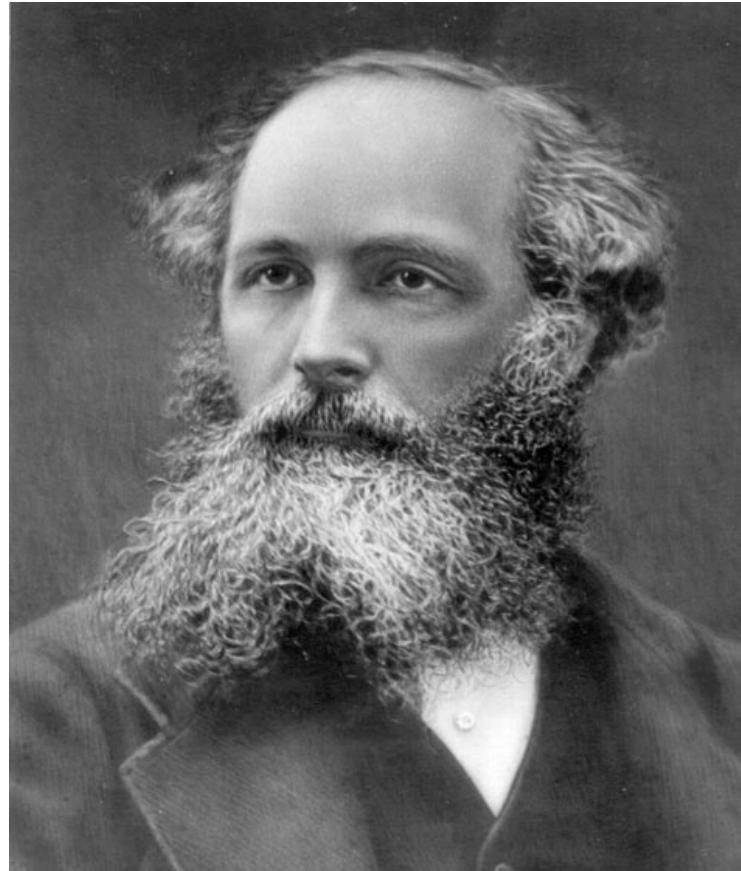


# Review of Electromagnetism



This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

## Maxwell's equations

(in integral form)

$$\oint \vec{H}(\vec{r}, t) \cdot d\vec{s} = \iint \vec{J}(\vec{r}, t) \cdot d\vec{A} + \frac{d}{dt} \iint \vec{D}(\vec{r}, t) \cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r}, t) \cdot d\vec{s} = -\frac{d}{dt} \iint \vec{B}(\vec{r}, t) \cdot d\vec{A}$$

$$\iint \vec{D}(\vec{r}, t) \cdot d\vec{A} = \iiint \rho(\vec{r}, t) dV$$

$$\iint \vec{B}(\vec{r}, t) \cdot d\vec{A} = 0$$

$\vec{E}, \vec{H}$  electric and magnetic field

$\vec{D}, \vec{B}$  electric displacement and magnetic induction

$\vec{J}$  electric current density

$\rho$  electric charge density

$\iint \vec{J}(\vec{r}, t) \cdot d\vec{A}$  stands for all currents going through the area A. It may consist of 3 parts

$$\vec{J}(\vec{r}, t) = \vec{J}_c(\vec{r}, t) + \vec{J}_{cv}(\vec{r}, t) + \vec{J}_i(\vec{r}, t)$$

$$\vec{J}_c(\vec{r}, t) = \kappa \vec{E}(\vec{r}, t) \quad \text{conduction current (Ohm's law)}$$

$$\vec{J}_{cv}(\vec{r}, t) = \rho(\vec{r}, t) \vec{v}(\vec{r}, t) \quad \text{convection current}$$

$$\vec{J}_i(\vec{r}, t) \quad \text{impressed current}$$

$\iiint \rho(\vec{r}, t) dV$  stands for all charges in the volume V

With Stokes' theorem:

$$\oint \vec{E} \cdot d\vec{s} = \iint (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = - \frac{d}{dt} \iint \vec{B} \cdot d\vec{A} = - \iint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$$\iint [\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}] \cdot d\vec{A} = 0 \quad \rightarrow \quad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (2)$$

correspondingly

$$\rightarrow \quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (1)$$

With Gauss' theorem:

$$\oint \vec{D} \cdot d\vec{A} = \iiint \vec{\nabla} \cdot \vec{D} dV = \iiint \rho dV$$

$$\iiint [\vec{\nabla} \cdot \vec{D} - \rho] dV = 0 \quad \rightarrow \quad \vec{\nabla} \cdot \vec{D} = \rho \quad (3)$$

correspondingly  $\rightarrow \vec{\nabla} \cdot \vec{B} = 0$  (4)

## Time-harmonic fields

Time-harmonic fields can be written as complex quantities

$$\begin{aligned}\vec{e}(\vec{r}, t) &= \vec{e}(\vec{r}) \cos(\omega t + \varphi) = \\ &= \Re[\vec{e}(\vec{r}) e^{i\varphi} e^{i\omega t}] = \Re[\tilde{\vec{E}}(\vec{r}) e^{i\omega t}] = \Re \tilde{\vec{E}}(\vec{r}, t)\end{aligned}$$

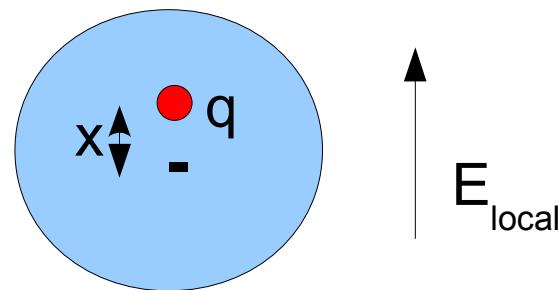
$\tilde{\vec{E}}(\vec{r})$  is called phasor.

Advantages are:

- $\frac{\partial}{\partial t} \rightarrow i\omega$ ,
- phasors are vectors in a coordinate system rotating with  $\omega t$ ,
- $e^{i\omega t}$  cancels out in the equations.

The effect of electric fields on matter can be described by a polarization P,  
the effect of magnetic fields by a magnetization M.

P and M result from averaging over atomic / molecular  
electric and magnetic dipoles induced by the fields, e.g.



$$p_e = qx \rightarrow \vec{P} = n \vec{p}_e = \epsilon_0 \chi_e \vec{E}$$

Linear materials:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \mu \vec{H}$$

There are losses due to changing polarisation

$$\epsilon = \epsilon' - i\epsilon'' = \epsilon'(1 - i\tan(\delta_\epsilon))$$

$$\tan(\delta_\epsilon) = \frac{\epsilon''}{\epsilon'}, \quad \delta_\epsilon \text{ electric loss angle}$$

and losses due to free charges

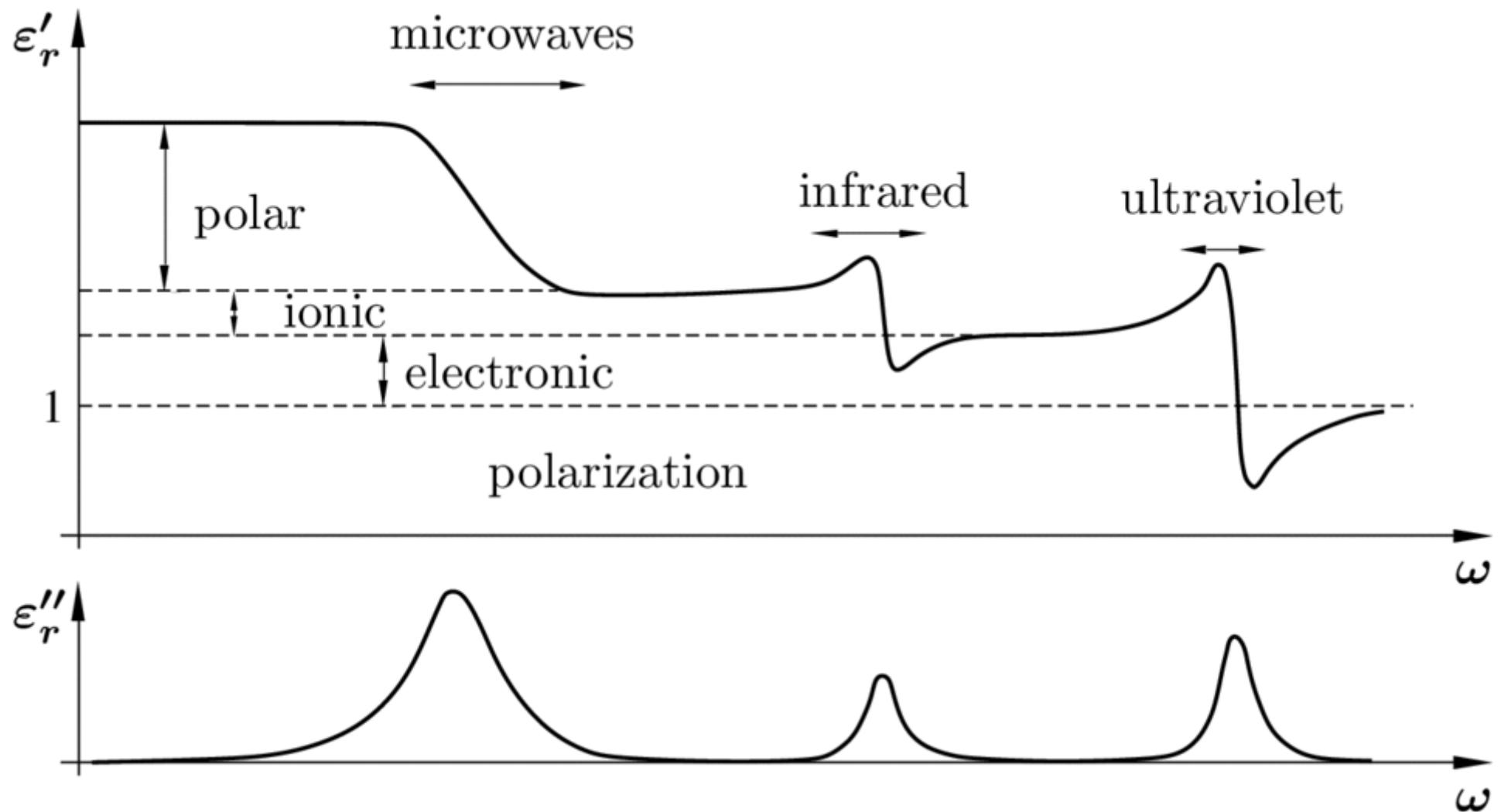
$$\vec{\nabla} \times \vec{H} = \vec{J} + i\omega\epsilon\vec{E} = \kappa\vec{E} + i\omega\epsilon\vec{E} = i\omega\epsilon\left(1 - \frac{\kappa}{i\omega\epsilon}\right)\vec{E}$$

$$\epsilon_c = \epsilon\left(1 - \frac{\kappa}{i\omega\epsilon}\right)$$

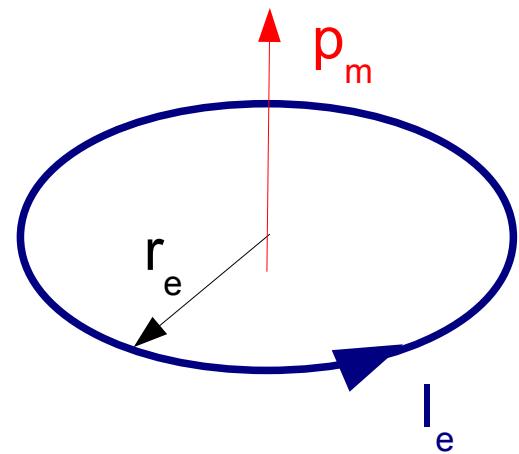
In most dielectrics is  $\tan(\delta_\epsilon) \ll 1$ .

In good conductors is  $\kappa/\omega \gg \epsilon \rightarrow \epsilon_c \approx \kappa/i\omega$ .





Magnetic reaction of material is due to circulating electrons and due to particle spins. It can be described by means of magnetic dipoles, i.e. by circulating elementary currents:

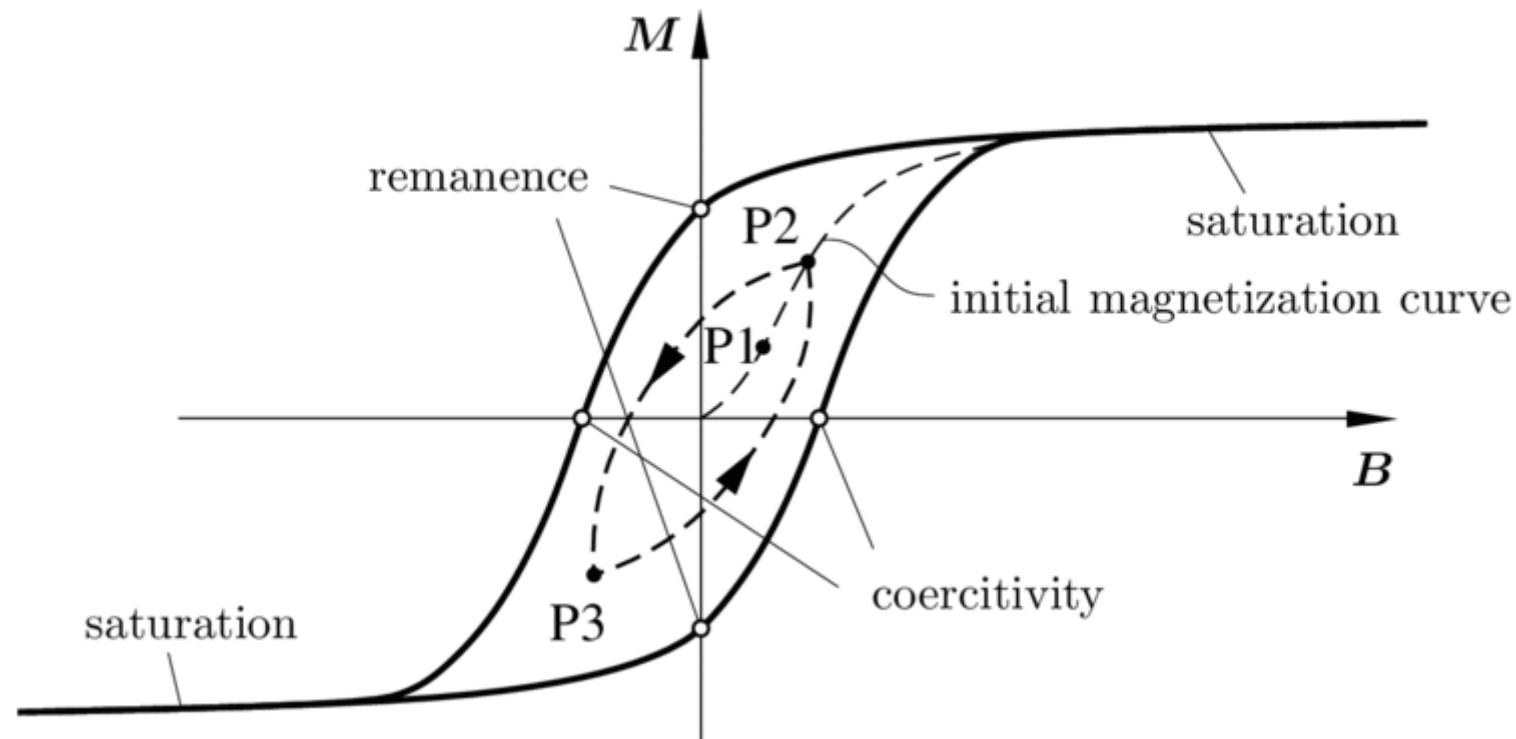


$$p_m = \pi r_e^2 I_e \rightarrow \vec{M} = n \vec{p}_m = \chi_m \vec{H}$$

Like  $P$ , the magnetization  $M$  is a dynamic process with losses due to rotating dipoles

$$\mu = \mu' - i\mu'' = \mu' (1 - i \tan(\delta_\mu)), \quad \delta_\mu \text{ magnetic loss angle}$$

For ferromagnetic materials the relation between the external field and the magnetization is non-linear and depends typically on the history of the material (hysteresis).



## Boundary / continuity conditions

Maxwell's theory is a continua theory. It requires continuous, double differentiable functions.

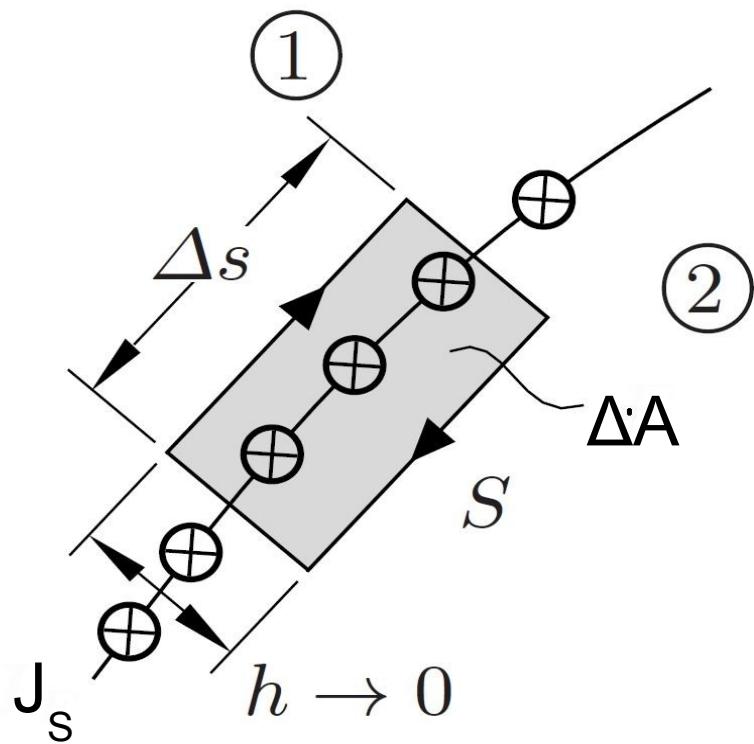
Solutions in different media have to be matched at the interface.

Take Maxwell's equs. in integral form

$$\oint \vec{H}(\vec{r}, t) \cdot d\vec{s} = \iint \vec{J}(\vec{r}, t) \cdot d\vec{A} + \frac{d}{dt} \iint \vec{D}(\vec{r}, t) \cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r}, t) \cdot d\vec{s} = -\frac{d}{dt} \iint \vec{B}(\vec{r}, t) \cdot d\vec{A}$$

and make an intelligent choice for the integration area



$\Delta s$  is finite but small, such that the fields are constant,  
 $h \rightarrow 0$  :

$$\begin{aligned}
 H_{t1} \Delta s - H_n h - H_{t2} \Delta s + H_n h &= \\
 &= J_s \Delta s + \frac{\partial}{\partial t} \iint_{\Delta A} \vec{D} \cdot \Delta \vec{A} \\
 \rightarrow H_{t1} - H_{t2} &= J_s \\
 E_{t1} - E_{t2} &= 0
 \end{aligned}$$

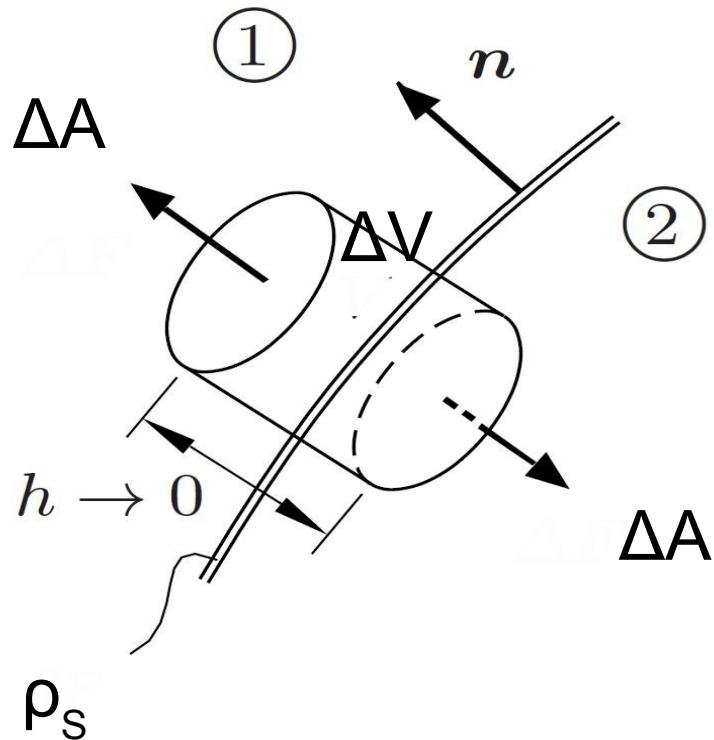
If medium 2 is perfectly electrically conducting (pec) :

$$E_{t1} = 0, \quad H_{t1} = J_s$$

An intelligent choice of the integration volume:

$$\oint \vec{D}(\vec{r}, t) \cdot d\vec{A} = \iiint \rho(\vec{r}, t) dV$$

$$\oint \vec{B}(\vec{r}, t) \cdot d\vec{A} = 0$$



$$D_{n1} \Delta A - D_{n2} \Delta A + \iint_{\Delta A_{zyl}} \vec{D} \cdot d\vec{A} =$$

$$= \rho_s \Delta A$$

$$D_{n1} - D_{n2} = \rho_s, \quad B_{n1} - B_{n2} = 0$$

$$D_{n1} = \rho_s, \quad B_{n1} = 0 \quad \text{if 2 is pec}$$

# Application of Maxwell's equations

Electrostatic fields

( $\delta/\delta t=0$ ,  $\epsilon=\text{const.}$ )

Maxwell's equations

$$\vec{\nabla} \times \vec{E} = 0$$

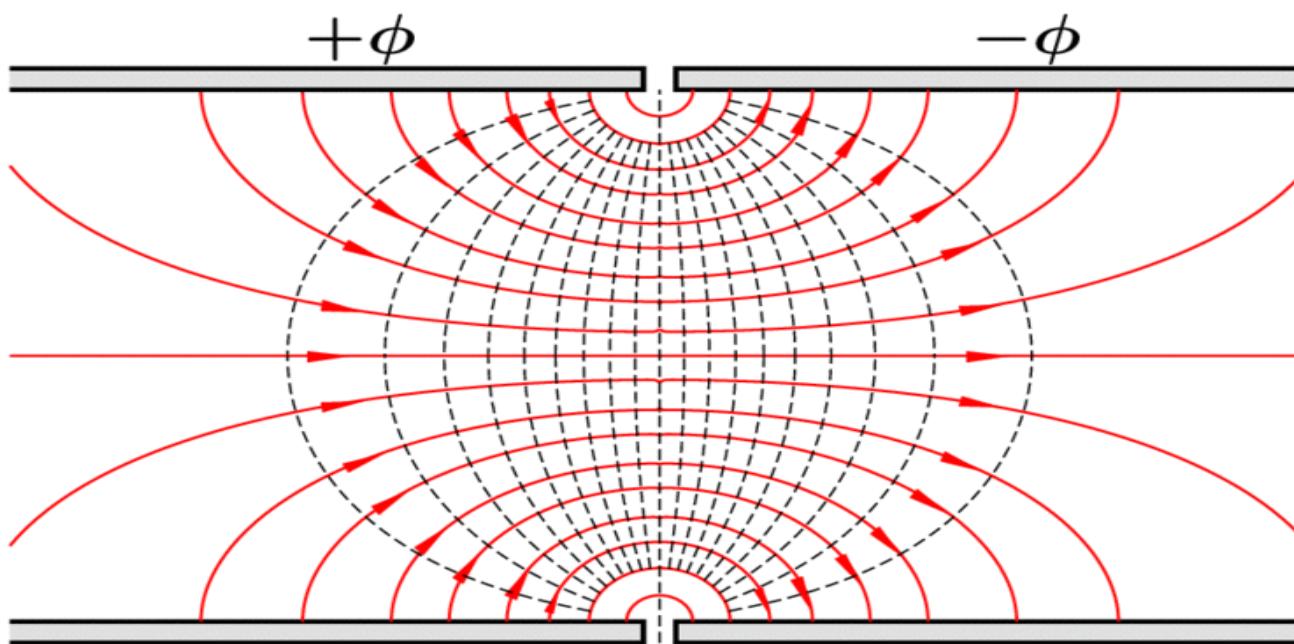
$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \rightarrow \quad \vec{\nabla} \times \vec{\nabla} \Phi \equiv 0 \quad \vec{E} = -\vec{\nabla} \Phi$$

Poisson equation:

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\epsilon \vec{E}) = \rho \quad \rightarrow \quad \vec{\nabla}^2 \Phi = -\frac{\rho}{\epsilon}$$

# Example: Circular electrostatic lens



E-field pattern

*Circular symmetric Laplace equation*

$$\vec{\nabla}^2 \Phi = \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad (i)$$

# Bernoulli ansatz

$$\Phi(\rho, z) = R(\rho)Z(z)$$

substituted in (i) and devide by RZ

$$\frac{1}{R} \frac{d^2 R}{d \rho^2} + \underbrace{\frac{1}{R \rho} \frac{d R}{d \rho}}_{k_z^2} + \frac{1}{Z} \frac{d^2 Z}{d z^2} = 0, \quad Z = \begin{cases} C_0 + D_0 z, & k_z = 0 \\ C e^{k_z z} + D e^{-k_z z}, & k_z \neq 0 \end{cases}$$

Bessel differential equation

$$\frac{d^2 R}{d \rho^2} + \frac{1}{\rho} \frac{d R}{d \rho} + k_z^2 R = 0, \quad R = \begin{cases} A_0 + B_0 \ln(\rho/\rho_0), & k_z = 0 \\ A J_0(k_z \rho) + B N_0(k_z \rho), & k_z \neq 0 \end{cases}$$

Boundary conditions

$$\Phi \text{ finite for } \rho \rightarrow 0: \quad B_0 = B = 0$$



# Boundary conditions

$$\Phi \text{ finite for } z = \pm\infty \rightarrow C_0 = D_0 = 0, \quad Z = A e^{-k_z |z|}$$

$$\Phi = \begin{cases} -\Phi_0 & \text{for } z > 0 \\ +\Phi_0 & \text{for } z < 0 \end{cases} \rightarrow A_0 = \Phi_0, \quad k_{zn} a = j_{0n}$$

$$\Phi = sign(z) [-\Phi_0 + \sum_{n=1}^{\infty} A_n J_0(j_{0n} \frac{\rho}{a}) e^{-j_{0n}|z|/a}]$$

Symmetry at  $z=0$

$$\Phi(z=0)=0$$
$$\Phi_0 = \sum_{n=1}^{\infty} A_n J_0(j_{0n} \frac{\rho}{a}) \quad (ii)$$

# Fourier-Bessel expansion

Multiplication of (ii) with  $\rho J_0(j_{0m} \rho/a)$  and integration over  $\rho$

$$\underbrace{\Phi_0 \int_0^a J_0(j_{0m} \frac{\rho}{a}) \rho d\rho}_{\frac{a^2}{j_{0m}} J_1(j_{0m})} = \sum_{n=1}^{\infty} \underbrace{A_n \int_0^a J_0(j_{0n} \frac{\rho}{a}) J_0(j_{0m} \frac{\rho}{a}) \rho d\rho}_{\delta_{m2}^n \frac{a^2}{j_{0m}} J_1^2(j_{0m})}$$

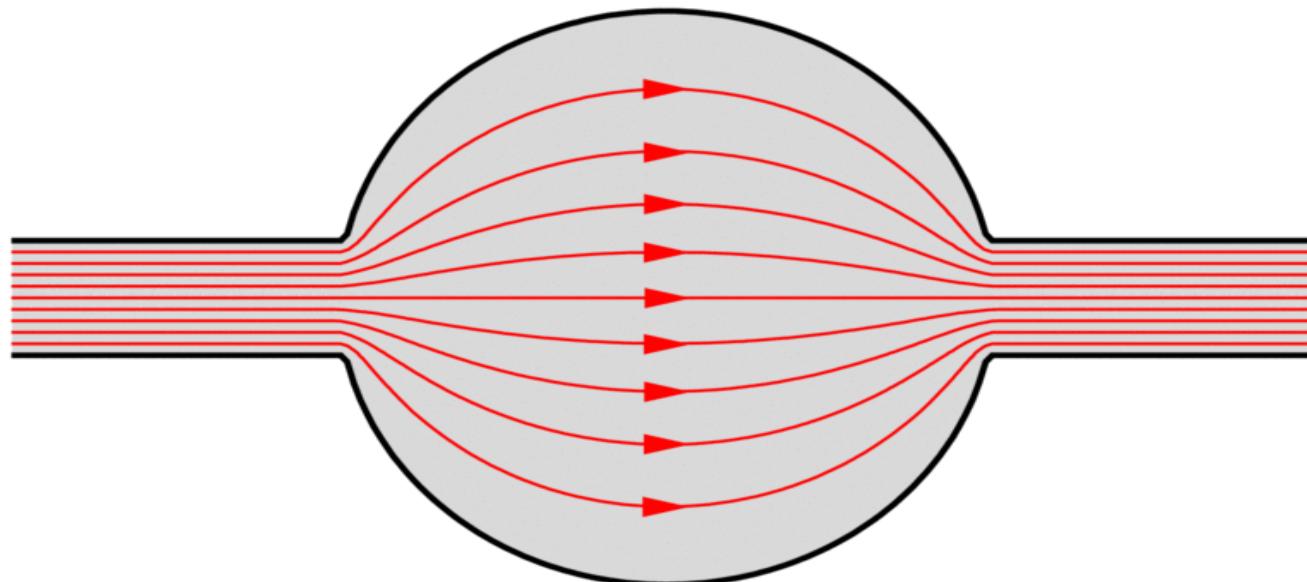
$$\Phi = sign(z) \Phi_0 \left[ -1 + 2 \sum_{n=1}^{\infty} \frac{J_0(j_{0n} \frac{\rho}{a})}{j_{0n} J_1(j_{0n})} e^{-j_{0n}|z|/a} \right]$$

## Stationary currents

( $\delta/\delta t=0$ ,  $\kappa=\text{const.}$ )

$$\vec{\nabla} \times \vec{E} = 0 \quad \rightarrow \quad \vec{E} = -\vec{\nabla} \Phi$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 = \vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\kappa \vec{E}) \quad \rightarrow \quad \vec{\nabla}^2 \Phi = 0$$



J-field lines

## Magnetostatic fields

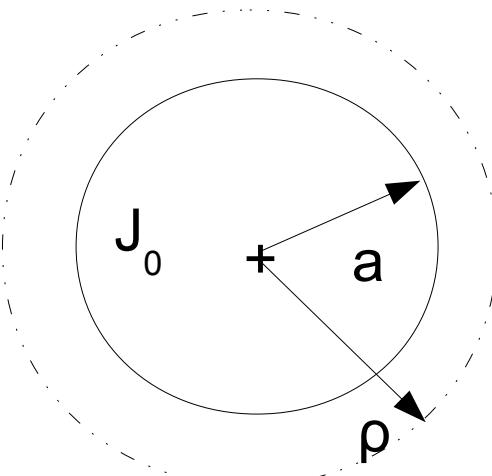
( $\delta/\delta t=0$ ,  $\mu=\text{const.}$ )

Maxwell's equations

$$\vec{\nabla} \times \vec{H} = \vec{J}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} = 0 &\rightarrow \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} \equiv 0 \rightarrow \vec{B} = \vec{\nabla} \times \vec{A} = \mu \vec{H} \\ \vec{\nabla} \times \vec{B} = \mu \vec{J} &\rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = -\vec{\nabla}^2 \vec{A} = \mu \vec{J}\end{aligned}$$

Example: Wire carrying a constant current



$$\begin{aligned}\vec{\nabla}^2 A_z &= \frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A_z}{\partial \rho} = \\ &= \begin{cases} 0 & \text{for } \rho \geq a \\ -\mu J_0 & \text{for } \rho \leq a \end{cases}\end{aligned}$$

$\rho \geq a$ :

$$\frac{d}{d\rho} \frac{dA_z^{(1)}}{d\rho} + \frac{1}{\rho} \frac{dA_z^{(1)}}{d\rho} = 0 \quad \rightarrow \quad \frac{dA_z^{(1)}}{d\rho} = \frac{C}{\rho}$$

$\rho \leq a$ :

$$\frac{d}{d\rho} \frac{dA_z^{(2)}}{d\rho} + \frac{1}{\rho} \frac{dA_z^{(2)}}{d\rho} = -\mu J_0 \quad \rightarrow \quad A_z^{(2)} = -\frac{\mu}{4} J_0 \rho^2$$

at  $\rho = a$ :  $B_\varphi^{(1)} = B_\varphi^{(2)}$

$$\vec{B} = \vec{\nabla} \times (A_z \vec{e}_z) = -\frac{dA_z}{d\rho} \vec{e}_\varphi \quad \rightarrow \quad C = -\frac{1}{2} \mu a^2 J_0$$

## Quasi-stationary fields

( $|D/\delta t| \ll |J|$ ,  $\epsilon, \mu, \kappa = \text{const.}$ )

Maxwell's equations

$$\vec{\nabla} \times \vec{H} = \vec{J}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{D} = \rho, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \rightarrow \quad \vec{\nabla} \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) \rightarrow \vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$$

good conductors:  $\rho = 0$

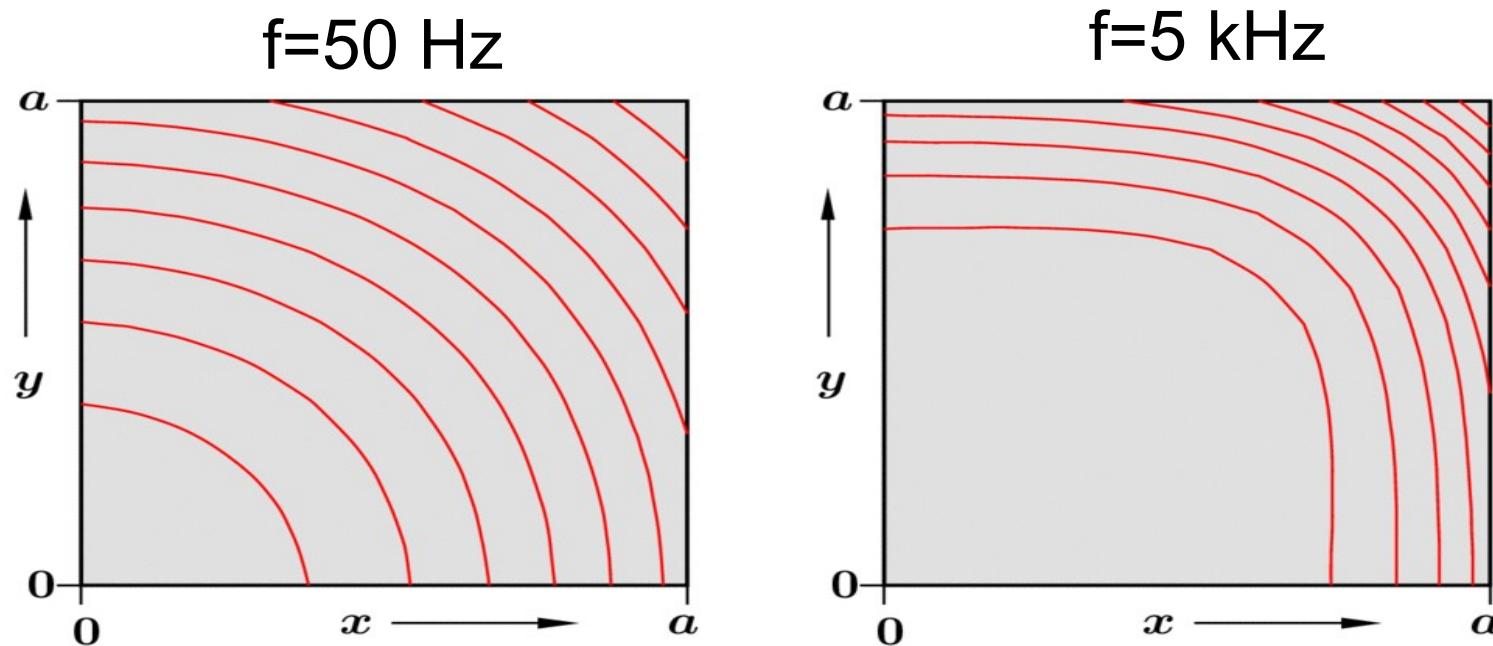
no impressed voltages:  $\Phi = 0$

$$\vec{\nabla} \cdot \vec{D} = -\epsilon \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = 0 \quad \rightarrow \quad \vec{\nabla} \cdot \vec{A} = 0$$

$$\begin{aligned}
 \vec{\nabla} \times \vec{H} &= \frac{1}{\mu} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{\mu} [\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}] = \\
 &= -\frac{1}{\mu} \vec{\nabla}^2 \vec{A} = \vec{J} = \kappa \vec{E} = -\kappa \frac{\partial \vec{A}}{\partial t} \rightarrow \vec{\nabla}^2 \vec{A} - \mu \kappa \frac{\partial \vec{A}}{\partial t} = 0
 \end{aligned}$$

diffusion equation

Example: Current distribution in aluminum bar  
 $\kappa = 17 \cdot 10^6 \Omega^{-1}m^{-1}$ ,  $a = 1\text{cm}$



## Poynting's theorem

( $\epsilon, \mu, \kappa = \text{const.}, J = \kappa E$ ,  
full set of Maxwell's equations)

A charge  $\rho dV$  is moved a distance  $\delta s$  by the fields.

Then the work done by the fields is

$$d \frac{\delta W}{\delta t} = d \vec{f} \cdot \frac{\delta \vec{s}}{\delta t} = \rho \frac{\delta \vec{s}}{\delta t} \cdot (\vec{E} + \vec{v} \times \vec{B}) dV = \rho \vec{v} \cdot \vec{E} dV = \vec{J} \cdot \vec{E} dV$$

using Maxwell's equations

$$\vec{E} \cdot \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$-\vec{H} \cdot \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

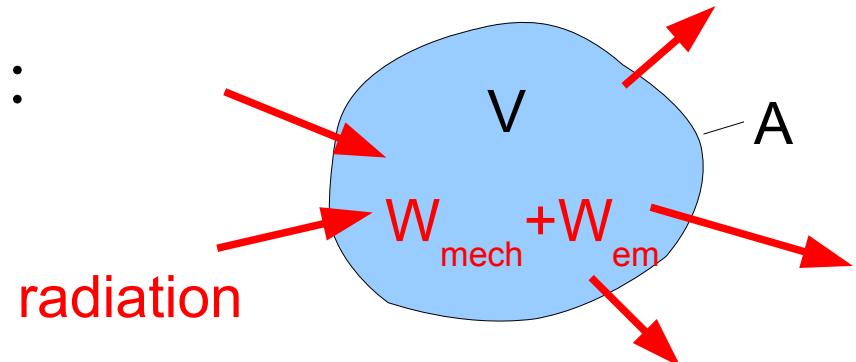
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$$\rightarrow \vec{E} \cdot \vec{J} = -\vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \frac{\partial}{\partial t} \left[ \frac{1}{2} \vec{H} \cdot \vec{B} + \frac{1}{2} \vec{E} \cdot \vec{D} \right]$$

After integration and Gauss' theorem:

$$-\oint (\vec{E} \times \vec{H}) \cdot d\vec{A} =$$

$$= \iiint \vec{E} \cdot \vec{J} dV + \frac{\partial}{\partial t} \iiint \left( \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{H} \cdot \vec{B} \right) dV$$



Energy radiated into  $V$  equals the dissipation in  $V$  and the increase of stored electromagnetic energy in  $V$ .

poynting vector (radiation flux)  $\vec{S} = \vec{E} \times \vec{H}$

dissipated power density  $p_d = \vec{E} \cdot \vec{J}$

electric energy density  $w_e = (1/2) \vec{E} \cdot \vec{D}$

magnetic energy density  $w_m = (1/2) \vec{H} \cdot \vec{B}$

## Poynting's theorem for time-harmonic fields

decompose e.g.  $\vec{E} = \Re[\tilde{\vec{E}} e^{i\omega t}] = \frac{1}{2} [\tilde{\vec{E}} e^{i\omega t} + \tilde{\vec{E}}^* e^{-i\omega t}]$

$$\begin{aligned} w_E &= \frac{1}{2} \vec{E} \cdot \vec{D} = \frac{1}{8} [\tilde{\vec{E}} \cdot \tilde{\vec{D}} e^{i2\omega t} + \tilde{\vec{E}}^* \cdot \tilde{\vec{D}}^* e^{-i2\omega t}] + \frac{1}{8} [\tilde{\vec{E}} \cdot \tilde{\vec{D}}^* + \tilde{\vec{E}}^* \cdot \tilde{\vec{D}}] \\ &= \frac{1}{4} \Re[\tilde{\vec{E}} \cdot \tilde{\vec{D}} e^{i2\omega t}] + \frac{1}{4} \Re[\tilde{\vec{E}} \cdot \tilde{\vec{D}}^*] \end{aligned}$$

then after time-averaging

$$\bar{w}_e = (1/4) \vec{E} \cdot \vec{D}^*$$

correspondingly  $\bar{w}_m = (1/4) \vec{H} \cdot \vec{B}^*$   $\bar{p}_d = (1/2) \vec{E} \cdot \vec{J}^*$

$$\vec{S}_c = (1/2) \vec{E} \times \vec{H}^* \quad \rightarrow \quad \vec{S} = (1/2) \Re[\vec{E} \times \vec{H}^*]$$

*using Maxwell 's equations*

$$\vec{E} \cdot \vec{\nabla} \times \vec{H}^* = \vec{J}^* - j\omega \vec{D}^*$$

$$-\vec{H}^* \cdot \vec{\nabla} \times \vec{E} = -j\omega \vec{B}$$

$$\rightarrow \frac{1}{2} \vec{E} \cdot \vec{J}^* = -\vec{\nabla} \cdot \left( \frac{1}{2} \vec{E} \times \vec{H}^* \right) - j2\omega \left( \frac{1}{4} \vec{H} \cdot \vec{B}^* - \frac{1}{4} \vec{E} \cdot \vec{D}^* \right)$$

*after integration and application of Gauss ' law*

$$-\oint \vec{S}_c \cdot d\vec{A} = \iiint \bar{p}_d dV + i2\omega \iiint (\bar{w}_m - \bar{w}_e) dV$$

*active power (time-averaged Joulean heat)*

$$\bar{P}_{act} = \bar{P}_d = -\oint \Re[\vec{S}_c] \cdot d\vec{A} = -\oint \vec{S} \cdot d\vec{A}$$

*reactive power*

$$\bar{P}_{react} = -\oint \Im[\vec{S}_c] \cdot d\vec{A} = 2\omega \iiint (\bar{w}_m - \bar{w}_e) dV$$

In good conductors is  $W_m \gg W_e$  ( $|E| \ll |H|$ )

$$-\oint \vec{S}_c \cdot d\vec{A} = \bar{P}_c = \bar{P}_d + i2\omega \bar{W}_m$$

This allows to calculate the resistance and internal inductance of a conductor. We define

$$\begin{aligned}\bar{I}^* &= \oint \bar{\vec{H}}^* \cdot d\vec{s} \\ \bar{U} &= \int_1^2 \bar{\vec{E}} \cdot d\vec{l} = \bar{I}(R + i\omega L_i)\end{aligned}$$

and obtain

$$\bar{P}_c = \frac{1}{2} \bar{U} \bar{I}^* = \frac{1}{2} |\bar{I}^2| (R + i\omega L_i) = \bar{P}_d + i2\omega \bar{W}_m$$

## Tut-Ex 1

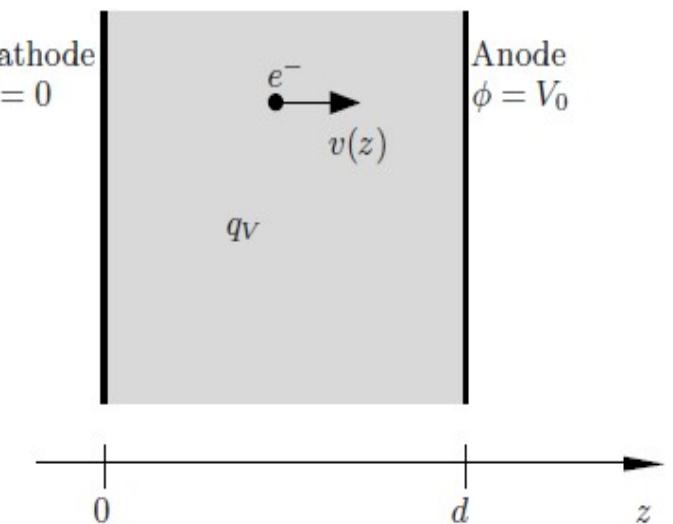
Given is a conducting hollow sphere carrying a charge  $Q$ . What is the field inside and outside and what is the electrostatic field energy?

## Tut-Ex 2

A capacitor is filled with a lossy dielectric and charged to a voltage  $V$ . What is the time constant for discharge?

## Tut-Ex 3

Given is a 1-dimensional planar diode. What is the current density at saturation?

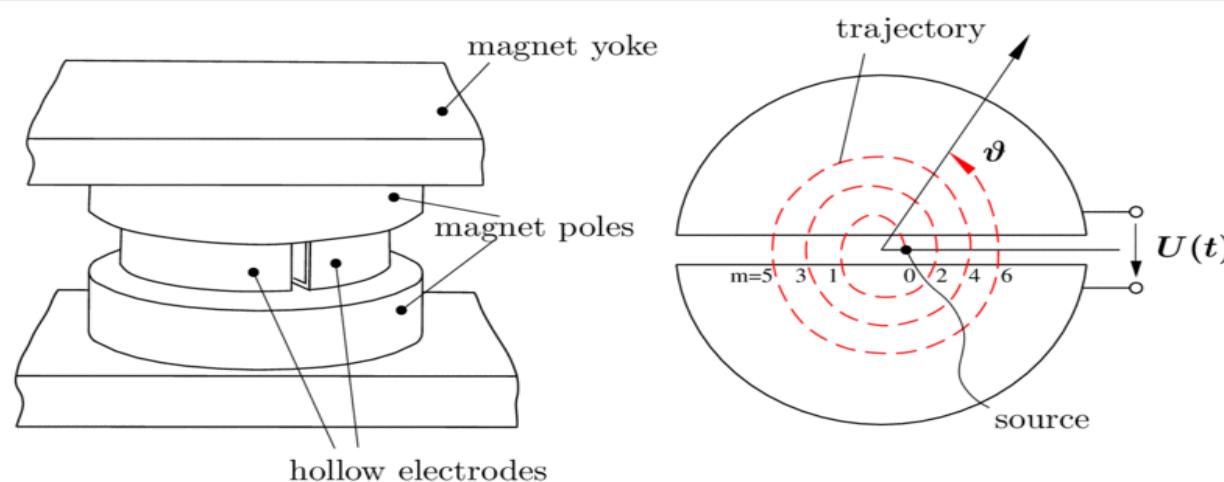


## Tut-Ex 4

Derive the magnetic vectorpotential for a given current density.

## Tut-Ex 5

Given is a non-relativistic cyclotron with a constant magnetic induction  $B$  and maximum radius  $R$ . What is the end energy?



## Tut-Ex 6

A long dipole magnet is excited by a coil with  $n$  windings and current  $I_0$ . Calculate the magnetic field in the air gap.

The simplest electromagnetic wave is a **plane wave**. It depends only on one space variable (direction of propagation) and on the time.

$$\vec{E} = \vec{E}(z, t), \quad \vec{H} = \vec{H}(z, t):$$

Maxwell's eqs. yield two sets of uncoupled equations:

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \epsilon \frac{\partial E_x}{\partial t} & \frac{\partial E_x}{\partial z} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial H_x}{\partial z} &= \epsilon \frac{\partial E_y}{\partial t} & -\frac{\partial E_y}{\partial z} &= -\mu \frac{\partial H_x}{\partial t} \end{aligned}$$

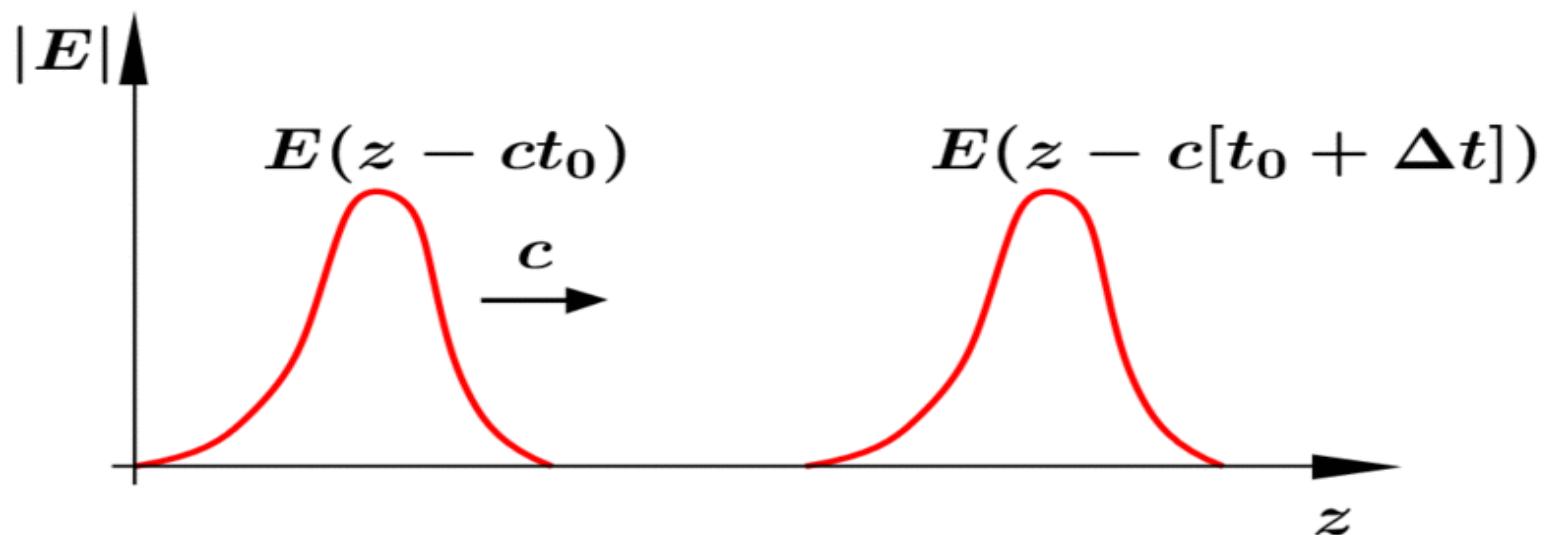
e.g. the red set gives the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = 0, \quad c = \frac{1}{\sqrt{\mu \epsilon}}$$

with d'Alembert's solution

$$E_x = f(z - ct) + g(z + ct)$$

$$H_y = \frac{1}{Z} [f(z - ct) - g(z + ct)], \quad Z = \sqrt{\frac{\mu}{\epsilon}}$$



*velocity of light:*

$$c = \frac{1}{\sqrt{\mu \epsilon}}$$

*wave impedance:*

$$Z = \sqrt{\frac{\mu}{\epsilon}}$$

$\approx 377 \Omega$  *in free space*

*field properties:*

$\vec{E} \perp \vec{H}$ ,  $\vec{E} \times \vec{H} \rightarrow$  direction of propagation

$\vec{E}, \vec{H} \perp$  direction of propagation

$$E^+ / H^+ = -E^- / H^- = Z$$

## Time-harmonic plane wave

$$\left( \frac{\partial}{\partial t} = i \omega \right)$$

Wave equation becomes Helmholtz equation:

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0, \quad k = \omega \sqrt{\mu \epsilon}$$

$$E_x = A e^{i(\omega t - kz)} + B e^{i(\omega t + kz)}$$

$$H_y = \frac{1}{Z} (A e^{i(\omega t - kz)} - B e^{i(\omega t + kz)})$$

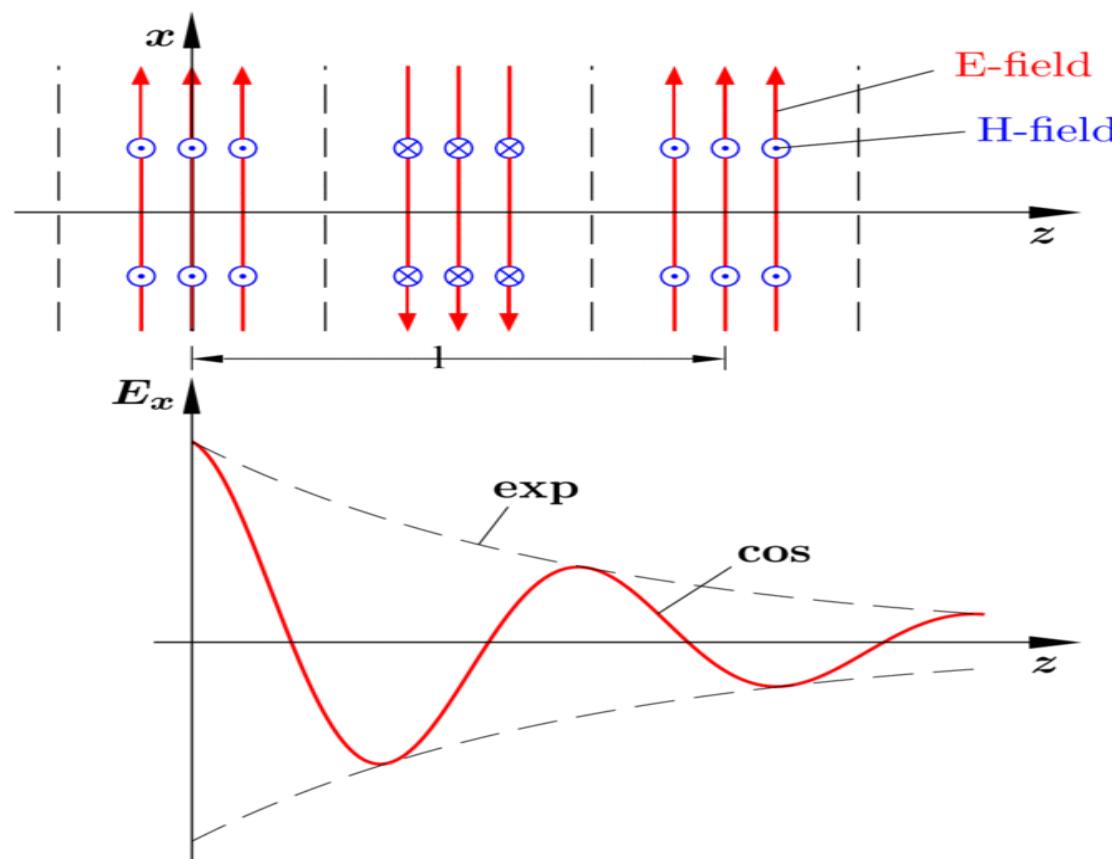
$$loss-free material: \quad k = \omega/c = 2\pi/\lambda$$

$$lossy material: \quad \epsilon_c = \epsilon_r \epsilon_0 \left( 1 - i \frac{\kappa}{\omega \epsilon_r \epsilon_0} \right) = \epsilon_0 (\epsilon_r' - i \epsilon_r'')$$

$$k = \omega \sqrt{\mu \epsilon_c} = \beta - i \alpha, \quad \beta = 2\pi/\lambda$$

$$\frac{\beta}{k_0} = \sqrt{\frac{\epsilon_{r'}}{2} + \frac{\epsilon_{r'}}{2} \sqrt{1 + \left(\frac{\epsilon_{r''}}{\epsilon_{r'}}\right)^2}}, \quad \frac{\alpha}{k_0} = \sqrt{-\frac{\epsilon_{r'}}{2} + \frac{\epsilon_{r'}}{2} \sqrt{1 + \left(\frac{\epsilon_{r''}}{\epsilon_{r'}}\right)^2}}$$

$$E_x = \Re A e^{i(\omega t - kz)} = A \cos(\omega t - \beta z) e^{-\alpha z}$$



## Phase velocity

$$\phi = \omega t - \beta z = \text{const.} \quad \rightarrow \quad \frac{d\phi}{dt} = \omega - \beta \frac{dz}{dt} = \omega - \beta v_{ph} = 0$$

$$v_{ph} = \frac{\omega}{\beta}$$

## Group velocity

Take two plane waves with  $\omega_1$  and  $\omega_2$

$$\omega_1 = \omega_0 + \delta\omega, \quad \omega_2 = \omega_0 - \delta\omega$$

$$\beta_1 = \beta_0 + \delta\beta, \quad \beta_2 = \beta_0 - \delta\beta$$

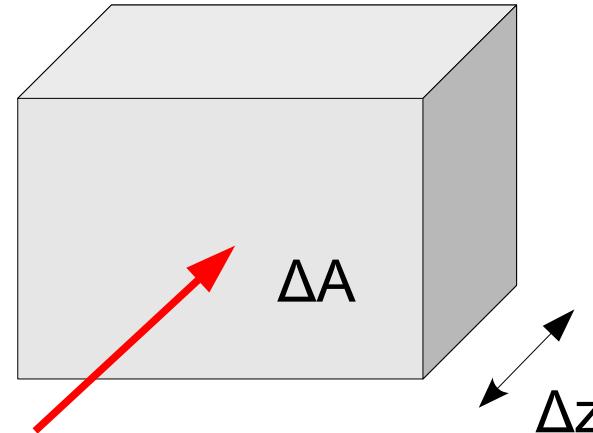
$$\Re [e^{i(\omega_1 t - \beta_1 z)} + e^{i(\omega_2 t - \beta_2 z)}] = 2 \cos(\delta\omega t - \delta\beta z) \cos(\omega_0 t - \beta_0 z)$$

$$v_g = \frac{\delta\omega}{\delta\beta} \quad \rightarrow \quad v_g = \frac{d\omega}{d\beta}$$



## Energy velocity

Energy transported by  $\Delta z$  in time  $\Delta t$ :



$$\frac{\bar{w} \Delta A \Delta z}{\Delta t} = \bar{S}_z \Delta A \quad \rightarrow \quad v_e = \frac{\Delta z}{\Delta t} = \frac{\bar{S}_z}{\bar{w}}$$

for plane waves

$$\bar{S}_z = \frac{1}{2} (\vec{E} \times \vec{H}^*)_z = \frac{|E_0|^2}{2Z}, \quad \bar{w} = \frac{1}{4} \vec{E} \cdot \vec{D}^* + \frac{1}{4} \vec{H} \cdot \vec{B}^* = \frac{1}{2} \epsilon |E_0|^2$$

$$v_e = \frac{1}{Z \epsilon} = \frac{1}{\sqrt{\mu \epsilon}} = c$$

Low-loss dielectrics:  $\epsilon'' \ll \epsilon'$

$$\beta \approx \sqrt{\epsilon_r'} k_0, \quad \alpha \approx \frac{1}{2} \frac{\epsilon_r''}{\sqrt{\epsilon_r'}} k_0, \quad Z \approx \frac{Z_0}{\sqrt{\epsilon_r'}} \left( 1 + \frac{i}{2} \frac{\epsilon_r''}{\epsilon_r'} \right)$$

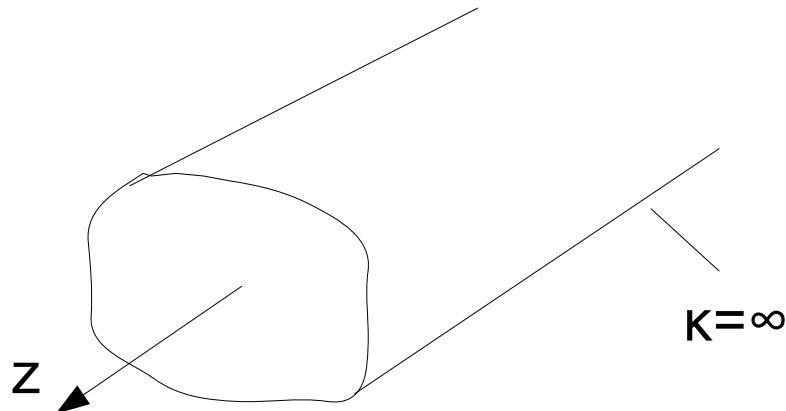
Example: Polyamide (nylon),  $\kappa = 10^{-8} \Omega^{-1}m^{-1}$ ,  $\epsilon_r = 3$ ,  $f = 10\text{MHz}$   
11% attenuation in 100km, arc  $Z \approx 10^{-4} \circ$

Very good conductors (metallic):  $\epsilon'' \approx -i\kappa/\omega \gg \epsilon'$

$$\beta \approx \alpha \approx \sqrt{\frac{\omega \mu \kappa}{2}}, \quad Z \approx (1+i) \frac{\alpha}{\kappa}, \quad \text{arc } Z = 45^\circ$$

Skin depth:  $e^{-\alpha \delta_s} = \frac{1}{e} \rightarrow \alpha \delta_s = 1 \rightarrow \delta_s = \sqrt{\frac{2}{\omega \mu \kappa}}$

# Cylindrical, ideal conducting waveguides



From Maxwell's 3<sup>d</sup> and 4<sup>th</sup> eq.  $\vec{\nabla} \cdot \vec{E} = 0 \rightarrow \vec{E}^{TE} = \vec{\nabla} \times \vec{A}^{TE}$   
 $\vec{\nabla} \cdot \vec{H} = 0 \rightarrow \vec{H}^{TM} = \vec{\nabla} \times \vec{A}^{TM}$

e.g. *TE waves*:

Using  $\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$  yields  $\vec{H} - \epsilon \frac{\partial \vec{A}}{\partial t} = \vec{\nabla} \Phi$

Next using  $\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$  gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = -\mu \vec{\nabla} \frac{\partial \Phi}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$\vec{A}, \Phi$  are not fully determined

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \psi, \quad \Phi \rightarrow \Phi - \epsilon \partial \psi / \partial t$$

give the same  $\vec{E}, \vec{H}$

use Lorenz' gauge

$$\vec{\nabla} \cdot \vec{A} = -\mu \frac{\partial \Phi}{\partial t}$$

yielding a vectorial wave equation

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$



*Similarly , we proceed for the TM – case and obtain the same equation.*

*Since only two independent functions are needed , we choose*

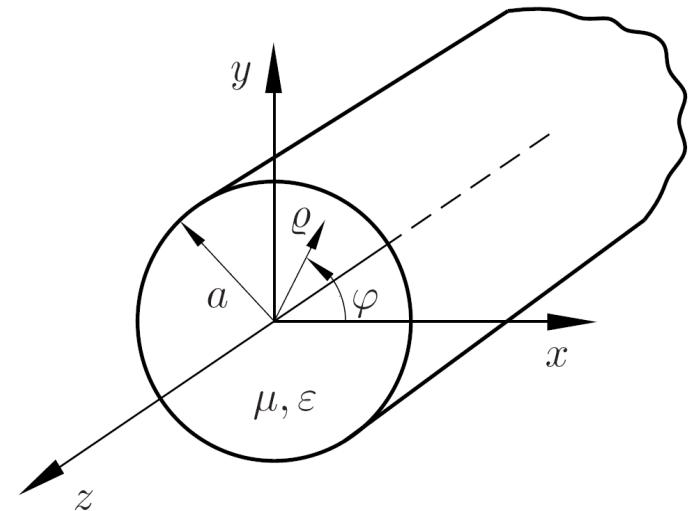
$$\vec{A}^{TE} = A^{TE} \vec{e}_z, \quad \vec{A}^{TM} = A^{TM} \vec{e}_z$$

*which for time – harmonic fields results in a scalar Helmholtz equation*

$$\vec{\nabla}^2 A^p + k^2 A^p = 0, \quad k = \frac{\omega}{c} = \omega \sqrt{\mu \epsilon}, \quad p = \begin{cases} TE \\ TM \end{cases}$$

## Circular waveguide

*Helmholtz equ.  
circular cylinder koordinates:*



$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial A}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A}{\partial \varphi^2} + \frac{\partial^2 A}{\partial z^2} + k^2 A = 0 \quad (1)$$

*Bernoulli ansatz:*  $A = R(\rho)\Phi(\varphi)Z(z)$

substituted in (1) and devision by  $R\Phi Z$

$$\frac{1}{\rho R} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial R}{\partial \rho} \right) + \frac{1}{\rho^2 \Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} + \underbrace{\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}}_{-k_z^2} + k^2 = 0 \quad (2)$$

$$\frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \rightarrow Z = C_1 e^{-ik_z z} + C_2 e^{ik_z z} \rightarrow C_1 e^{-ik_z z}$$

for waves propagating in +z-direction

(2) becomes

$$\frac{\rho}{R} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial R}{\partial \rho} \right) + \underbrace{\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \varphi^2}}_{-k_u^2} + \rho^2 (k^2 - k_z^2) = 0 \quad (3)$$

$$\frac{d^2 \Phi}{d \varphi^2} + k_u^2 \Phi = 0 \rightarrow \Phi = C_3 \cos(k_u \varphi) + C_4 \sin(k_u \varphi)$$

$$\rightarrow k_u = m \rightarrow \Phi = C_3 \cos(m \varphi)$$

because of rotational symmetry  
and  $2\pi$ -periodicity

(2) becomes with  $k_z$  and  $m$

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R}{\partial \rho} + \left[ K^2 - \frac{m^2}{\rho^2} \right] R = 0, \quad K = \sqrt{k^2 - k_z^2}$$

$$R = C_5 J_m(K\rho) + C_6 N_m(K\rho) \rightarrow R = C_5 J_m(K\rho)$$

because Neumann function  
is infinite at  $\rho=0$

*Vector potential:*

$$A = C \cos(m\varphi) J_m(K\rho) e^{-ik_z z}$$

*TE-waves:*  $\vec{E} = \vec{\nabla} \times A \vec{e}_z$

$$E_\varphi = -\frac{\partial A}{\partial \rho} \sim J_m'(K\rho)$$

$$E_\varphi(\rho=a)=0 \rightarrow K_{mn} a = j'_{mn}$$



$$E_{\rho} = \frac{1}{\rho} \frac{\partial A}{\partial \phi} = -\frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$E_{\varphi} = -\frac{\partial A}{\partial \rho} = -\frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$\vec{\nabla} \times \vec{E} = -i\omega\mu H :$$

$$H_{\rho} = \frac{k_z}{\omega\mu} \frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$H_{\varphi} = -\frac{k_z}{\omega\mu} \frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$H_z = -\frac{j'^2_{mn}/a^2}{i\omega\mu} C_{mn} \cos(m\varphi) J_m(j'_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$TM-waves: \quad \vec{H} = \vec{\nabla} \times A \vec{e}_z$$

$$E_z = \frac{K^2}{i\omega\epsilon} A \sim J_m(K\rho), \quad E_z(\rho=a)=0 \rightarrow K_{mn}a=j_{mn}$$

$$H_\rho = -\frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(j_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$H_\varphi = -\frac{j_{mn}}{a} D_{mn} \cos(m\varphi) J'_m(j_{mn} \frac{\rho}{a}) e^{-ik_z z}, \quad H_z = 0$$

$$\vec{\nabla} \times \vec{H} = i\omega\epsilon \vec{E}:$$

$$E_\rho = -\frac{k_z}{\omega\epsilon} \frac{j_{mn}}{a} D_{mn} \cos(m\varphi) J'_m(j_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$E_\varphi = \frac{k_z}{\omega\epsilon} \frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(j_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

$$E_z = \frac{j_{mn}^2/a^2}{i\omega\epsilon} D_{mn} \cos(m\varphi) J_m(j_{mn} \frac{\rho}{a}) e^{-ik_z z}$$

The ratio of the transverse field components is the field (wave) impedance

$$Z_F = \left\{ \begin{array}{l} Z_F^{TE} = \frac{E_\rho}{H_\varphi} = - \frac{E_\varphi}{H_\rho} = \frac{\omega \mu}{k_z} \\ Z_F^{TM} = \frac{E_\rho}{H_\varphi} = - \frac{E_\varphi}{H_\rho} = \frac{k_z}{\omega \epsilon} \end{array} \right\}$$

The dependence of the propagation constant on frequency is the dispersion relation

$$K_{mn}^2 = k^2 - k_{zmn}^2 \rightarrow k_{zmn} = \sqrt{k^2 - K_{mn}^2} = \sqrt{k^2 - k_{cmn}^2}$$

$$k_{zmn} = \begin{cases} \text{real} & k > k_{cmn} \\ 0 & \text{for } k = k_{cmn} \\ \text{imaginary} & k < k_{cmn} \end{cases}$$



*critical wavenumber:*  $k_{cmn} = K_{mn} = \begin{cases} j_{mn}'/a & \text{for } TE \\ j_{mn}/a & \text{for } TM \end{cases}$

*cutoff frequency:*  $f_{cmn} = c k_{cmn} / 2\pi$

*cutoff wavelength:*  $\lambda_{cmn} = 2\pi/k_{cmn}$

*guide wavelength:*  $\lambda_{zmn} = 2\pi/k_{zmn} = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_{cmn})^2}}$

*energy flux density:*

$$\bar{S}_{cz} = \frac{1}{2} (\vec{E} \times \vec{H}^*)_z = \frac{1}{2} Z_F [ |H_x|^2 + |H_y|^2 ]$$

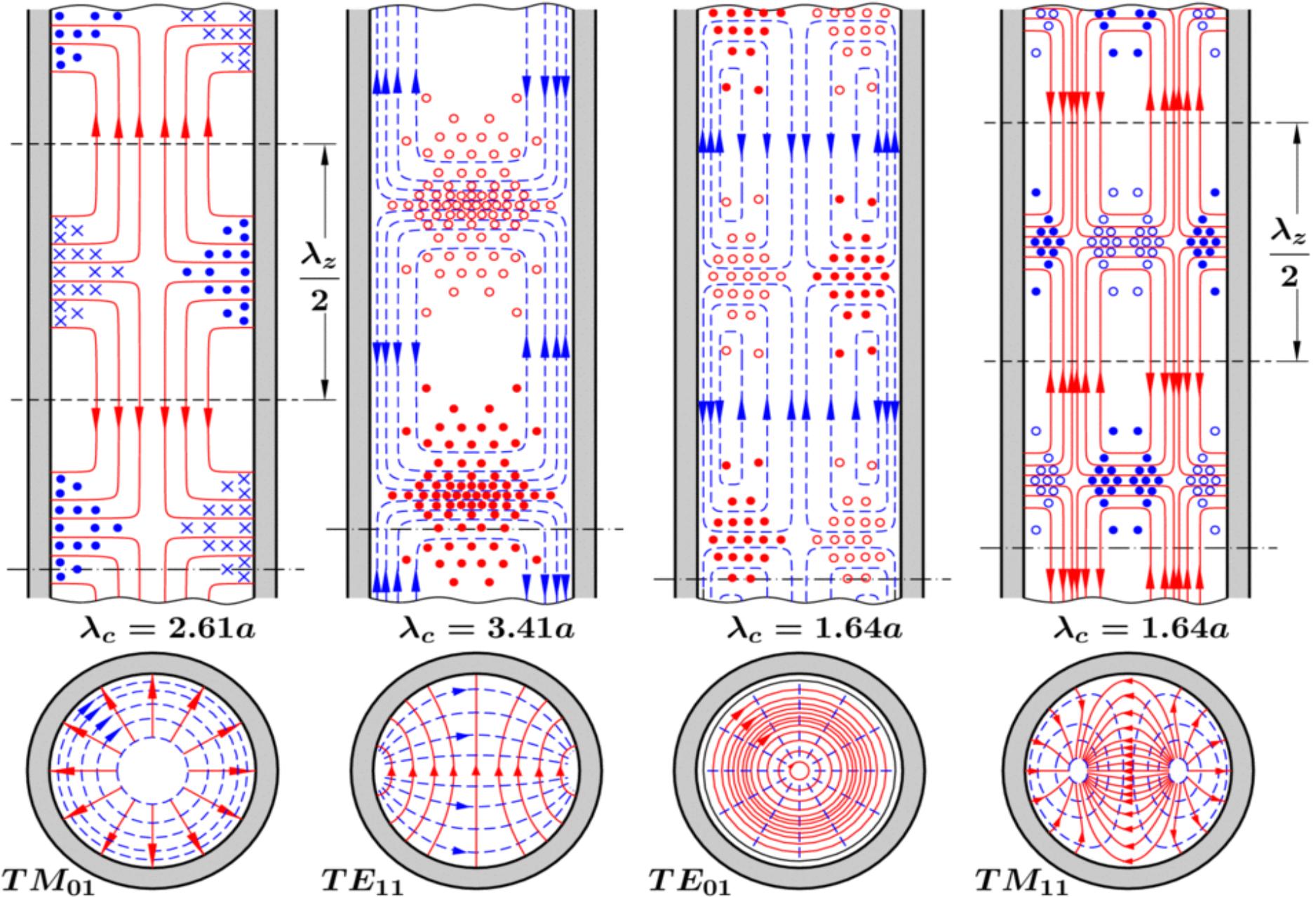
$$= \begin{cases} \text{imaginary} & k < k_c \\ 0 & \text{for } k = k_c \\ \text{real} & k > k_c \end{cases}$$

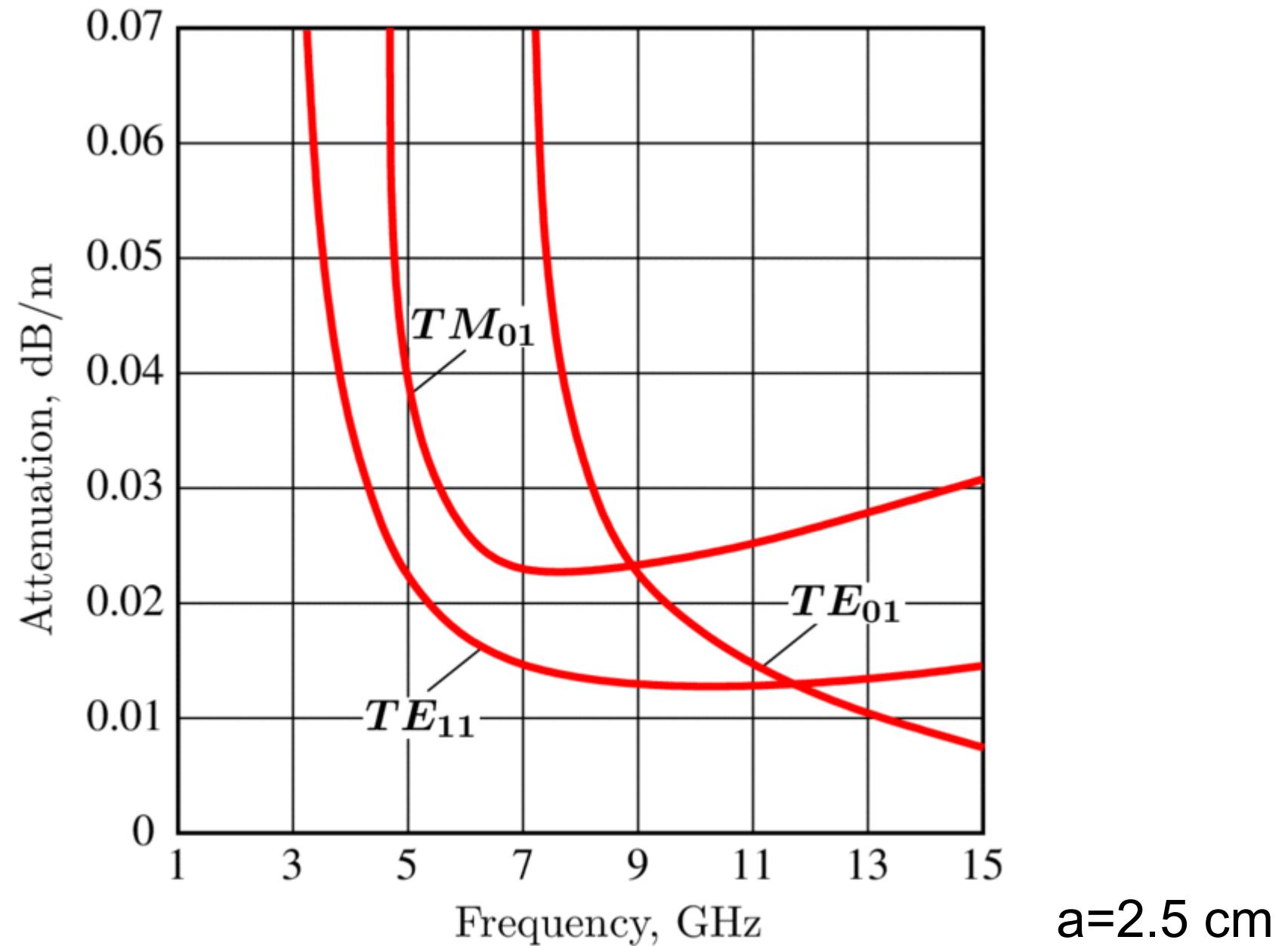
Each  $mn$  defines a certain (eigen-) mode. The general solution is the linear combination of all modes

$$\vec{E} = \sum (\vec{E}_{mn}^{TE} + \vec{E}_{mn}^{TM}), \quad \vec{H} = \sum (\vec{H}_{mn}^{TE} + \vec{H}_{mn}^{TM})$$

Modes are normally sorted referring to their cutoff frequency:

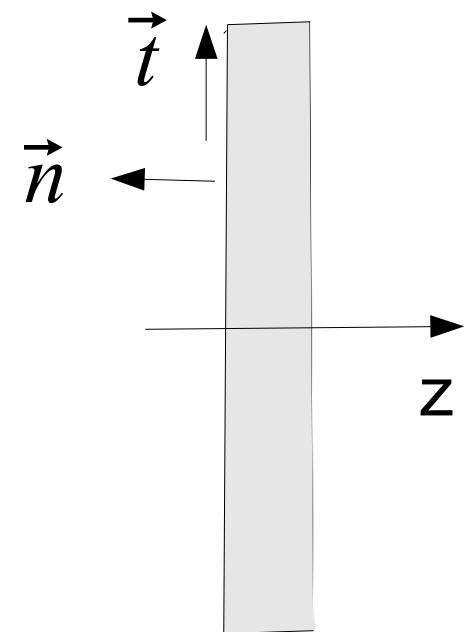
| type  | m   | n   | ( $f_c$ / GHz)(a/cm) |
|-------|-----|-----|----------------------|
| TE    | 1   | 1   | 8.78                 |
| TM    | 0   | 1   | 11.46                |
| TE    | 2   | 1   | 14.56                |
| TE/TM | 0/1 | 1/1 | 18.29                |
| TE    | 3   | 1   | 20.05                |





## Impedance boundary condition on good conductors

On metallic surfaces:  $E \approx \text{perp}$ ,  $H \approx \text{parallel}$



$$\vec{\nabla} \times \vec{H} = J = \kappa \vec{E}$$

$$(\vec{\nabla}_t + \vec{e}_z \frac{\partial}{\partial z}) \times (\vec{H}_t + H_z \vec{e}_z) = \kappa (\vec{E}_t + E_z \vec{e}_z)$$

$$-\vec{e}_z \times \vec{\nabla}_t H_z + \vec{e}_z \times \frac{\partial \vec{H}_t}{\partial z} = \kappa \vec{E}_t$$

$$\left| \frac{\partial}{\partial z} \right| \approx \frac{1}{\delta_s} \gg \left| \vec{\nabla}_t \right| \approx \frac{1}{\lambda_0} \rightarrow \vec{E}_t \approx \frac{1}{\kappa} \vec{e}_z \times \frac{\partial \vec{H}_t}{\partial z} \quad (1)$$

$$\vec{\nabla} \times \vec{H} = \kappa \vec{E}, \quad \vec{\nabla} \times \vec{E} = -i\omega\mu \vec{H} \quad \rightarrow$$

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \vec{\nabla}^2 \vec{H} = -i\omega\mu\kappa \vec{H} \\ \vec{\nabla}^2 \vec{H} - i\omega\mu\kappa \vec{H} &= 0\end{aligned}$$

and since  $H$  is essentially transverse

$$\vec{\nabla}^2 \vec{H}_t - i\omega\mu\kappa \vec{H}_t \approx 0 \quad \rightarrow \quad \vec{H}_t \approx \vec{H}_{t0} e^{-(1+i)z/\delta_s} \quad (2)$$

from (1) and (2)

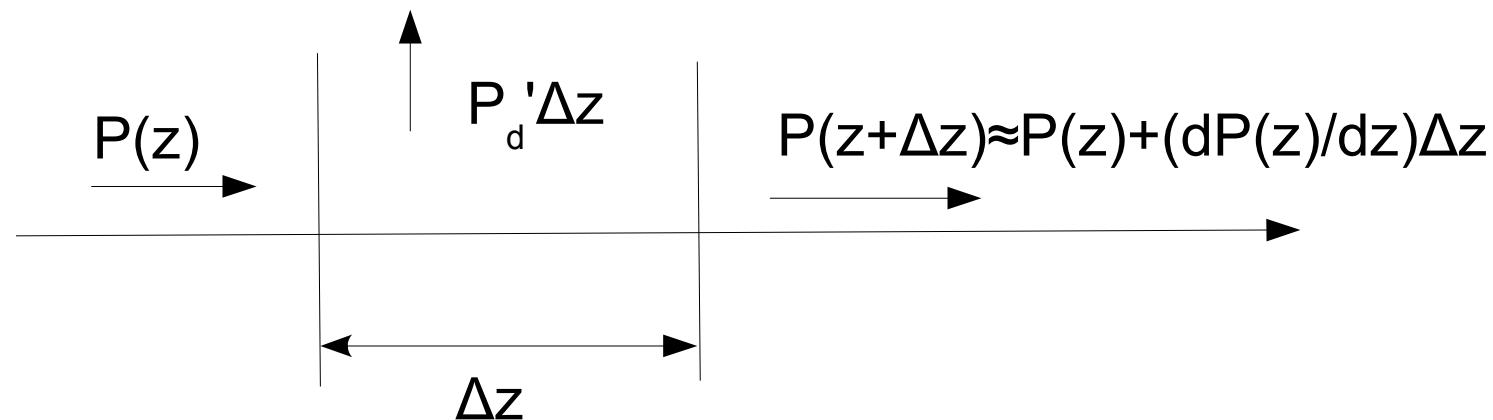
$$\vec{E}_{t0} = \frac{1}{\kappa} \vec{e}_z \times \frac{\partial \vec{H}_{t0}}{\partial z} = Z_W (\vec{n} \times \vec{H}_{t0})$$

$$Z_W = \frac{1+i}{\kappa \delta_s}, \quad \delta_s = \sqrt{\frac{2}{\omega\mu\kappa}}$$



# Attenuation in waveguides

(power-loss method)



*conservation of power:*

$$\frac{dP(z)}{dz} = -P_d'$$

$$\vec{E}, \vec{H} \sim e^{-\alpha z}, P(z) \sim e^{-2\alpha z} \rightarrow \frac{dP(z)}{dz} = -2\alpha P(z) = -P_d'$$

dissipation per surface area:

$$\frac{\Delta P_d}{\Delta A} = -\vec{n} \cdot \Re(\vec{S}_c) = -\frac{1}{2} \Re(\vec{n} \cdot (\vec{E} \times \vec{H}^*)) = \frac{1}{2} \Re(Z_w) |\vec{H}_{t0}|^2$$

$$\frac{\Delta P_d}{\Delta A} = \frac{1}{2 \kappa \delta_s} |\vec{H}_{t0}|^2$$

*dissipation per length:*

$$P_d' = \frac{1}{2 \kappa \delta} \oint |\vec{H}_{t0}|^2 ds$$

*transported power:*

$$P(z) = \iint \Re(\vec{S}_c) \cdot d\vec{A} = \frac{1}{2} \Re(Z_F) \iint |\vec{H}_{trans}|^2 dA$$

$$= \frac{1}{2} Z_F \iint |\vec{H}_{trans}|^2 dA$$

*attenuation:*     $\alpha = \frac{1}{2} \frac{P_d'}{P(z)}$

## Resonant cavities

Example: Cylindrical cavity, radius a, length g, TM-modes

*Forward plus backward traveling wave from transparency 48*

$$E_\varphi = \frac{k_z}{\omega \epsilon} \frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(K_{mn}\rho) [e^{-ik_z z} - r e^{ik_z z}]$$

*Boundary conditions*

$$E_\varphi(z=0) = 0 \quad \rightarrow \quad r_{mn} = 1, \quad E_\varphi \sim \sin(k_z z)$$

$$E_\varphi(z=g) = 0 \quad \rightarrow \quad k_{zp} g = p\pi, \quad p=0, 1, 2, \dots$$

*Fields:*

$$H_{\rho} = -2 \frac{m}{\rho} D_{mnp} \sin(m\varphi) \cos(k_{zp} z) J_m(K_{mn} \rho)$$

$$H_{\varphi} = -2 K_{mn} D_{mnp} \cos(m\varphi) \cos(k_{zp} z) J_m'(K_{mn} \rho), \quad H_z = 0$$

$$E_{\rho} = i 2 \frac{k_{zp}}{\omega \epsilon} K_{mn} D_{mnp} \cos(m\varphi) \sin(k_{zp} z) J_m'(K_{mn} \rho)$$

$$E_{\varphi} = -i 2 \frac{k_{zp}}{\omega \epsilon} \frac{m}{\rho} D_{mnp} \sin(m\varphi) \sin(k_{zp} z) J_m(K_{mn} \rho)$$

$$E_z = -i 2 \frac{K^2}{\omega \epsilon} D_{mnp} \cos(m\varphi) \cos(k_{zp} z) J_m(K_{mn} \rho)$$

$$K_{mn} = \sqrt{k^2 - k_{zp}^2} = \frac{j_{mn}}{a}$$

Example: TM<sub>010</sub>-resonator (m=0, n=1, p=0)

$$H_\varphi = 2 \frac{j_{01}}{a} D_{010} J_1\left(j_{01} \frac{\rho}{a}\right)$$

$$E_z = -i \frac{2}{\omega \epsilon} \left(\frac{j_{01}}{a}\right)^2 D_{010} J_0\left(j_{01} \frac{\rho}{a}\right)$$

*Resonance frequency*

$$k_{010} = \frac{\omega_{010}}{c} = K_{01} = \frac{j_{01}}{a}$$

$$f_{010} = \frac{j_{01} c_0}{2 \pi a}$$

*Stored energy*

$$\bar{W} = \bar{W}_e + \bar{W}_m = 2 \bar{W}_e = \frac{1}{2} \iiint \vec{E} \cdot \vec{D}^* dV = \frac{\epsilon}{2} \iiint |E_z|^2 dV$$

$$\bar{W} = \frac{2\pi g}{\omega^2 \epsilon} \frac{j_{01}^4}{a^2} |D_{010}|^2 J_1^2(j_{01})$$

*Dissipation per unit area*

$$\bar{P}_d'' = \frac{1}{2\kappa\delta_s} |\vec{H}_{\tan}|^2$$

$$\bar{P}_d = \iint \bar{P}_d'' dA = \frac{4\pi}{\kappa\delta_s} j_{01}^2 \left(1 + \frac{g}{a}\right) |D_{010}|^2 J_1^2(j_{01})$$

*Quality factor (Q-value)*

$$Q_0 = \frac{\omega_0 \bar{W}}{\bar{P}_d} = \frac{1}{\delta_s} \frac{g}{1+g/a} \quad \rightarrow \quad \delta_s Q_0 = 2 \frac{V}{S} \quad \sim \frac{V}{S}$$

$Q_0$  gives the decay rate of the stored energy

$$-\frac{d \bar{W}}{dt} = \bar{P}_d = \frac{\omega_0}{Q_0} \bar{W} \quad \rightarrow \quad \bar{W} = \bar{W}_0 e^{-2t/T_f}, \quad T_f = 2 \frac{Q_0}{\omega_0}$$

Example: 3 Ghz copper cavity,  $g=\lambda_0/2=5$  cm

$$j_{01}=2.405, \quad J_1(j_{01})=0.5191, \quad \kappa=58 \cdot 10^6 \Omega^{-1}m^{-1}$$

$$a=3.83 \text{ cm}, \quad \delta_s=1.21 \mu\text{m}, \quad Q_0=17963, \quad T_f=1.9 \mu\text{s}$$



## Resonance behaviour of a cavity mode

Instead of lossy walls assume lossy dielectric filling. That preserves the ideal mode but allows for studying losses.

The cavity is driven by a current  $J$  passing through it.  $J$  splits into a conduction current  $J_c = \kappa E$ , responsible for the losses in the dielectric, and an enforced current  $J_0$  as driving term.

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\mu \frac{\partial}{\partial t} \vec{\nabla} \times \vec{H} = \\ &= -\mu \frac{\partial}{\partial t} (\vec{J}_0 + \kappa \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}) \\ \vec{\nabla}^2 \vec{E} - \mu \kappa \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} &= \mu \frac{\partial \vec{J}_0}{\partial t} \quad (1)\end{aligned}$$

We expand  $\mathbf{E}$  in (eigen-)modes

$$\vec{E} = \sum_n a_n(t) \vec{e}_n(x, y, z) \quad (2)$$

where  $\vec{\nabla}^2 \vec{e}_n + k_n^2 \vec{e}_n = 0$

$$\vec{\nabla} \cdot \vec{e}_n = 0 \text{ in volume}, \quad \vec{n} \times \vec{e}_n = 0 \text{ on walls}$$
$$\iiint \vec{e}_n \cdot \vec{e}_m dV = \delta_m^n$$

Substituting (2) in (1) and deviding by  $\mu\epsilon$

$$\sum_n \left[ \frac{d^2 a_n}{dt^2} + \frac{\kappa}{\epsilon} \frac{da_n}{dt} + \frac{k_n^2}{\mu \epsilon} a_n \right] \vec{e}_n = -\frac{1}{\epsilon} \frac{\partial \vec{J}_0}{\partial t} \quad (3)$$

Multiplying (3) with  $e_m$  and integrating over  $V$

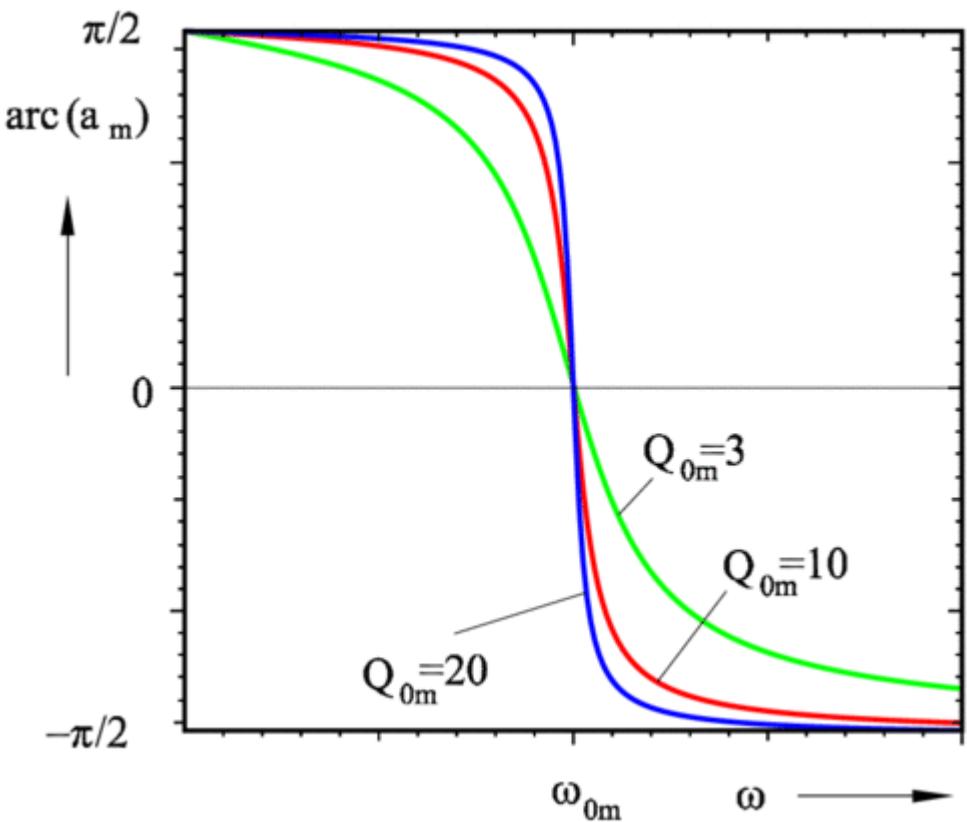
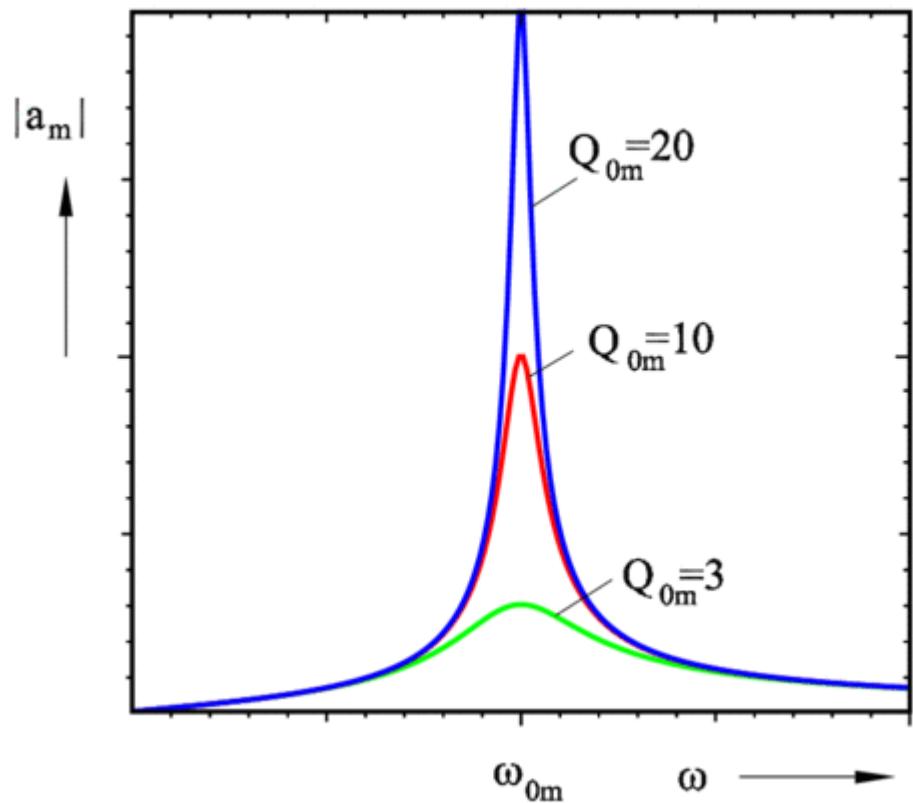
$$\frac{d^2 a_m}{dt^2} + \frac{\kappa}{\epsilon} \frac{da_m}{dt} + \frac{k_n^2}{\mu \epsilon} a_m = -\frac{1}{\epsilon} \iiint \frac{\partial \vec{J}_0}{\partial t} \cdot \vec{e}_m dV = \frac{\partial f_m}{\partial t}$$

Time-harmonic excitation

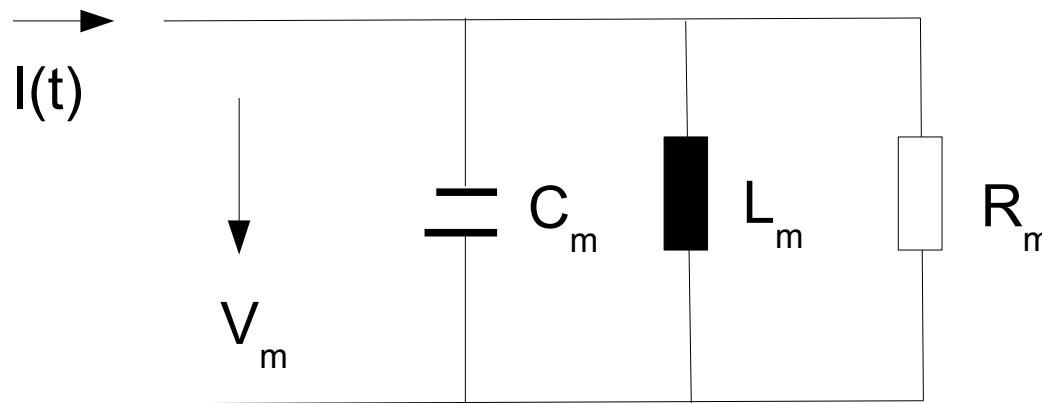
$$[-\omega^2 + i \frac{\kappa}{\epsilon} \omega + \frac{k_m^2}{\mu \epsilon}] a_m = i \omega f_m$$

$$a_m = \frac{Q_{0m}}{\omega_{0m}} \frac{f_m}{1 + i Q_{0m} \left[ \frac{\omega}{\omega_{0m}} - \frac{\omega_{0m}}{\omega} \right]}$$

$$\text{with } \quad \omega_{0m} = c k_m, \quad Q_{0m} = \epsilon \omega_{0m} / \kappa$$



Well separated modes can be represented by a lumped element resonator



$$\omega_{0m} = \frac{1}{\sqrt{L_m C_m}}, \quad Q_{0m} = \frac{\omega_{0m} W_m}{P_{dm}} = \omega_{0m} R_m C_m$$

$$B_m = \frac{(\omega_{0m} + \delta\omega) - (\omega_{0m} - \delta\omega)}{\omega_{0m}} = 2 \frac{\delta\omega}{\omega_{0m}} = \frac{1}{Q_{0m}}$$

*Bandwidth*

$$T_{fm} = 2 \frac{Q_{0m}}{\omega_{0m}} = \frac{1}{\delta\omega}$$

*Filling time*

Accelerating voltage for a particle passing the cavity on-axis

$$V_m = \left| \int_0^g a_m \vec{e}_m \cdot \vec{e}_z e^{i\omega t} dz \right|, \quad z = vt$$

Shunt impedance (amplitude independent)

$$R_{shm} = \frac{V_m^2}{P_{dm}} = 2R_m$$

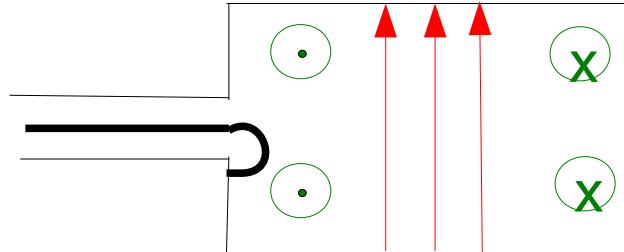
$\omega_{0m}$ ,  $B_m$  and  $R_{shm}$  define  $R_m$ ,  $L_m$ ,  $C_m$ .

R-upon-Q (accelerating voltage for a given stored energy, loss independent)

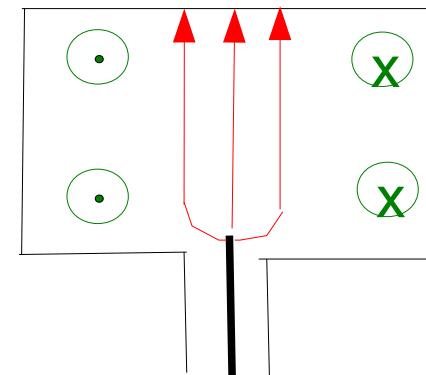
$$\frac{R_{shm}}{Q_{0m}} = \frac{V_m^2}{\omega_{0m} W_m} = \frac{2}{\omega_{0m} C_m}$$

# Coupling to a cavity

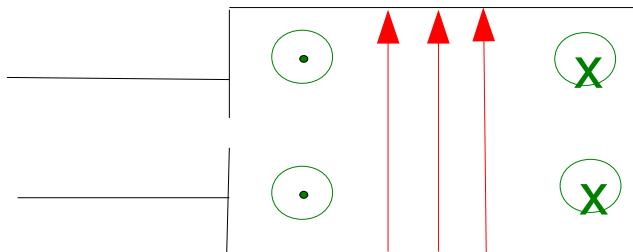
Loop / magnetic coupling



Probe / electric coupling



Electromagnetic coupling



## Tut-Ex 7

Give the E- and H-field of a z-polarized plane wave which propagates in x-direction.

What is the time-averaged radiated power density?

## Tut-Ex 8

Derive the longitudinal vector potential for TM-waves in a rectangular waveguide.

What is the equation for the separation constants?

## Tut-Ex 9

Give the longitudinal wavelength and phase and group velocity of a  $TE_{10}$ -mode in a rectangular waveguide?

## Tut-Ex 10

What is the lowest mode in a circular waveguide?  
Show the field pattern.  
In which frequency range is mono-mode operation  
possible?

## Tut-Ex 11

Calculate the accelerating voltage, shunt impedance  
and R-upon-Q of a  $\text{TM}_{010}$ -mode pill-box cavity.