Introduction to Transverse Beam Dynamics

Lecture 4: Dispersion / Errors in fields and gradient / Chromaticity correction

Andrea Latina (CERN)

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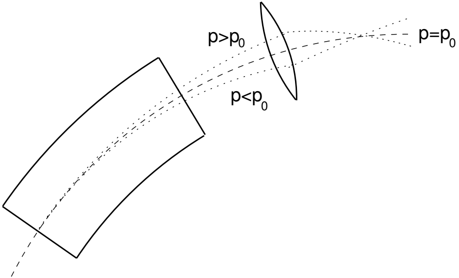
Re-Introducing the dispersion

So far we have studied monochromatic beams of particles, but this is slightly unrealistic: We always have some (small) momentum spread among all particles: $\Delta p = p - p_0 \neq 0$.

Consider three particles with p respectively: less than, greater than, and equal to p_0 , traveling through a dipole. Remembering $B\rho = \frac{p}{q}$:



The system introduces a correlation of momentum with transverse position. This correlation is known as **dispersion** (an intrinsic property of the dipole magnets).

Dispersion function and orbit

We need to study the motion for particles with $\Delta p = p - p_0 \neq 0$:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The general solution of this equation is:

$$x(s) = x_\beta(s) + x_D(s) \qquad \begin{cases} x_\beta''(s) + K(s)\,x_\beta(s) = 0 \\ D''(s) + K(s)\,D(s) = \dfrac{1}{\rho} \end{cases}$$

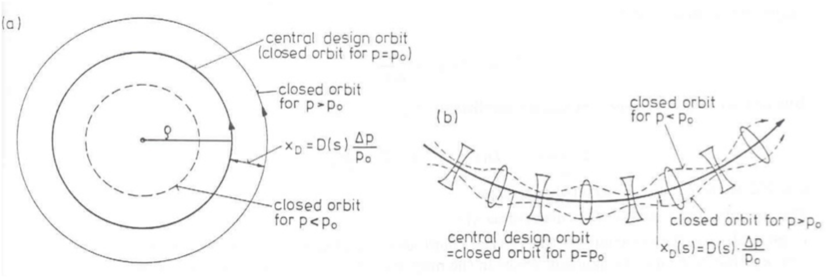
with $x_D(s) = D(s)\,\frac{\Delta p}{p_0}$.

Remarks

- $D(s)$ is that special orbit that a particle would have for $\Delta p/p = 1$
- $x_D(s)$ describes the deviation from the new **closed orbit** for an off-momentum particle with a certain Δp
- the orbit of a generic particle is the sum of the well known $x_\beta(s)$ and $x_D(s)$

Understanding the solution $x(s) = x_\beta(s) + x_D(s)$

with $x_D(s) = D(s)\frac{\Delta p}{p_0}$.



Closed orbit for particles with momentum $p \neq p_0$ in
a weakly (a) and strongly (b) focusing circular accelerator.

- $x_D(s) = D(s)\frac{\Delta p}{p_0}$ describes the deviation of the closed orbit for off-momentum particles with a fixed Δp from the reference orbit
- $x_\beta(s)$ describes the betatron oscillation around this dispersive closed orbit

Dispersion and orbit propagation

Inside a magnet, the dispersion trajectory is solution of $D''(s) + K(s)\,D(s) = \frac{1}{\rho}$:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

Now the orbit:

$$x(s) = x_\beta(s) + x_D(s)$$
$$x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\frac{\Delta p}{p}$$

In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D_0' = 0$.

Dispersion examples

Let's study, for different magnetic elements, the solution of:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

at the exit of the element: that is, $D(s)$ with $s = L_{\text{magnet}}$

- Drift space:

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$C(t) = 1,\ S(t) = L,\ \rho(t) = \infty \quad \Rightarrow \text{the integrals cancel}$$

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function: sector dipole

- Sector dipole:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

which gives

$$D(s) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$

$$D'(s) = \sin\frac{L}{\rho}$$

therefore

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & \rho\left(1-\cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion function: quadrupole

- Focusing quadrupole, $K > 0$:

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Defocusing quadrupole, $K < 0$:

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Dispersion propagation through the machine

- The equation:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

  shows **the dispersion inside a magnet**, which does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length L_m the dispersion reaches the value $D(L_\mathrm{m})$

- ...then it propagates from there on through the rest of the machine, just like any other particle:

$$\begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_0$$

(to avoid confusion, $\tilde{D}$ and $\tilde{D}'$, are often called η and η')

Closed orbit for an off-momentum particle

In a periodic lattice, also the dispersion must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we want:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta\prime \\ 1 \end{pmatrix}$$

Let's rewrite this in 2×2 form:

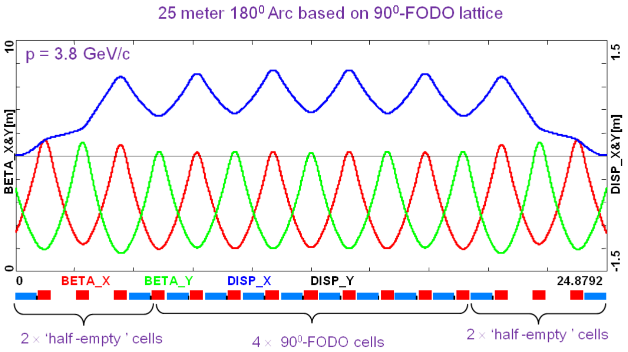
$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1-C & -S \\ -C' & 1-S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1-C)(1-S')-C'S} \begin{pmatrix} 1-S' & S \\ C' & 1-C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

Example of dispersion function in a FODO lattice



Aperture radius: r = 15 cm

12 × Dipoles:	field: 3.9 Tesla	length: 85 cm
15 × Quads:	gradient : 25 Tesla/m (3.8 Tesla at the pole)	length: 50 cm

Dispersion in a FODO cell: $D^{\pm} = \frac{L\phi\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}}$. The dispersion has a maximum at the focusing quadrupoles.

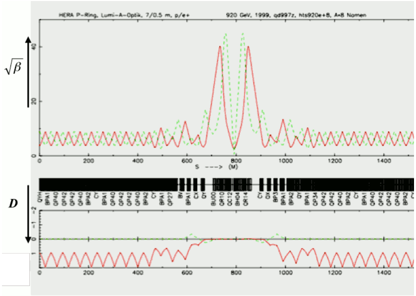
Example of dispersion function

In this example from the HERA storage ring (DESY) we see the twiss parameters and the dispersion near the interaction point. In the periodic region,

$$x_\beta(s) = 1 \ldots 2 \text{ mm}$$
$$D(s) = 1 \ldots 2 \text{ m}$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta p}{p}$$



Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \left(D \cdot \frac{\Delta p}{p}\right)^2} = \sqrt{\epsilon\beta + \left(D \cdot \frac{\Delta p}{p}\right)^2}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*

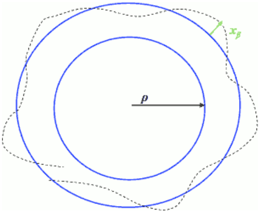
The momentum compaction factor

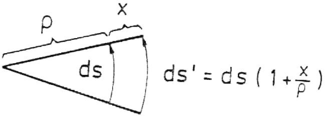
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$x\,(s) = x_\beta\,(s) + D\,(s)\,\frac{\Delta p}{p}$$

The dispersion changes also the length of the off-energy orbit.





particle with offset x w.r.t. the design orbit:

$$\frac{\mathrm{d}s'}{\mathrm{d}s} = \frac{\rho + x}{\rho} \qquad \rightarrow \qquad \mathrm{d}s' = \left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

The circumference change is ΔC, that is $C' = \oint \left(1 + \frac{x}{\rho}\right)\mathrm{d}s = C + \Delta C$

We define the "momentum compaction factor", α, such that:

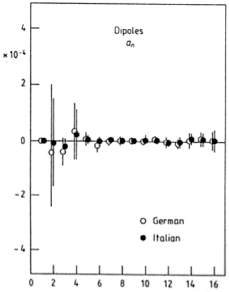
$$\frac{\Delta C}{C} = \alpha\frac{\Delta p}{p} \qquad \rightarrow \text{a rough estimate is } \alpha = \frac{1}{Q_x^2}$$

Magnetic imperfections

HIgh order multipolar components and misalignments

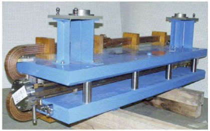
Taylor expansion of the B field:

$$B_y(x) = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}}_{\text{quad}} x + \frac{1}{2}\underbrace{\frac{\partial^2 B_y}{\partial x^2}}_{\text{sextupole}} x^2 + \frac{1}{3!}\underbrace{\frac{\partial^3 B_y}{\partial x^3}}_{\text{octupole}} x^3 + \ldots \qquad \text{divide by } B_{y0}$$



There can be undesired multipolar components, due to small fabrication defects

Or also errors in the windings, in the gap h, ... remember: $B = \dfrac{\mu_0 n I}{h}$



Moreover: **"feed-down"** effect $\Rightarrow$ a misalign magnet of order n, behaves like a magnet of order n + a magnet of order $n-1$ overlapped

Dipole magnet errors

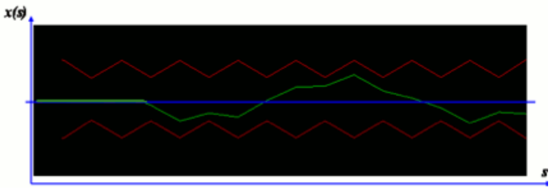
Let's imagine to have a magnet with $B_x = B_0 + \Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$F_x = ev\,(B_0 + \Delta B)\,; \qquad \Delta x' = \Delta B ds / B\rho$$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

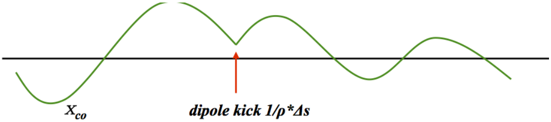
Example: 1 single dipole error

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{\text{lattice}} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_0$$



In order to have bounded motion the tune Q must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum p_0 an integer Q value is forbidden, since small field errors are always present.

Orbit distortion for a single dipole field error



We consider a single thin dipole field error at the location $s = s_0$, with a kick angle $\Delta x'$.

$$X_- = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}, \quad X_+ = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

are the phase space coordinates before and after the kick located at s_0. The closed-orbit condition becomes

$$M_{\text{Lattice}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}$$

The resulting closed orbit at s_0 is

$$x_0 = \frac{\beta_0 \Delta x'}{2 \sin \pi Q} \cos \pi Q; \quad x_0' = \frac{\Delta x'}{2 \sin \pi Q} (\sin \pi Q - \alpha_0 \cos \pi Q)$$

where Q is the tune. The orbit at any other location s is

$$x(s) = \frac{\sqrt{\beta_s \beta_0}}{2 \sin \pi Q} \cos \left(\pi Q - |\mu_s - \mu_0| \right) \Delta x'$$

(see the references for a demonstration)

Orbit distortion for distributed dipole field errors

One single dipole field error

$$x(s) = \frac{\sqrt{\beta(s)\,\beta(0)}}{2\sin\pi Q}\cos\left(\pi Q - |\mu(s) - \mu(0)|\right)\Delta x'$$

Distributed dipole field errors

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint\sqrt{\beta(t)}\cos\left(\pi Q - |\mu(s) - \mu(t)|\right)\Delta x' \mathrm{d}t$$

- orbit distortion is visible at any position s in the ring, even if the dipole error is located at one single point s_0
- the β function describes the sensitivity of the beam to external fields
- the β function acts as amplification factor for the orbit amplitude at the given observation point
- there is a singularity at the denominator when Q integer $\Rightarrow$ it's called resonance

The resonances

Closed orbit distortion due to distributed dipole field errors:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint\sqrt{\beta(t)}\cos\left(\pi Q - |\mu(s) - \mu(t)|\right)\Delta x' \mathrm{d}t$$

Remember the definition of tune:

$$Q = \frac{\mu_L}{2\pi}$$

$\Rightarrow$ it is the phase advance for a revolution μ_L in units of 2π.

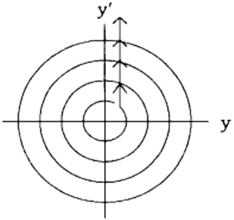
Extremely important:

- In case of imperfections the orbit becomes unstable for Q integer
- Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error!

Tunes and resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.



ν_y = integer

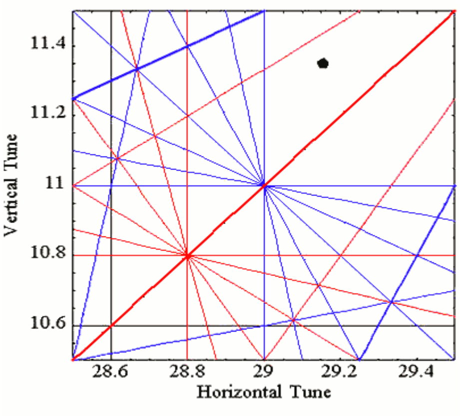
The cure:

1. **avoid** large magnet errors
2. **avoid forbidden tune values** in both planes

$$\mathrm{m}\cdot Q_x + \mathrm{n}\cdot Q_y \neq \mathrm{p}$$

with m, n, p integer numbers

Resonance diagram



A resonance diagram for the Diamond light source. The lines shown are the resonances and the black dot shows a suitable place where the machine could be operated.

Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$M_{\mathrm{Perturbed}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k \mathrm{d}s & 1 \end{pmatrix}}_{\text{perturbation}} \underbrace{\begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}}_{\text{ideal ring}}$$

Let's see how the tunes changes: one-turn map

$$M_{\mathrm{Perturbed}} = \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ \Delta k \mathrm{d}s\left(\cos\mu_0 + \alpha\sin\mu_0\right) - \gamma\sin\mu_0 & \Delta k \mathrm{d}s \beta\sin\mu_0 + \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}$$

Remember the rule for computing the tune:

$$2\cos\mu = \mathrm{trace}\left(M\right) = 2\cos\mu_0 + \Delta k \mathrm{d}s \beta\sin\mu_0$$

Quadrupole errors: tune shift (cont.)

We rewrite $\cos\mu = \cos(\mu_0 + \Delta\mu)$

$$\cos(\mu_0 + \Delta\mu) = \cos\mu_0 + \frac{1}{2}\Delta k \mathrm{d}s \beta \sin\mu_0$$

from which we can compute that

$$\Delta\mu = \frac{\Delta k \mathrm{d}s \beta}{2} \quad \text{shift in the phase advance}$$

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta k(s)\beta(s)\,\mathrm{d}s}{4\pi} \quad \text{tune shift}$$

Important remarks:

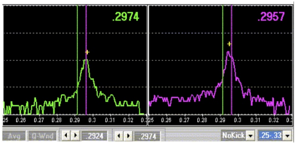
- the tune shift if proportional to the β-function at the quadrupole
  - field quality, power supply tolerances etc. are much tighter at places where β is large
- β is a measurement of the sensitivity of the beam

Quadrupole errors: tune shift example

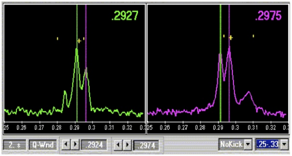
Deliberate change of a quadrupole strength in a synchrotron:

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta K(s)\,\beta(s)\,\mathrm{d}s}{4\pi} \approx \frac{\Delta K(s)\; L_{\mathrm{quad}}\,\overline{\beta}}{4\pi}$$

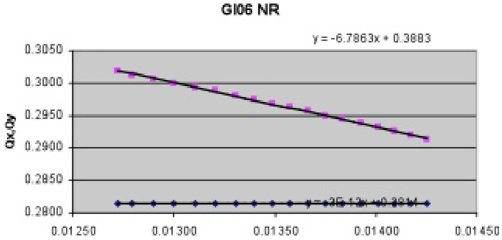
$$\Rightarrow$$



the tune is measured permanently



After changing the strength of a quad: we get a second peak

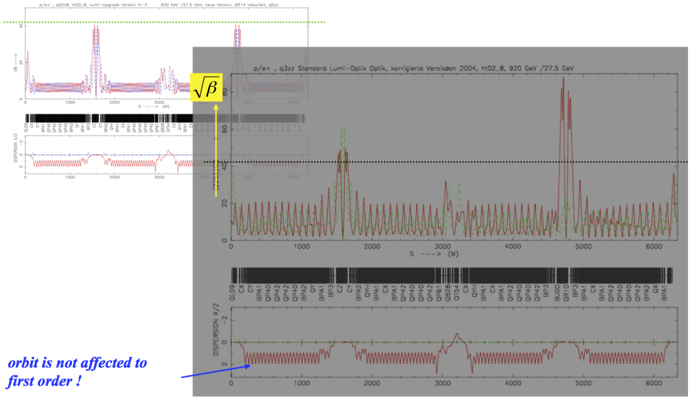


Quadrupole errors: beta beat

A quadrupole error at s_0 causes distortion of β-function at s: $\Delta\beta(s)$ due to the errors of all quadrupoles:

$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2\sin 2\pi Q} \oint \beta(t)\,\Delta k(t) \cos\left(2\pi Q - 2\left(\mu(t) - \mu(s)\right)\right) \mathrm{d}t$$

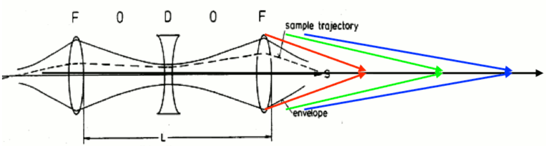
affects the element M_{12} of the M matrix.



orbit is not affected to first order !

Quadrupole errors: chromaticity, ξ

Is an error (optical aberration) that happens in quadrupoles when $\Delta p/p \neq 0$:



The chromaticity ξ is the variation of tune ΔQ with the relative momentum error:

$$\Delta Q = \xi \frac{\Delta p}{p_0} \quad \Rightarrow \quad \xi = \frac{\Delta Q}{\Delta p/p}$$

Remember the quadrupole strength:

$$k = \frac{g}{p/e} \qquad \text{with } p = p_0 + \Delta p$$

then

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

Quadrupole errors: chromaticity (cont.)

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a *tune spread:*

$$\Delta Q_{\text{one quad}} = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s)\, \mathrm{d}s \qquad \Rightarrow \Delta Q_{\text{all quads}} = -\frac{1}{4\pi} \frac{\Delta p}{p_0} \oint k(s)\, \beta(s)\, \mathrm{d}s$$

Therefore the definition of chromaticity ξ is

$$\xi = -\frac{1}{4\pi} \oint_{\text{quads}} k(s)\, \beta(s)\, \mathrm{d}s$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

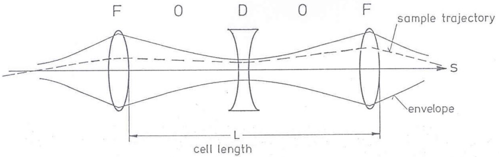
Remarks:

- ξ is a number indicating the size of the tune spot in the working diagram
- ξ is always created by the focusing strength k of **all** quadrupoles

In other words, because of chromaticity the tune is not a point, but it is pancake

Example: Chromaticity of the FODO cell

Consider a ring composed by N_{cell} FODO cells like in figure, with two thin quads separated by length $L/2$,



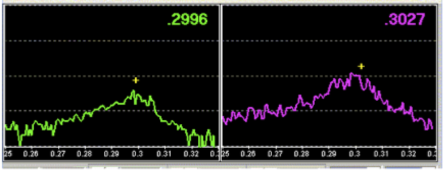
The natural chromaticity ξ_N for the N_{cell} cells is:

$$
\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi}N_{\text{cell}}\int_{\text{cell}} \beta(s)\,\underbrace{k(s)d}_{\frac{1}{f}} \\
&= -\frac{1}{4\pi}N_{\text{cell}}\left[\beta^{+}\left(\frac{1}{f_F}\right) - \beta^{-}\left(\frac{1}{f_D}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4\pi\sin\mu}N_{\text{cell}}\left[\left(L+\frac{L^2}{4f_D}\right)\frac{1}{f_F} - \left(L-\frac{L^2}{4f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{1}{4\pi\sin\mu}N_{\text{cell}}\left[\frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_F f_D}\right] \\
&\simeq -\frac{1}{8\pi\sin\mu}N_{\text{cell}}\frac{L^2}{f_F f_D}
\end{aligned}
$$

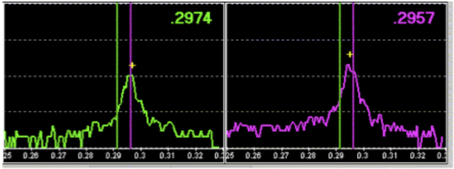
The chromaticity of the ring is N_{cell} times the chromaticity of each cell.

Quadrupole errors: chromaticity



Tune signal for a nearly uncompensated cromaticity ($Q' \approx 20$)

Ideal situation: cromaticity well corrected, ($Q' \approx 1$)



Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k(s)\,\beta(s)\,ds}{4\pi}$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1/f = \Delta k_1 L$
  - it changes both Q_x and Q_y:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1}$$

- We need to use two independent correcting quadrupoles:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2}$$
$$\Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}$$

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \beta_{1x} & \beta_{2x} \\ \beta_{1y} & \beta_{2y} \end{pmatrix} \begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix}$$

- Solve by inversion:

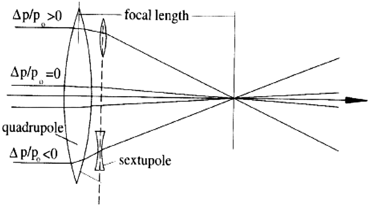
$$\begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix} = \frac{4\pi}{\beta_{1x}\beta_{2y} - \beta_{2x}\beta_{1y}} \begin{pmatrix} \beta_{2y} & -\beta_{2x} \\ -\beta_{1y} & \beta_{1x} \end{pmatrix} \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix}$$

Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances

Cure: we need additional energy dependent focusing. This is given by sextupoles



- The sextupole magnetic field rises quadratically:

$$B_x = \tilde{g}xy \qquad \Rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x \quad \text{a "gradient"}$$
$$B_y = \frac{1}{2}\tilde{g}\left(x^2 - y^2\right)$$

it provides a linearly increasing quadrupole gradient

Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$k = \frac{g}{p/e} \ [\mathrm{m}^{-2}]$$

- Sextupoles are characterised by a normalised sextupole strength k_2, which carries a focusing quadrupolar component k_1:

$$k_2 = \frac{\tilde{g}}{p/e} \ [\mathrm{m}^{-3}]; \qquad \tilde{k}_1 = \frac{\tilde{g}x}{p/e} \ [\mathrm{m}^{-2}]$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x \to x + D \cdot \frac{\Delta p}{p}$

$$\tilde{k}_1 = \frac{\tilde{g}\left(x + D\frac{\Delta p}{p}\right)}{p/e} \; [\mathrm{m}^{-2}]$$

- for $x = 0$ it corresponds to an energy-dependent focal length

$$\frac{1}{f_{\mathrm{sext}}} = \tilde{k}_1 L_{\mathrm{sext}} = \overbrace{\underbrace{\frac{\tilde{g}}{p/e}}_{k_2} D\frac{\Delta p}{p}}^{\tilde{k}_1} \cdot L_{\mathrm{sext}} = k_2 D \cdot \frac{\Delta p}{p} \cdot L_{\mathrm{sext}}$$

Now the formula for the chromaticity rewrites:

$$\xi = \underbrace{-\frac{1}{4\pi}\oint k(s)\,\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to quadrupoles}} + \underbrace{\frac{1}{4\pi}\oint k_2(s)\,D\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to sextupoles}}$$

Design rules for sextupole scheme

- Chromatic aberrations must be corrected in both planes ⇒ you need at least two sextupoles

- In each plane the sextupole fields contribute with different signs to the chromaticity ξ_x and ξ_y:

$$\xi_x = -\frac{1}{4\pi}\oint\left[k - S_F D_x + S_D D_x\right]\beta_x(s)\,\mathrm{d}s$$
$$\xi_y = \frac{1}{4\pi}\oint\left[k - S_F D_x + S_D D_x\right]\beta_y(s)\,\mathrm{d}s$$

- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_x D_x$ and $\beta_y D_x$ are maximum.

- Important remark: for offset orbits, the sextupoles introduce *geometric aberrations*. They can be reduced by adopting a $-I$ transformation scheme:
  - put sextupoles in $(2n+1)\pi$ phase advance apart in a periodic lattice to compensate the effect

Summary

orbit for an off-momentum particle	$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p}$
dispersion trajectory	$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$
equations of motion with dispersion	$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$
definition of momentum compaction	$\frac{\Delta C}{C} = \alpha \frac{\Delta p}{p}$
stability condition	$m \cdot Q_x + n \cdot Q_y \neq p$ with n, m, p integers
tune shift	$\Delta Q = \frac{1}{4\pi}\oint_{\text{quads}} \Delta k(s)\,\beta(s)\,\mathrm{d}s$
beta beat	$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2\sin 2\pi Q} \cdot \oint \beta(t)\,\Delta k(t)\cos(2\pi Q - 2(\mu(t) - \mu(s)))\,\mathrm{d}t$
chromaticity	$\xi = \frac{\Delta Q}{\Delta p/p} = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$

References

Derivation of the equation of the orbit distortion for a dipole field errors:

1. Shyh-Yuan Lee: Accelerator Physics, World Scientific, 2004
2. The CERN Accelerator School (CAS) Proceedings