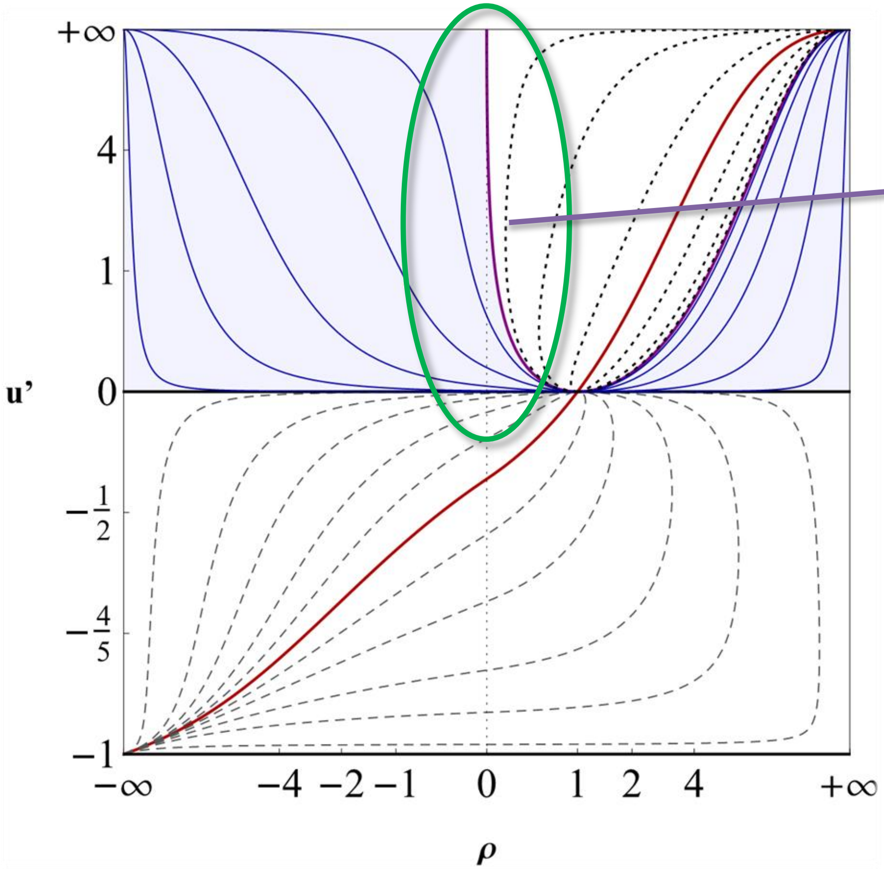
$$\delta u' \propto \epsilon e^{\theta t} \rho^{\theta + 2}$$

$$\nu = 1/2 \quad \text{(mean-field)}$$

The fixed point structure

BARDEEN-MOSHE -BANDER PH.

The BMB fixed point solution has a singularity at 0 ⟶ demanding analyticity is useless



BMB fixed point

$$u'_* \propto \frac{1}{\sqrt{5}}\frac{1}{\sqrt{\rho}} \quad \text{for} \quad \rho \to 0$$

$$\frac{U'_*}{k^2} \propto \frac{1}{\sqrt{5}}\frac{\sqrt{k}}{\sqrt{\bar{\rho}}} \xrightarrow{m^2 \equiv U'_*(0)} \frac{m^2}{k^2} \propto \frac{1}{\sqrt{5}}\frac{\sqrt{k}}{\sqrt{\bar{\rho}}}$$

$(\bar{\rho})^{1/2} \propto \frac{k^{5/2}}{m^2}$ ⟶ Arbitrarily large mass

Breaking of scale invariance!

Conclusion

Non-perturbative solution to a 3d, *O(N)* symmetric quantum field theory theory in the large *N* limit

Study of the fixed point solutions and phase transitions (WF, Tricrit., BMB)

Critical exponents recovered

BMB: UV fixed point with breaking of the scale invariance

LITERATURE

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[5] Daniel F. Litim, Marianne C. Mastaler, Franziska Synatschke-Czerwonka, Andreas Wipf *,Critical behavior of supersymmetric O(N) models in the large-N limit*

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THANK YOU FOR YOUR ATTENTION.