

Introduction to Transverse Beam Dynamics

Lecture 2: Particle trajectories & beams

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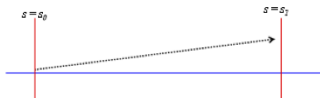
Reminder of lecture 1

Equation of motion:

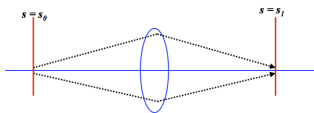
$$x'' + Kx = 0 \quad \begin{array}{ll} K = 1/\rho^2 - k & \dots \text{horiz. plane} \\ K = k & \dots \text{vert. plane} \end{array}$$

Solution of trajectory equations:

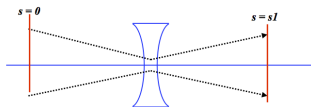
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



$$M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$



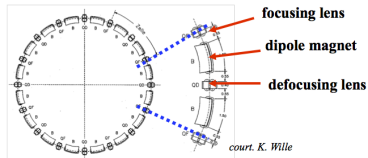
$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ \sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix}$$

Transformation through a system of lattice elements

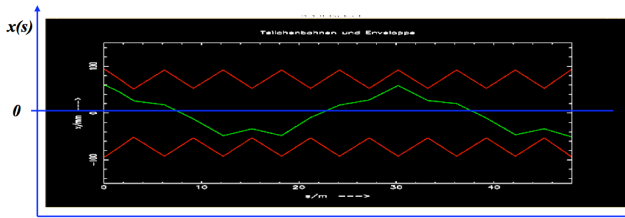
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \dots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \rightarrow s_2} \cdot M_{s_0 \rightarrow s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

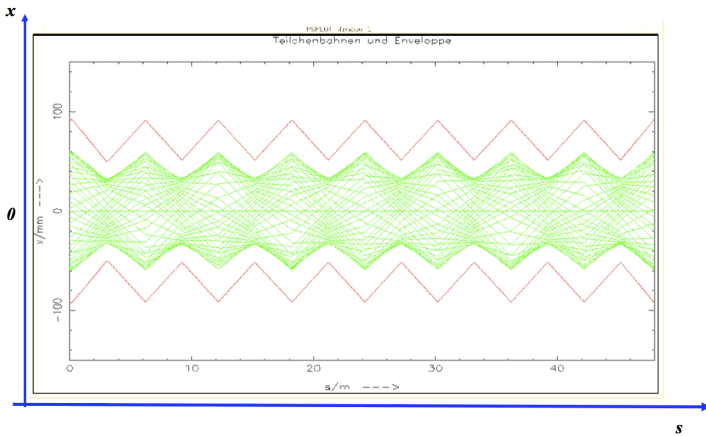
$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

Envelope

Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



The Hill's equation

In 19th century George William Hill, one of the greatest master of celestial mechanics of his time, studied the differential equation for “motions with periodic focusing properties”: the “Hill's equation”

$$x''(s) + K(s)x(s) = 0$$

with:

- ▶ a restoring force $\neq \text{const}$
- ▶ $K(s)$ depends on the position s
- ▶ $K(s + L) = K(s)$ periodic function, where L is the “lattice period”

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The beta function

General solution of Hill's equation:

$$x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \quad (1)$$

ϵ, ϕ = integration constants determined by initial conditions

$\beta(s)$ is a periodic function given by the focusing properties of the lattice \leftrightarrow quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$\psi(s)$ is the “phase advance” of the oscillation between the points 0 and s in the lattice. For one complete revolution, $\psi(s)$ is the number of oscillations per turn, the “tune”

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

ϵ is the “emittance” also known as Courant-Snyder invariant.

Beam emittance and phase-space ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) & (1) \\ x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \} & (2) \end{cases}$$

From Eq. (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\epsilon} \sqrt{\beta(s)}} \quad \alpha(s) = -\frac{1}{2} \beta'(s)$$
$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

Insert into Eq. (2) and solve for ϵ

$$\epsilon = \gamma(s) x(s)^2 + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

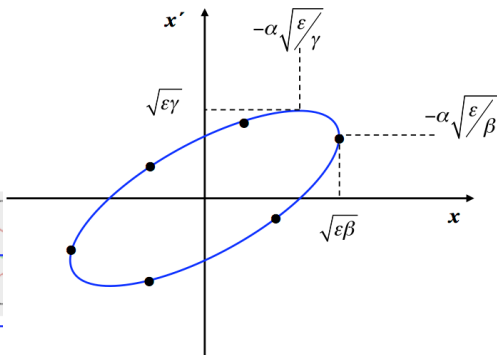
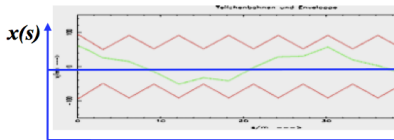
- ▶ ϵ is a constant of the motion, independent of s
- ▶ it is a parametric representation of an ellipse in the xx' space
- ▶ the shape and the orientation of the ellipse are given by α , β , and $\gamma \Rightarrow$ these are the Twiss parameters

Beam emittance and phase-space ellipse

$$\epsilon = \gamma(s) x(s)^2 + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

Liouville: in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

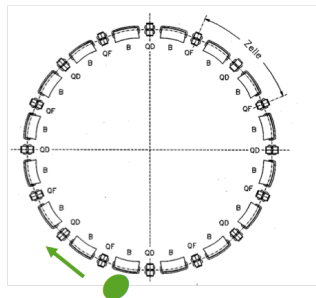
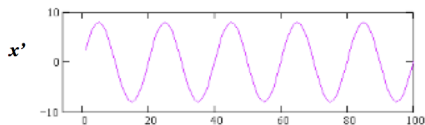
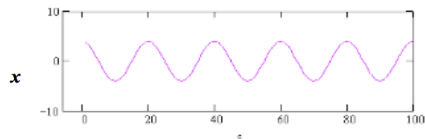
$$A = \pi \cdot \epsilon = \text{const}$$



ϵ beam emittance = area of ellipse. It is an intrinsic beam parameter and cannot be changed by the focal properties.

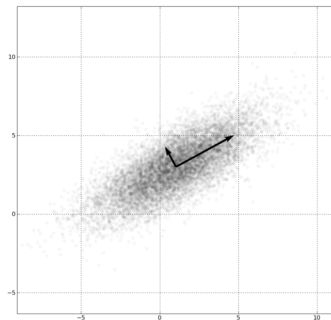
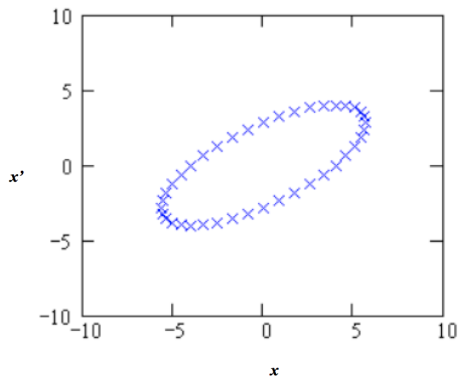
Particle tracking in a storage ring

Computation of x and x' for each linear element, according to matrix formalism. We plot x and x' as a function of s



Particle tracking and the beam ellipse

For each turn x, x' at a given position s_1 and plot in the phase-space diagram



Plane: $x - x'$

The emittance and the phase-space ellipse

Given the particle trajectory:

$$x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi)$$

- ▶ the max. amplitude is:

$$\hat{x}(s) = \sqrt{\epsilon\beta}$$

- ▶ the corresponding angle, in $\hat{x}(s)$, can be found putting $\hat{x}(s) = \sqrt{\epsilon\beta}$ in Eq.

$$\epsilon = \gamma(s) x(s)^2 + 2\alpha(s) x(s) x'(s) + \beta(s) x'(s)^2$$

and solving for x' :

$$\epsilon = \gamma \cdot \epsilon\beta + 2\alpha\sqrt{\epsilon\beta} \cdot x' + \beta x'^2$$

$$\rightarrow \hat{x}' = -\alpha \sqrt{\frac{\epsilon}{\beta}} \leftarrow$$

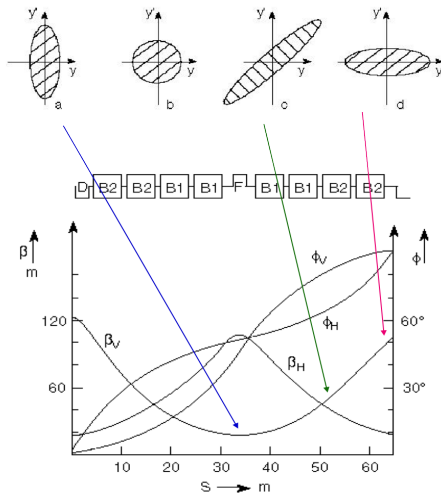
Important remarks:

- ▶ A large β -function corresponds to a large beam size and a small beam divergence
- ▶ In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$

The emittance and the phase-space ellipse

Let's repeat the remarks:

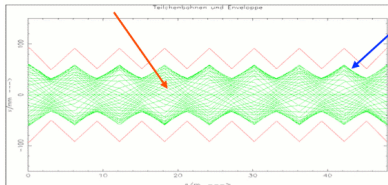
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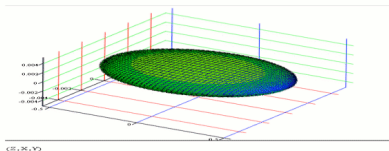
Emittance of an ensemble of particles

$$x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi)$$

$$\hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch



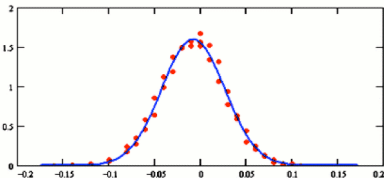
Gauss Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

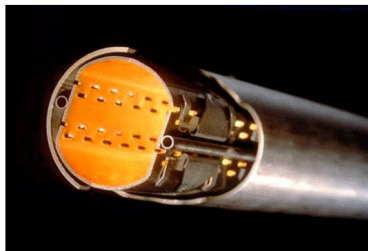
particle at distance 1σ from centre \leftrightarrow 68.3 % of all beam particles

vertical:

$$\sigma_{\text{fit}} = 24.376 \cdot \mu\text{m}$$

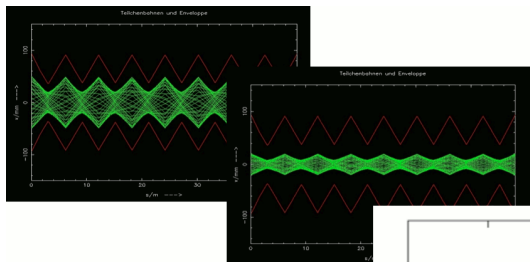


LHC: $\sigma = \sqrt{\varepsilon \cdot \beta} = \sqrt{5 \cdot 10^{-10} \text{ m} \cdot 180 \text{ m}} = 0.3 \text{ mm}$



aperture requirements: $r_0 \geq 10 \cdot \sigma$

Emittance of an ensemble of particles



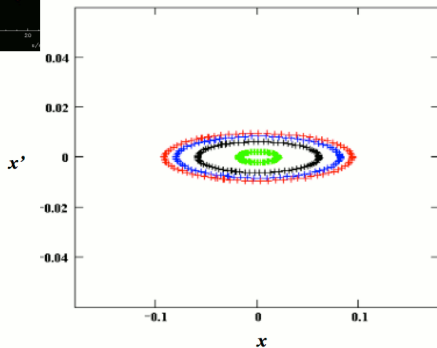
Example: LHC

beam parameters in the arc

$$\beta(x) \approx 180 \text{ m}$$

$$\varepsilon \approx 5 \cdot 10^{-10} \text{ rad} \cdot \text{m} \quad (\Leftrightarrow 1\sigma)$$

$$\sigma = \sqrt{\varepsilon\beta} \approx 0.3 \text{ mm}$$



The beam matrix

To track a beam of particles, let's assume with Gaussian distribution, the beam ellipse can be characterised by a "beam matrix" Σ

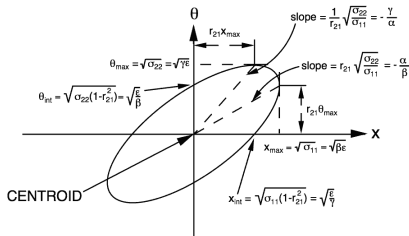
The equation of an ellipse can be written in matrix form:

$$X^T \Sigma X = 1$$

with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$,

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

Σ is the covariance matrix of the particles distribution



- The area of the ellipse is

$$A = \pi \sqrt{\det \Sigma} = \pi \epsilon$$

with slope $r_{21} = \sigma_{21} / \sqrt{\sigma_{11}\sigma_{22}}$.

- ϵ is again the beam emittance. For an ensemble of particles, ϵ is the area covered by the particles in the transverse x - x' phase-space, and it is preserved due to the Liouville's theorem.

The transfer matrix M

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\epsilon\beta(s)} \cos(\psi(s) + \phi)$$

$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}} [\alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)]$$

Let's remember some trigonometric formulæ:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b, \dots$$

then,

$$x(s) = \sqrt{\epsilon\beta(s)} (\cos \psi(s) \cos \phi - \sin \psi(s) \sin \phi)$$

$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}} [\alpha(s) (\cos \psi(s) \cos \phi - \sin \psi(s) \sin \phi) + \sin \psi(s) \cos \phi + \cos \psi(s) \sin \phi]$$

At the starting point, $s(0) = s_0$, we put $\psi(0) = 0$. Therefore we have

$$\cos \phi = \frac{x_0}{\sqrt{\epsilon \beta_0}}$$

$$\sin \phi = -\frac{1}{\sqrt{\epsilon}} \left(x'_0 \sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}} \right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x}(s) = \sqrt{\frac{\beta_s}{\beta_0}} \{ \cos \psi_s + \alpha_0 \sin \psi_s \} \underline{x_0} + \left\{ \sqrt{\beta_s \beta_0} \sin \psi_s \right\} \underline{x'_0}$$

$$\underline{x}'(s) = \frac{1}{\sqrt{\beta_s \beta_0}} \{ (\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s \} \underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}} \{ \cos \psi_s - \alpha_s \sin \psi_s \} \underline{x'_0}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

- We can compute the single particle trajectories between two locations in the ring, if we know the α , β , and γ at these positions!

Periodic lattices

The transfer matrix for a particle trajectory

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

simplifies considerably if we consider one complete turn...



$$M = \begin{pmatrix} \cos \psi_L + \alpha_s \sin \psi_L & \beta_s \sin \psi_L \\ -\gamma_s \sin \psi_L & \cos \psi_L - \alpha_s \sin \psi_L \end{pmatrix}$$

where ψ_L is the phase advance per period

$$\psi_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember: the tune is the phase advance in units of 2π :

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} = \frac{\psi_L}{2\pi}$$

Stability criterion

We have seen that if our particles perform one complete turn, the transport map is

$$\begin{aligned} M &= \begin{pmatrix} \cos \psi_L + \alpha_s \sin \psi_L & \beta_s \sin \psi_L \\ -\gamma_s \sin \psi_L & \cos \psi_L - \alpha_s \sin \psi_L \end{pmatrix} = \\ &= \cos \psi_L \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin \psi_L \underbrace{\begin{pmatrix} \alpha_s & \beta_s \\ -\gamma_s & -\alpha_s \end{pmatrix}}_{\mathbf{J}} \end{aligned}$$

What happens if we consider N turns?

$$M^N = (\mathbf{I} \cos \psi_L + \mathbf{J} \sin \psi_L)^N = \mathbf{I} \cos N\psi_L + \mathbf{J} \sin N\psi_L$$

\Rightarrow The motion for N turns remains bounded if the elements of M^N remain bounded:

$$\psi_L \in \mathbf{R} \quad |\cos \psi_L| < 1 \quad |\text{trace}(M)| < 2$$

Stability criterion ... demonstrated

Matrix for 1 turn:

$$M = \underbrace{\cos \psi_L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \underbrace{\sin \psi_L \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for 2 turns:

$$\begin{aligned} M^2 &= (\mathbf{I} \cos \psi_1 + \mathbf{J} \sin \psi_1) (\mathbf{I} \cos \psi_2 + \mathbf{J} \sin \psi_2) \\ &= \mathbf{I}^2 \cos \psi_1 \cos \psi_2 + \mathbf{IJ} \cos \psi_1 \sin \psi_2 + \mathbf{JI} \sin \psi_1 \cos \psi_2 + \mathbf{J}^2 \sin \psi_1 \sin \psi_2 \end{aligned}$$

now

$$\mathbf{I}^2 = \mathbf{I}$$

$$\mathbf{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$

$$\mathbf{JI} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$

$$\mathbf{J}^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

which brings us to

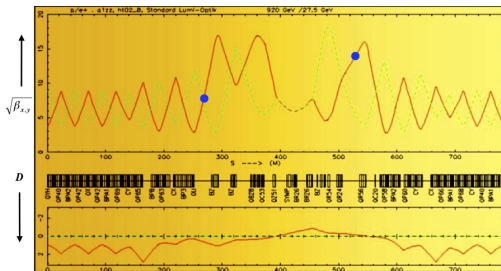
$$\begin{aligned} M^2 &= \mathbf{I} \cos(\psi_1 + \psi_2) + \mathbf{J} \sin(\psi_1 + \psi_2) = \mathbf{I} \cos(2\psi) + \mathbf{J} \sin(2\psi) \\ \Rightarrow M^N &= \mathbf{I} \cos(N\psi) + \mathbf{J} \sin(N\psi) \end{aligned}$$

The transformation for α , β , and γ

Consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with} \quad M = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \dots$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \quad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$



Since the Liouville theorem holds, $\epsilon = \text{const}$:

$$\epsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\epsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x'_0 as a function of x and x' :

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_s \Rightarrow \begin{aligned} x_0 &= S'x - Sx' \\ x'_0 &= -C'x + Cx' \end{aligned}$$

Inserting into ϵ we obtain:

$$\epsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\epsilon = \beta_0 (-C'x + Cx')^2 + 2\alpha_0 (S'x - Sx') (-C'x + Cx') + \gamma_0 (S'x - Sx')^2$$

We need to sort by x and x' :

$$\beta(s) = C^2 \beta_0 - 2SC\alpha_0 + S^2 \gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2 \beta_0 - 2S'C'\alpha_0 + S'^2 \gamma_0$$

The transformation for α , β , and γ

In matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

1. this expression is important, and useful
2. given the twiss parameters α , β , γ at any point in the lattice we can transform them and compute their values at any other point in the ring
3. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories
4. go back to point 1.

Beam matrix and Twiss parameters

The beam matrix is the covariance matrix of the particle distribution

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

this matrix can be also expressed in terms of Twiss parameters α , β , γ and ϵ :

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

We can transport the beam matrix, or the twiss parameters, from 0 to s by two equivalent ways:

- Twiss 3×3 transport matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

- Using the transfer matrix $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \rightarrow s}$:

$$\Sigma_s = M \Sigma_0 M^T$$

Summary

equation of motion: $x''(s) + K(s)x(s) = 0, \quad K = 1/\rho - k$

general solution of the

Hill's equation: $x(s) = \sqrt{\epsilon\beta(s)} \cos(\psi(s) + \phi)$

phase advance & tune: $\psi_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$

emittance: $\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$

transfer matrix $s_1 \rightarrow s_2$:
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

matrix for 1 turn:
$$M = \begin{pmatrix} \cos \psi_L + \alpha_s \sin \psi_L & \beta_s \sin \psi_L \\ -\gamma_s \sin \psi_L & \cos \psi_L - \alpha_s \sin \psi_L \end{pmatrix}$$

stability criterion: $|\text{trace}(M)| < 2$

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