Transverse Beam Dynamics

JUAS tutorial 2

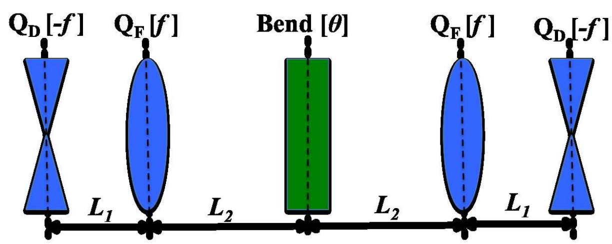
15 January 2014

1 Exercise: geometry of a storage ring, thin lens, tune, dispersion

Consider a proton synchrotron accumulator made of identical cells. The relevant ring parameters are given in the following table:

Proton kinetic energy	2 GeV
Cell type	Symmetric triplet$^{(*)}$
Ring circumference	960 m
Integrated quadrupole gradient ($\int Gdl$)	1.5 T

$^{(*)}$Note: A thin lens symmetric triplet cell consists of a thin lens defocusing quadrupole of focal length $-f$, followed with a drift space of length L_1, a thin lens focusing quadrupole of focal length f, a drift of length L_2, a thin lens dipole of horizontal bending angle θ, a drift of length L_2, a thin lens focusing quadrupole of focal length f, a drift of length L_1, and a thin lens defocusing quadrupole of focal length $-f$ (see Figure below).



1. Compute the focal length f of the quadrupoles. The proton rest mass is 938 MeV.
2. Given the numerical values $L_1 = 1.5$ m and $L_2 = 6$ m:

- Compute the horizontal and vertical transfer matrices of a triplet cell (take into account that the sign of the focal length changes when going from horizontal to vertical plane).
- Compute the horizontal and vertical machine tunes.
- Compute the horizontal and vertical betatron functions at the entrance of a triplet cell.
- Compute the horizontal and vertical dispersion functions at the entrance of a triplet cell.

Hint:

The 3×3 horizontal transfer matrix for one symmetric triplet cell is (in the thin lens approximation):

$$M_{triplet} = \begin{pmatrix} \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{2(f-L_1)(L_1f+L_2f-L_1L_2)}{f^2} & (L_1+L_2-\frac{L_1L_2}{f})\theta \\ \frac{2L_1(L_1L_2-L_1f-f^2)}{f^4} & \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{(f^2+L_1f-L_1L_2)}{f^2}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

for the transport of a vector $\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}$, where $\Delta p/p_0$ is the momentum offset with respect to the design momentum p_0.

2 Exercise: emittance

From the solution of the trajectory equation,

$$x = \sqrt{\beta\epsilon}\cos[\phi + \phi_0]$$

- Derive the following relation:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

where ϵ is the emittance, and β, α, and γ are the so-called Twiss parameters:

$$\alpha \equiv -\frac{\beta'}{2} \text{ , and } \gamma \equiv \frac{1+\alpha^2}{\beta}$$

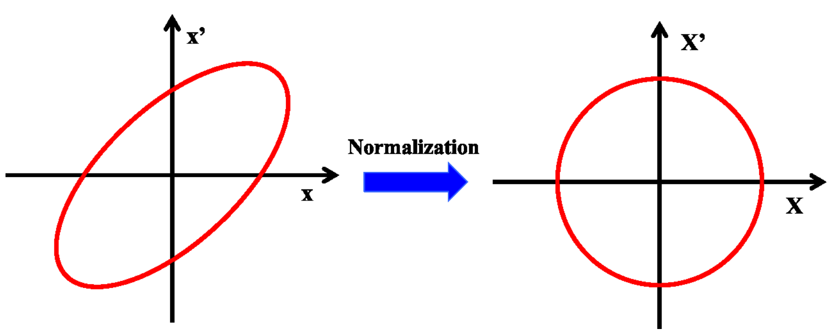
(DO NOT CONFUSE this γ with the relativistic Lorentz factor!).

3 Exercise: normalized phase space

Let us consider the following phase space vector: (x, x'). The transformation to a normalized phase space (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalization process of the phase space is illustrated in the figure below:



If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{12} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x - (1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalized phase space.