

# Synchrotron radiation

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and

Diamond Light Source

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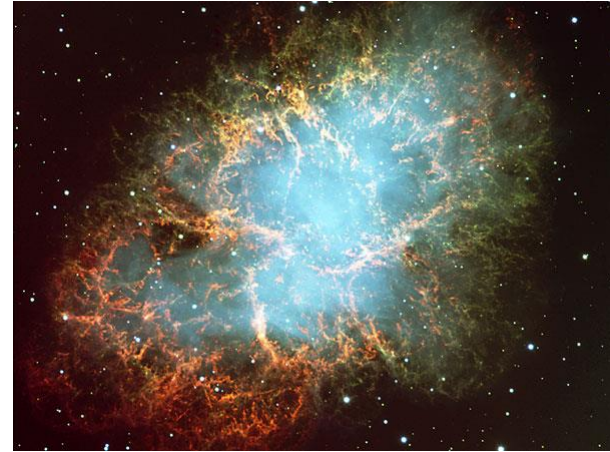
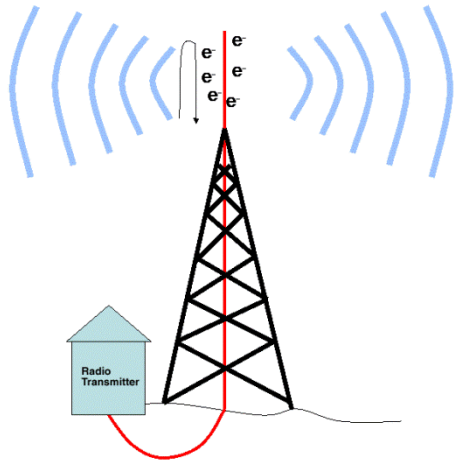
**emittance and brilliance**

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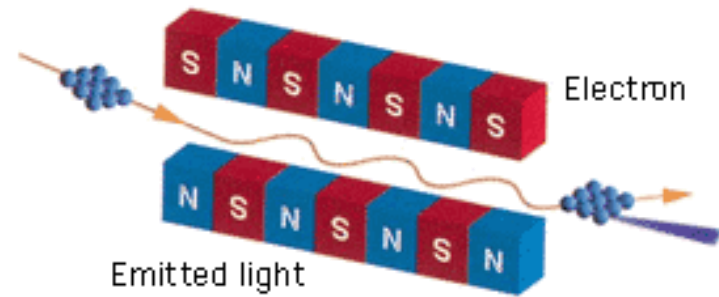
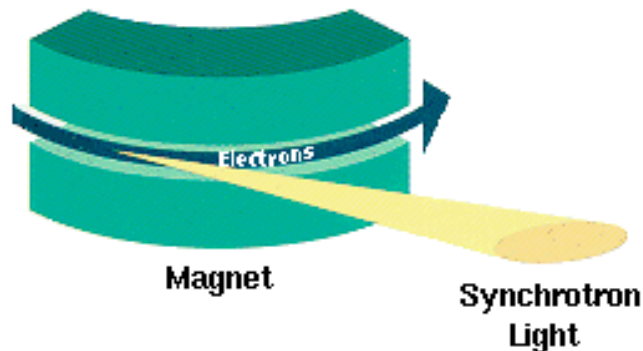
# What is synchrotron radiation

Electromagnetic radiation is emitted by charged particles when accelerated



The electromagnetic radiation emitted when the charged particles are accelerated radially ( $\mathbf{v} \perp \mathbf{a}$ ) is called **synchrotron radiation**

It is produced in the synchrotron radiation sources using bending magnets undulators and wigglers



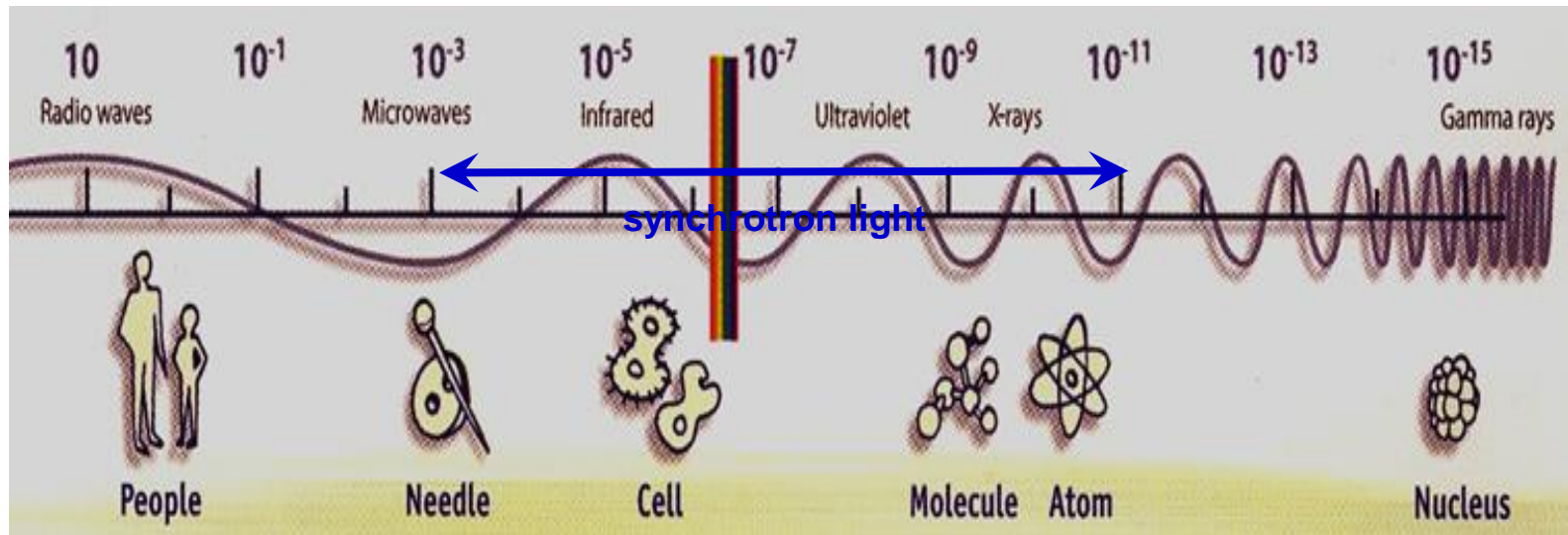
# Synchrotron radiation sources properties (I)

**Broad Spectrum** which covers from microwaves to hard X-rays:

the user can select the wavelength required for experiment;

either with a monochromator

or adjusting the emission wavelength of insertion devices



# Synchrotron radiation sources properties (II)

**High Flux:** high intensity photon beam, allows rapid experiments or use of weakly scattering crystals;

$$\text{Flux} = \text{Photons} / (\text{s} \bullet \text{BW})$$

**High Brilliance (Spectral Brightness):** highly collimated photon beam generated by a small divergence and small size source

$$\text{Brilliance} = \text{Photons} / (\text{s} \bullet \text{mm}^2 \bullet \text{mrad}^2 \bullet \text{BW})$$

Partial coherence in SRs

Full T coherence in FELs

**Polarisation:** both linear and circular (with IDs)

10s ps in SRs

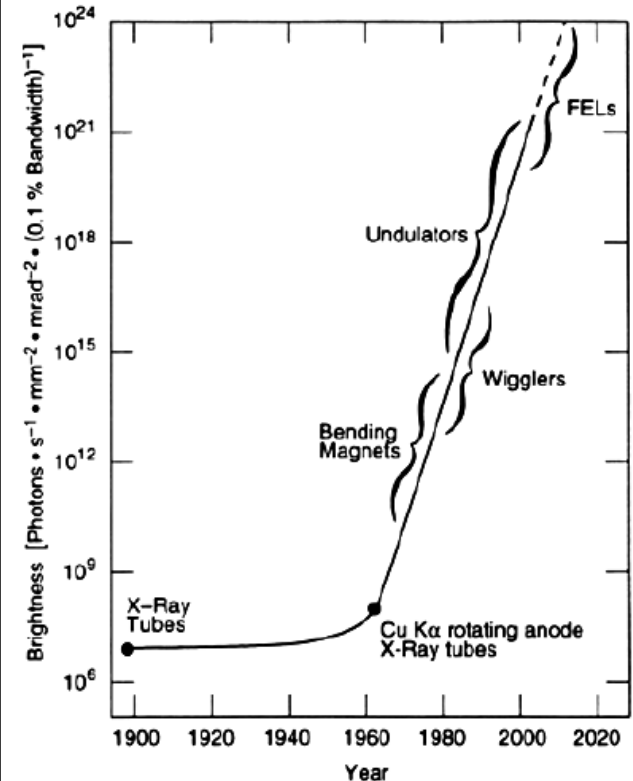
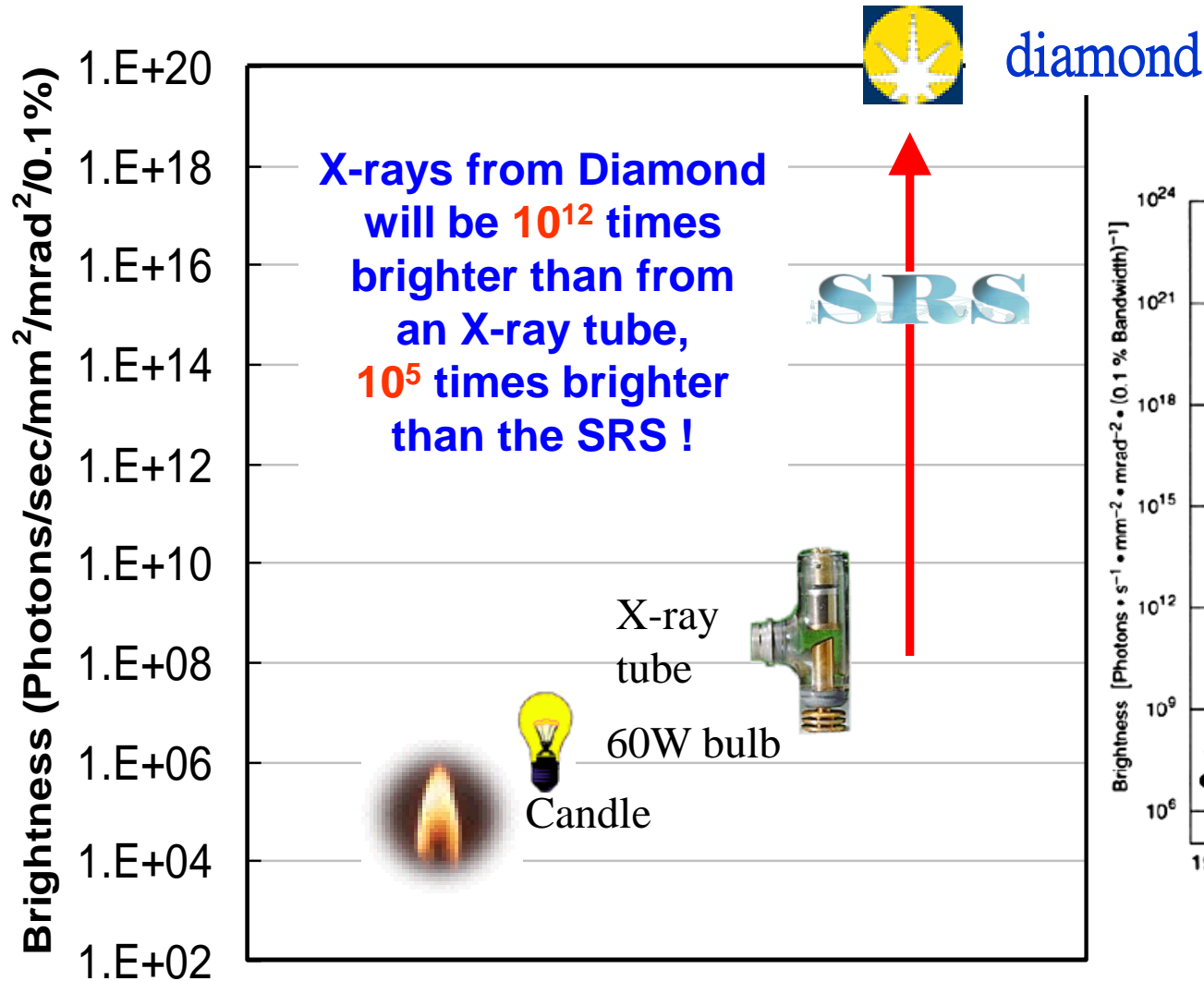
**Pulsed Time Structure:** pulsed length down to

10s fs in FELs

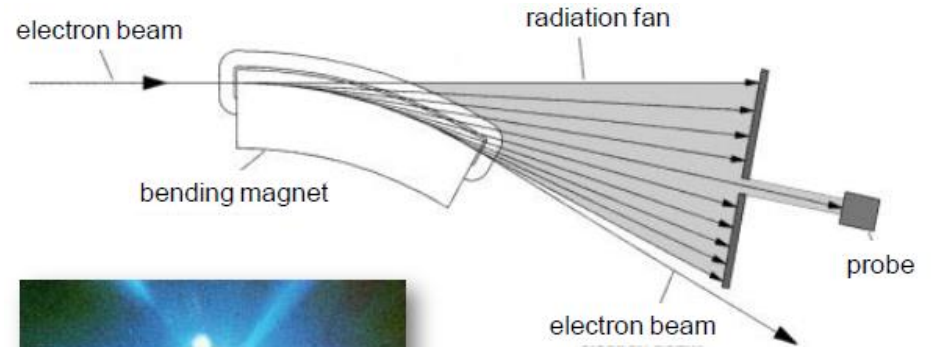
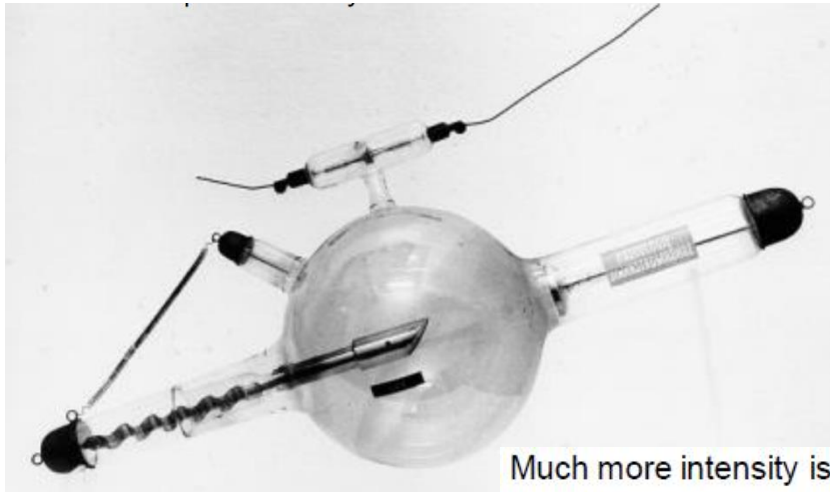
**High Stability:** submicron source stability in SR

... and **it can be computed!**

# Peak Brilliance

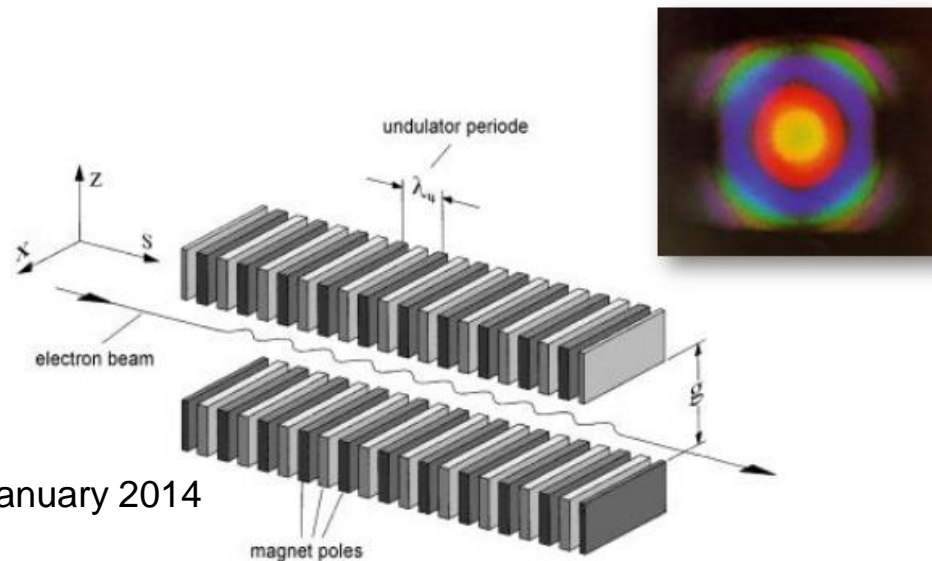


# X-ray sources



the synchrotron radiation from a bending magnet is horizontally spread out over a wide radiation fan.

Much more intensity is provided by **wigglers** and **undulators**

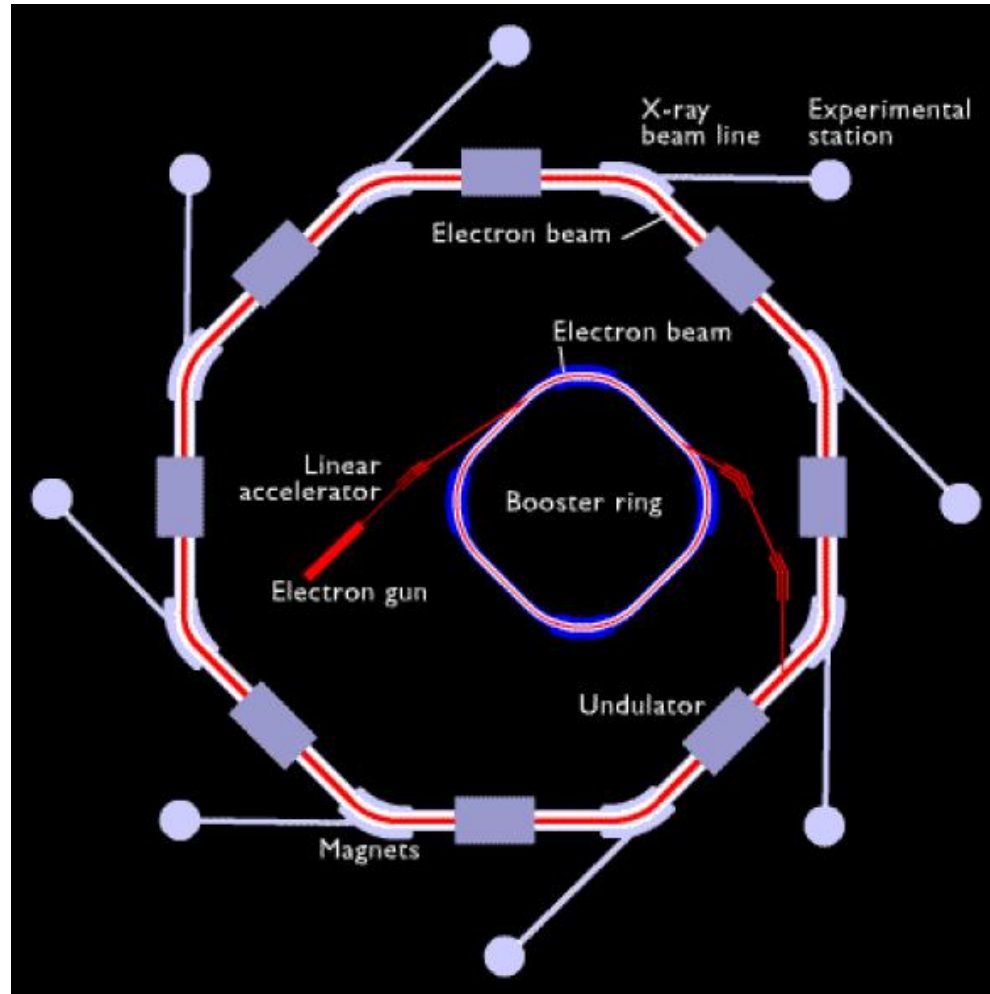
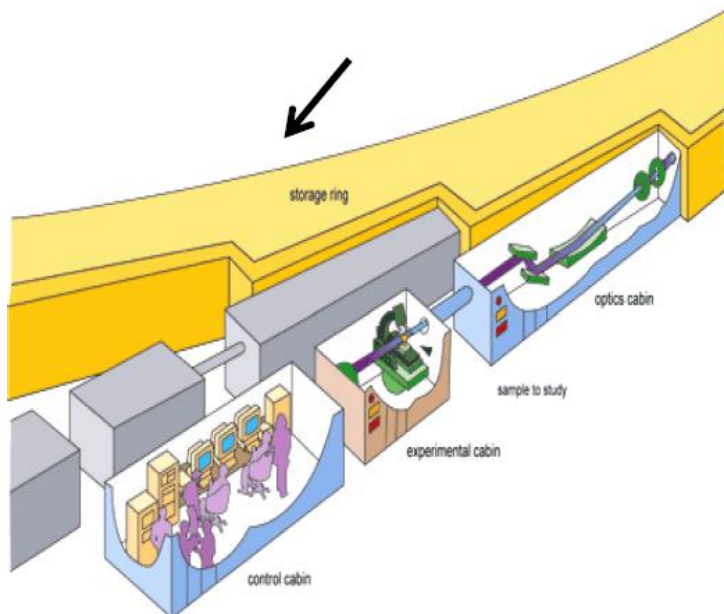




# Layout of a synchrotron radiation source (I)

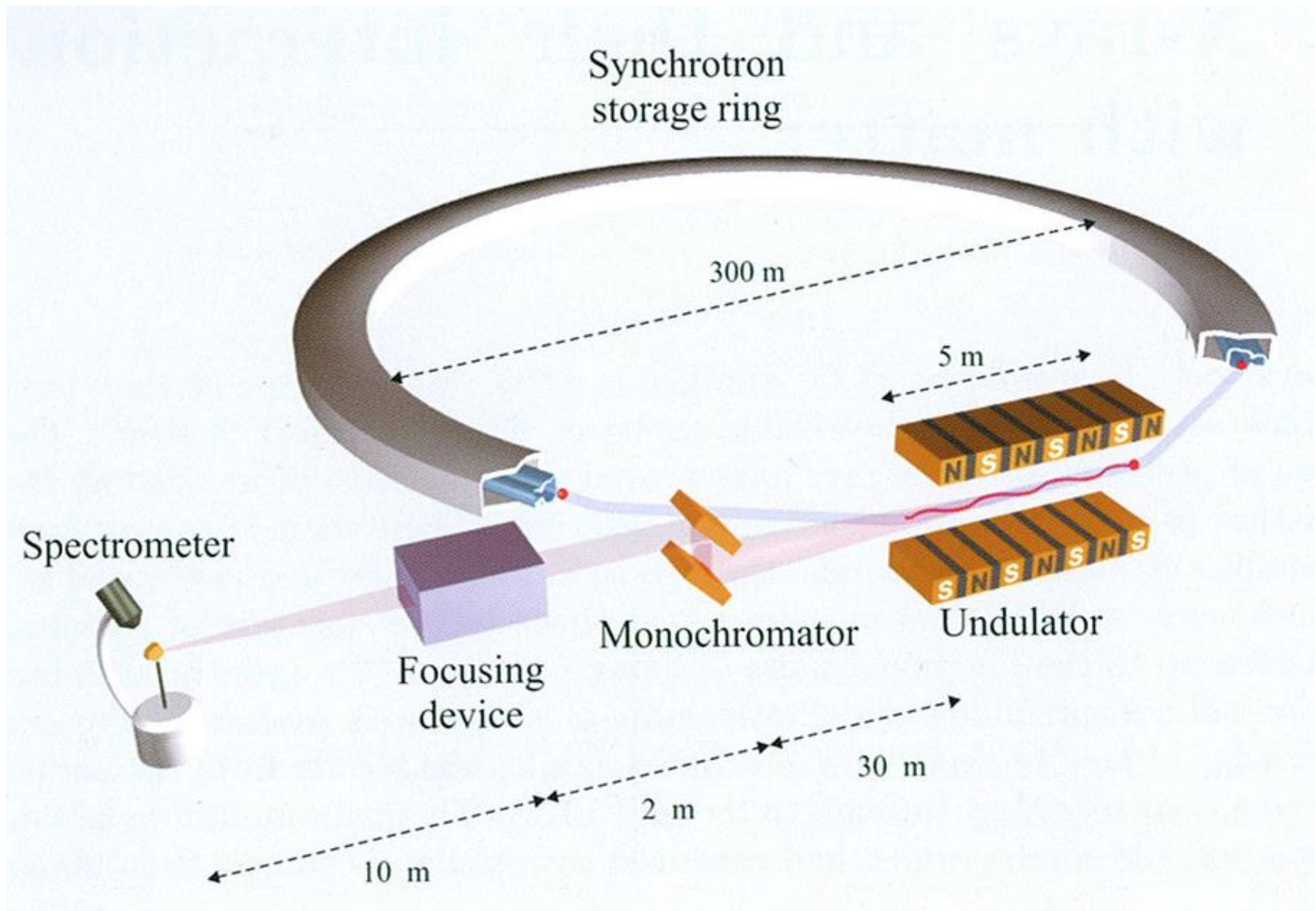
Electrons are generated and accelerated in a linac, further accelerated to the required energy in a booster and injected and stored in the storage ring

The circulating electrons emit an intense beam of synchrotron radiation which is sent down the beamline





# Layout of a synchrotron radiation source (II)



# Evolution of synchrotron radiation sources (I)

- **First observation:**

1947, General Electric, 70 MeV synchrotron

- **First user experiments:**

1956, Cornell, 320 MeV synchrotron

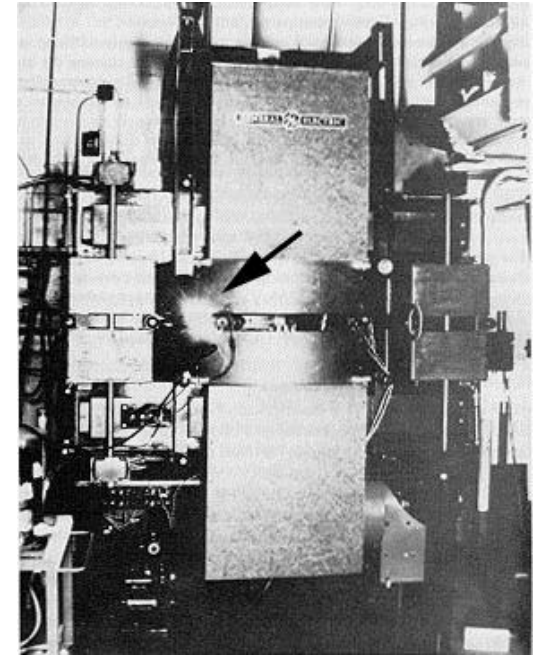
- **1<sup>st</sup> generation light sources:** machine built for High Energy Physics or other purposes used parasitically for synchrotron radiation

- **2<sup>nd</sup> generation light sources:** purpose built synchrotron light sources, SRS at Daresbury was the first dedicated machine (1981 – 2008)

- **3<sup>rd</sup> generation light sources:** optimised for high brilliance with low emittance and Insertion Devices; ESRF, Diamond,

...

R. Bartolini, JUAS, 27-31 January 2014



# Evolution of synchrotron radiation sources (II)

- **4<sup>th</sup> generation light sources:** photoinjectors LINAC based Free Electron Laser sources;

FLASH (DESY) 2007

LCLS (SLAC) 2009

SACLA (Japan) 2011

Elettra (Italy) 2012

and in the near(?) future

- **4<sup>th</sup> generation light sources storage ring based:** diffraction limited storage rings
- ...and even a **5<sup>th</sup> generation** with more compact and advanced accelerator technologies e.g. based on laser plasma wakefield accelerators

# 3<sup>rd</sup> generation storage ring light sources

1992	<b>ESRF</b> , France (EU)	6 GeV
	<b>ALS</b> , US	1.5-1.9 GeV
1993	<b>TLS</b> , Taiwan	1.5 GeV
1994	<b>ELETTRA</b> , Italy	2.4 GeV
	<b>PLS</b> , Korea	2 GeV
	<b>MAX II</b> , Sweden	1.5 GeV
1996	<b>APS</b> , US	7 GeV
	<b>LNLS</b> , Brazil	1.35 GeV
1997	<b>Spring-8</b> , Japan	8 GeV
1998	<b>BESSY II</b> , Germany	1.9 GeV
2000	<b>ANKA</b> , Germany	2.5 GeV
	<b>SLS</b> , Switzerland	2.4 GeV
2004	<b>SPEAR3</b> , US	3 GeV
	<b>CLS</b> , Canada	2.9 GeV
2006:	<b>SOLEIL</b> , France	2.8 GeV
	<b>DIAMOND</b> , UK	3 GeV
	<b>ASP</b> , Australia	3 GeV
	<b>MAX III</b> , Sweden	700 MeV
	<b>Indus-II</b> , India	2.5 GeV
2008	<b>SSRF</b> , China	3.4 GeV
2009	<b>PETRA-III</b> , Germany	6 GeV
2011	<b>ALBA</b> , Spain	3 GeV





# 3<sup>rd</sup> generation storage ring light sources

under construction or planned

> 2013

**NSLS-II**, US

3 GeV

**MAX-IV**, Sweden

1.5-3 GeV

**SOLARIS**, Poland

1.5 GeV

**SESAME**, Jordan

2.5 GeV

**TPS**, Taiwan

3 GeV

**CANDLE**, Armenia

3 GeV

**PEP-X**, USA

4.5 GeV

**Spring8-II**, Japan

6 GeV



# Diamond aerial views

**June 2003**



**Oct 2006**

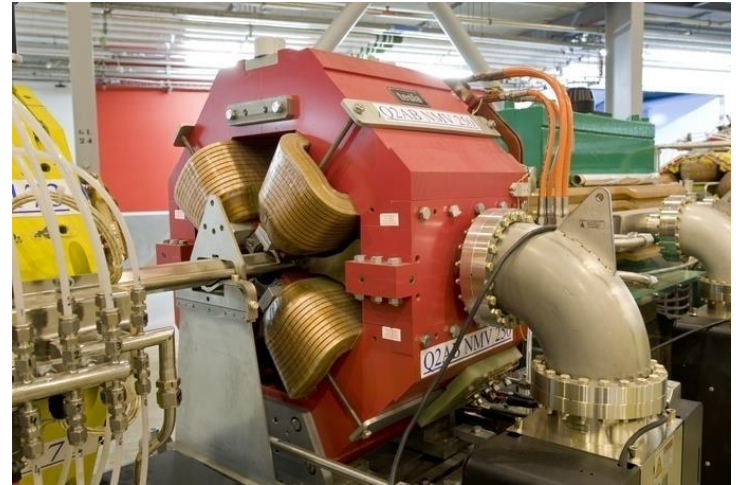


# Main components of a storage ring

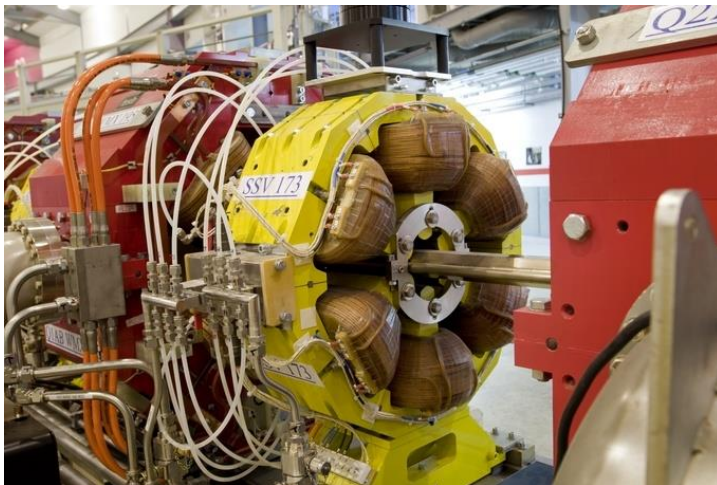
**Dipole magnets** to bend the electrons



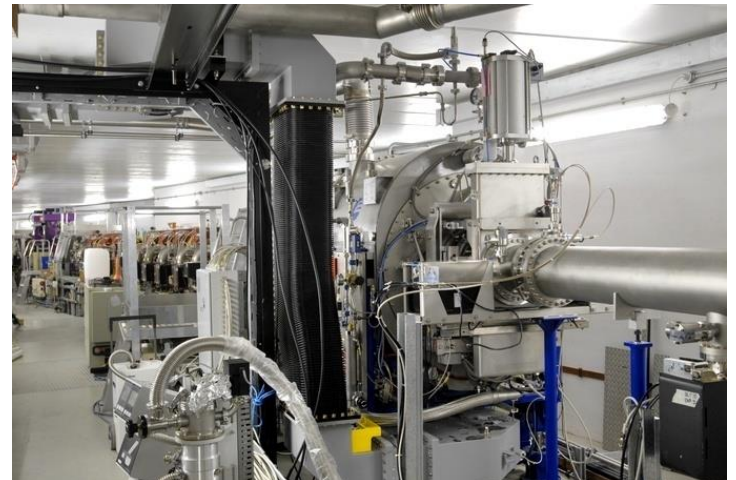
**Quadrupole magnets** to focus the electrons



**Sextupole magnets** to focus off-energy electrons (mainly)



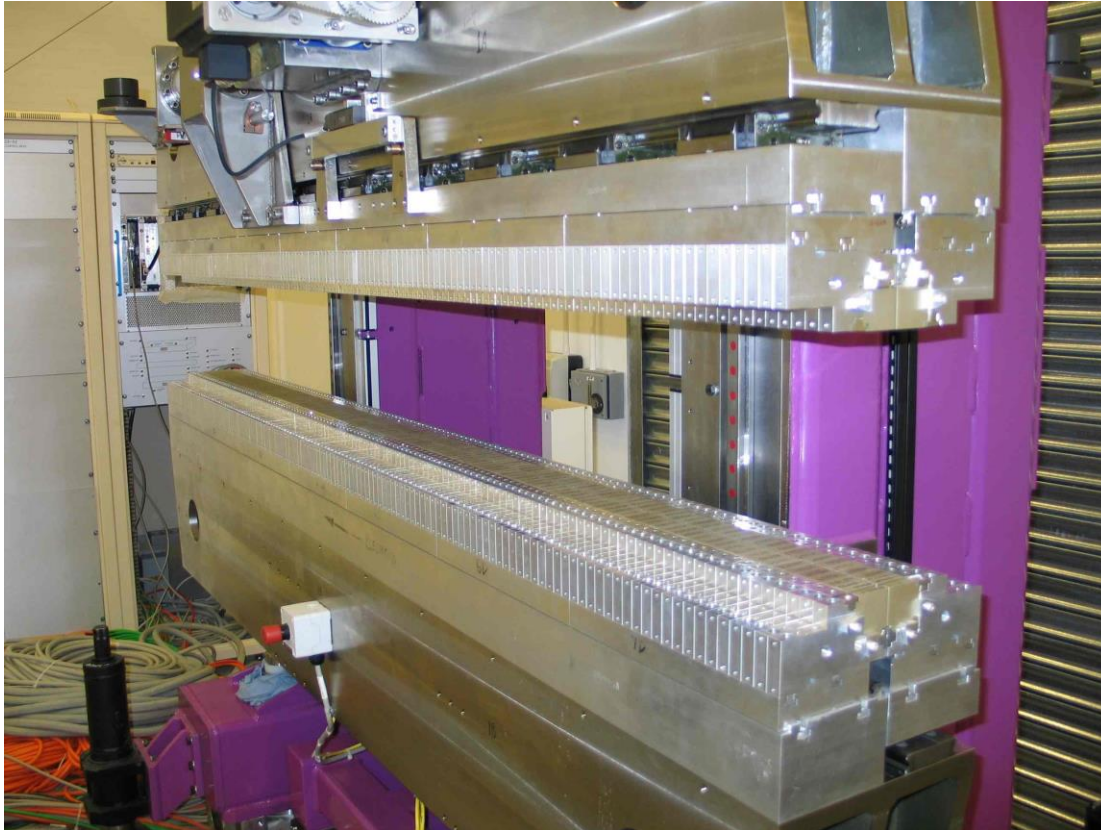
**RF cavities** to replace energy losses due to the emission of synchrotron radiation





# Main components of a storage ring

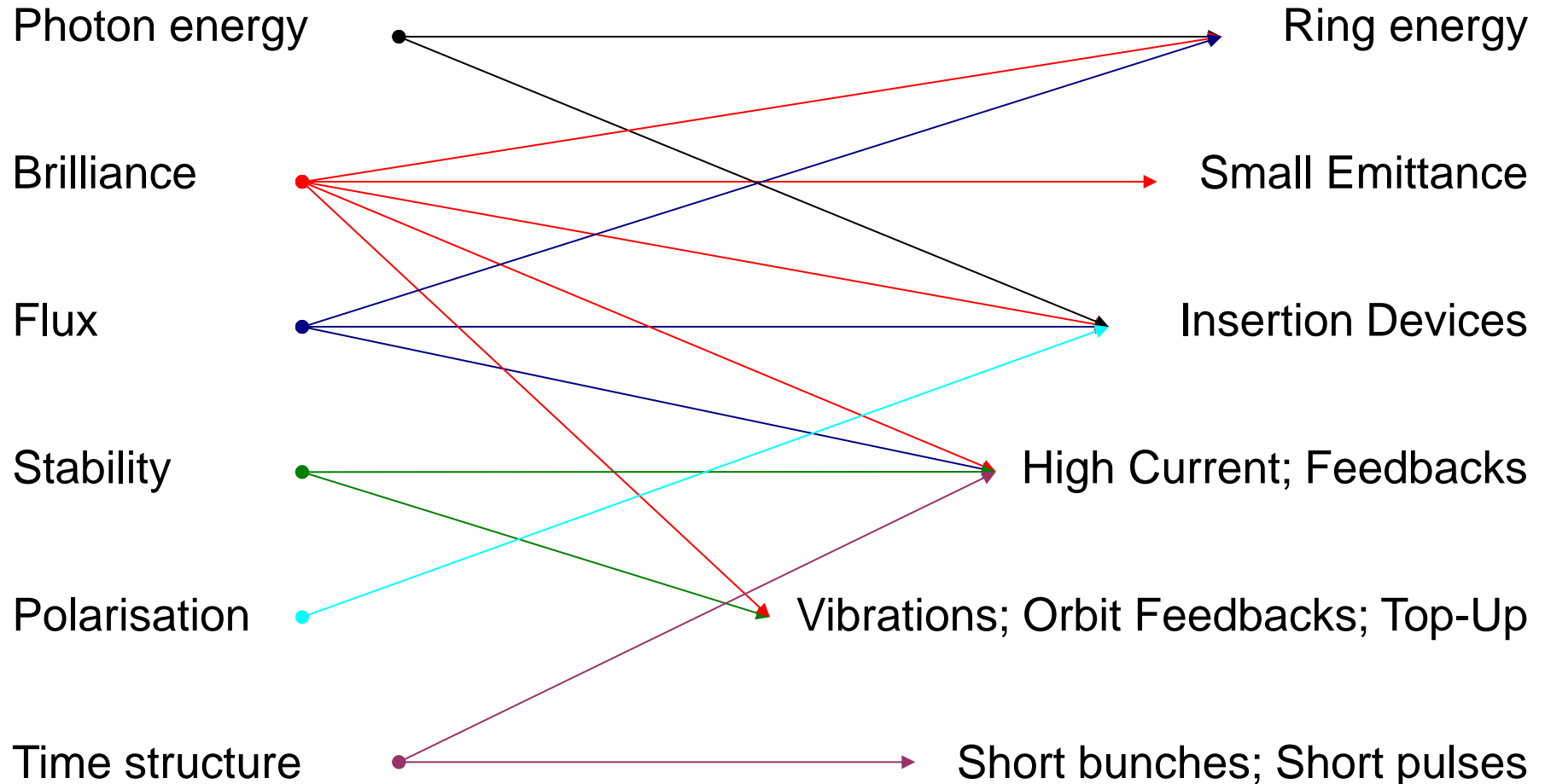
**Insertion devices (undulators) to generate high brilliance radiation**



**Insertion devices (wiggler) to reach high photon energies**

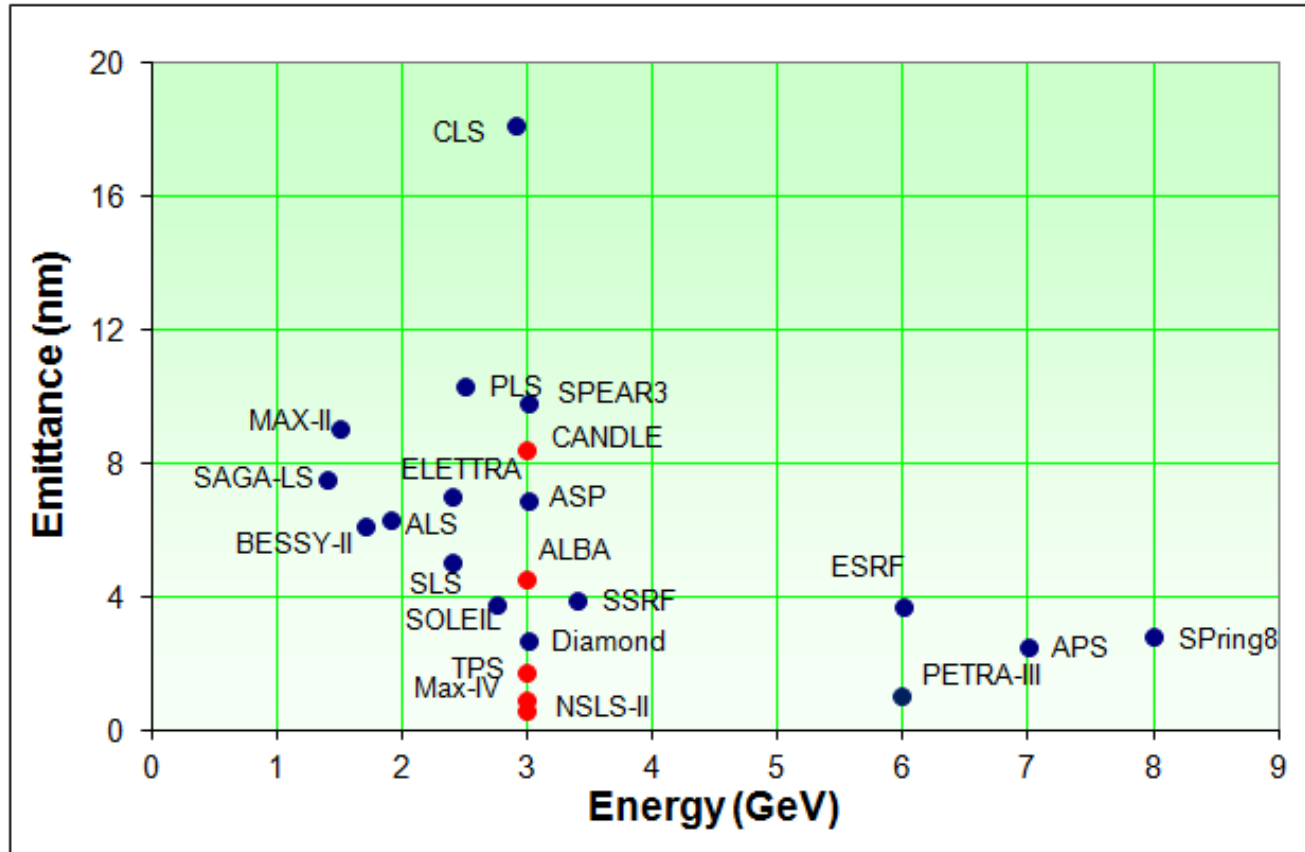


# Accelerator physics and technology challenges



# Brilliance and low emittance

The brilliance of the photon beam is determined (mostly) by the electron beam



$$\text{brilliance} = \frac{\text{flux}}{4\pi^2 \Sigma_x \Sigma_{x'} \Sigma_y \Sigma_{y'}}$$

$$\Sigma_x = \sqrt{\sigma_{x,e}^2 + \sigma_{ph,e}^2}$$

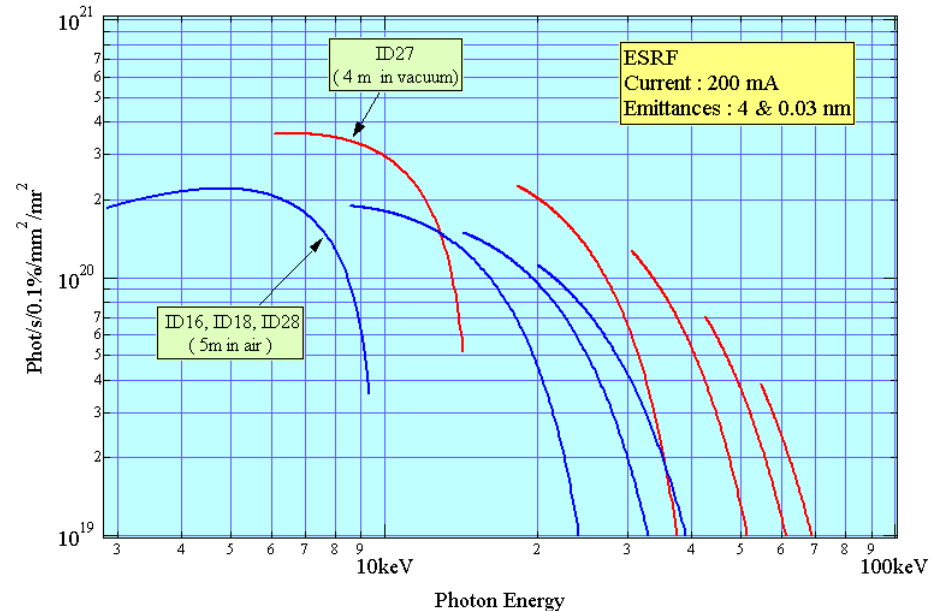
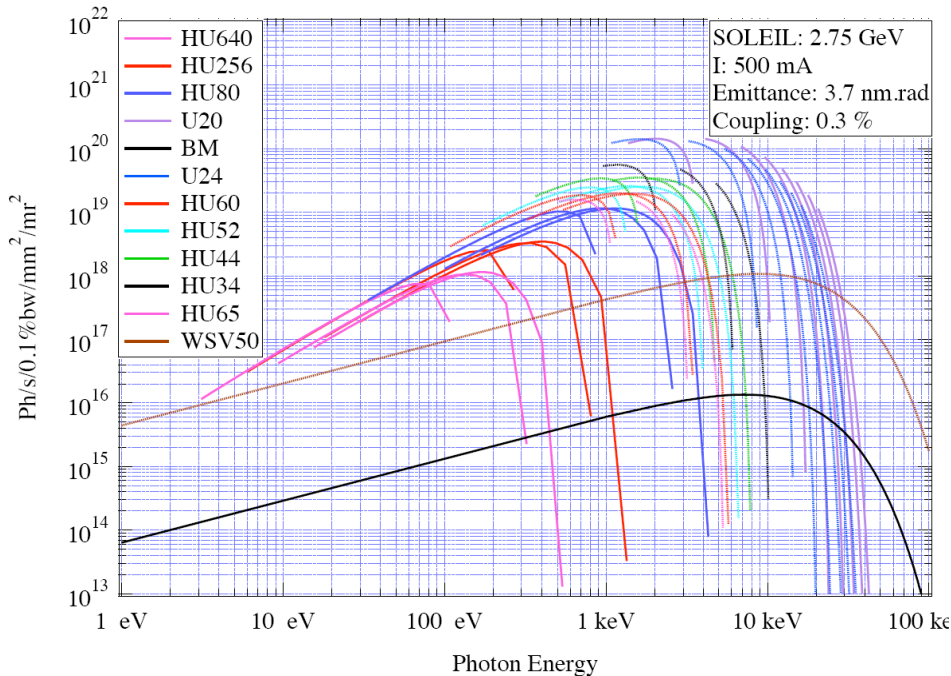
$$\Sigma_{x'} = \sqrt{\sigma_{x',e}^2 + \sigma_{ph,e}'^2}$$

$$\sigma_x = \sqrt{\varepsilon_x \beta_x + (D_x \sigma_\varepsilon)^2}$$

$$\sigma_{x'} = \sqrt{\varepsilon_x \beta_x + (D'_x \sigma_\varepsilon)^2}$$

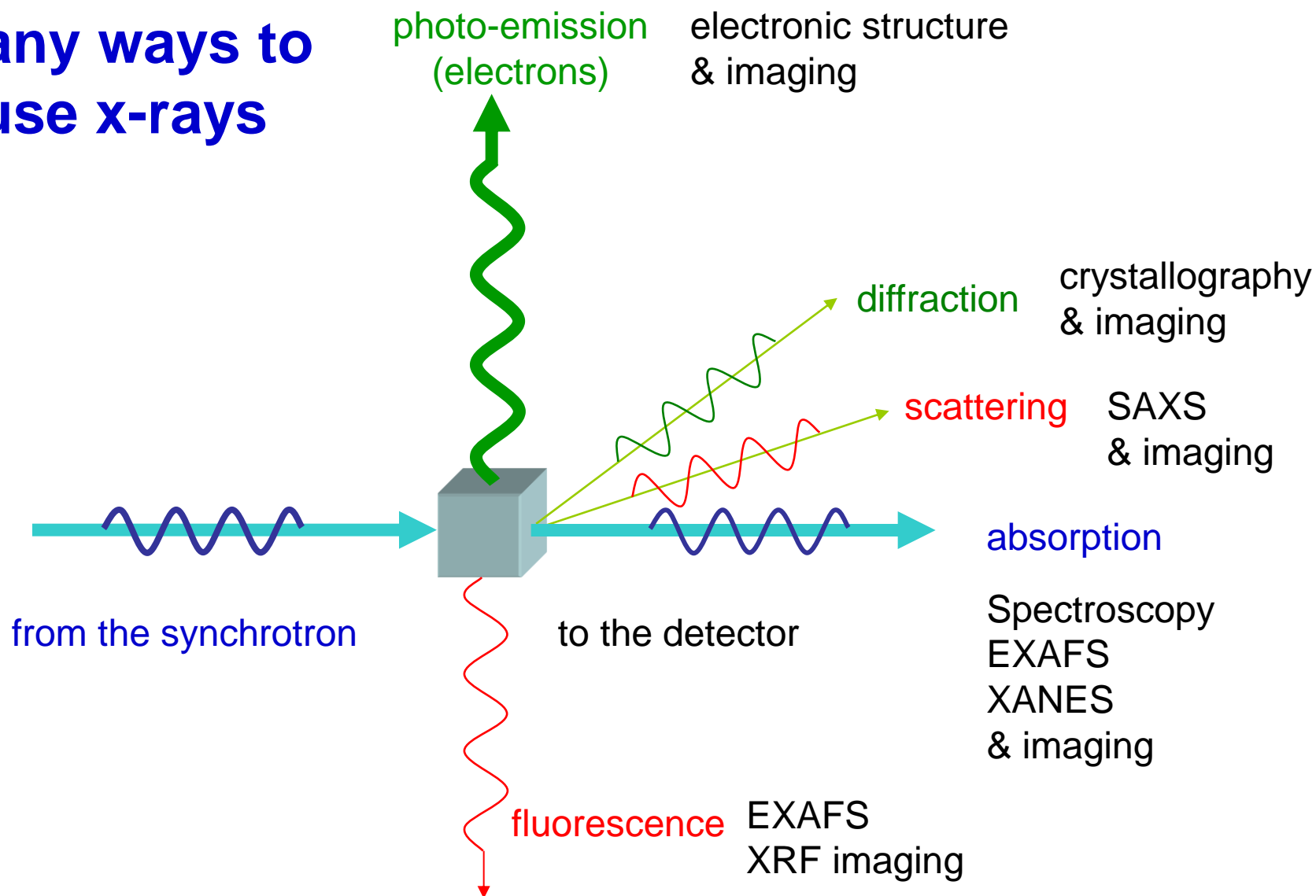
# Brilliance with IDs

Thanks to the progress with IDs technology storage ring light sources can cover a photon range from few tens of eV to tens 10 keV or more with high brilliance



Medium energy storage rings with In-vacuum undulators operated at low gaps (e.g. 5-7 mm) can reach 10 keV with a brilliance of  $10^{20}$  ph/s/0.1%BW/mm<sup>2</sup>/mrad<sup>2</sup>

# Many ways to use x-rays



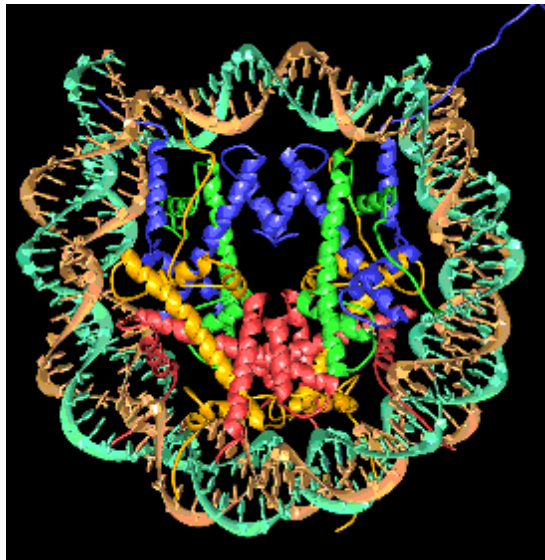


# Applications

**Medicine, Biology, Chemistry, Material Science, Environmental Science and more**

## Biology

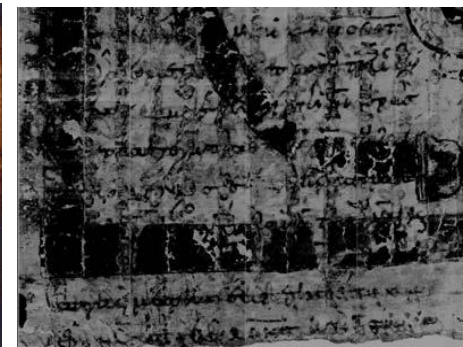
Reconstruction of the 3D structure of a nucleosome with a resolution of 0.2 nm



The collection of precise information on the molecular structure of chromosomes and their components can improve the knowledge of how the genetic code of DNA is maintained and reproduced

## Archeology

A synchrotron X-ray beam at the SSRL facility illuminated an obscured work erased, written over and even painted over of the ancient mathematical genius Archimedes, born 287 B.C. in Sicily.



X-ray fluorescence imaging revealed the hidden text by revealing the iron contained in the ink used by a 10th century scribe. This x-ray image shows the lower left corner of the page.

# Life science examples: DNA and myoglobin

Franklin and Gosling used a X-ray tube:

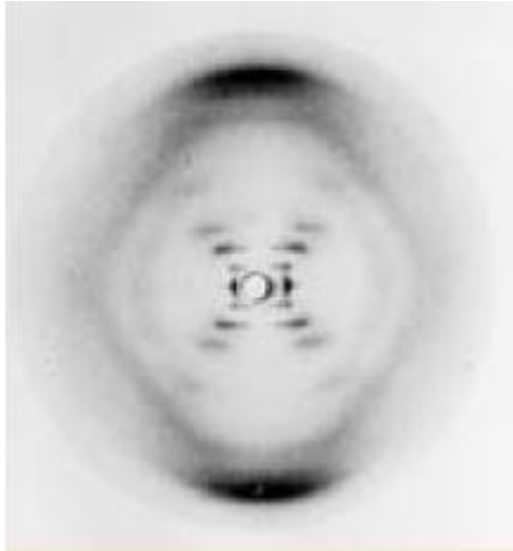
Brilliance was  $10^8$  (ph/sec/mm<sup>2</sup>/mrad<sup>2</sup>/0.1BW)

Exposure times of 1 day were typical ( $10^5$  sec)

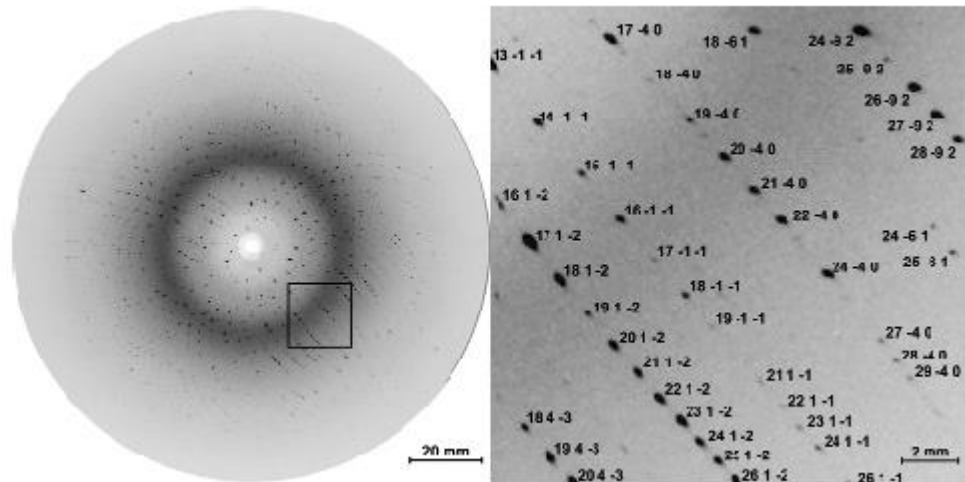
e.g. Diamond provides a brilliance of  $10^{20}$

100 ns exposure would be sufficient

Nowadays pump probe experiment in life science are performed using 100 ps pulses from storage ring light sources: e.g. ESRF myoglobin in action



**Photograph 51**  
**Franklin-Gosling**  
**DNA (form B)**  
**1952**





## Lienard-Wiechert potentials (II)

The general solutions for the wave equation driven by a time varying charge and current density read (in the Lorentz gauge)

$$\Phi(\bar{x}, t) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\bar{x} - \bar{x}', t_{\text{ret}})}{|\bar{x} - \bar{x}'|} d^3\bar{x}', \quad \bar{A}(\bar{x}, t) = \frac{\mu_0}{4\pi} \iiint_V \frac{\bar{J}(\bar{x} - \bar{x}', t_{\text{ret}})}{|\bar{x} - \bar{x}'|} d^3\bar{x}'$$

where ret means retarded  $t_{\text{ret}} = t - \frac{|\bar{x}(t) - \bar{x}(t_{\text{ret}})|}{c}$  (see next slide)

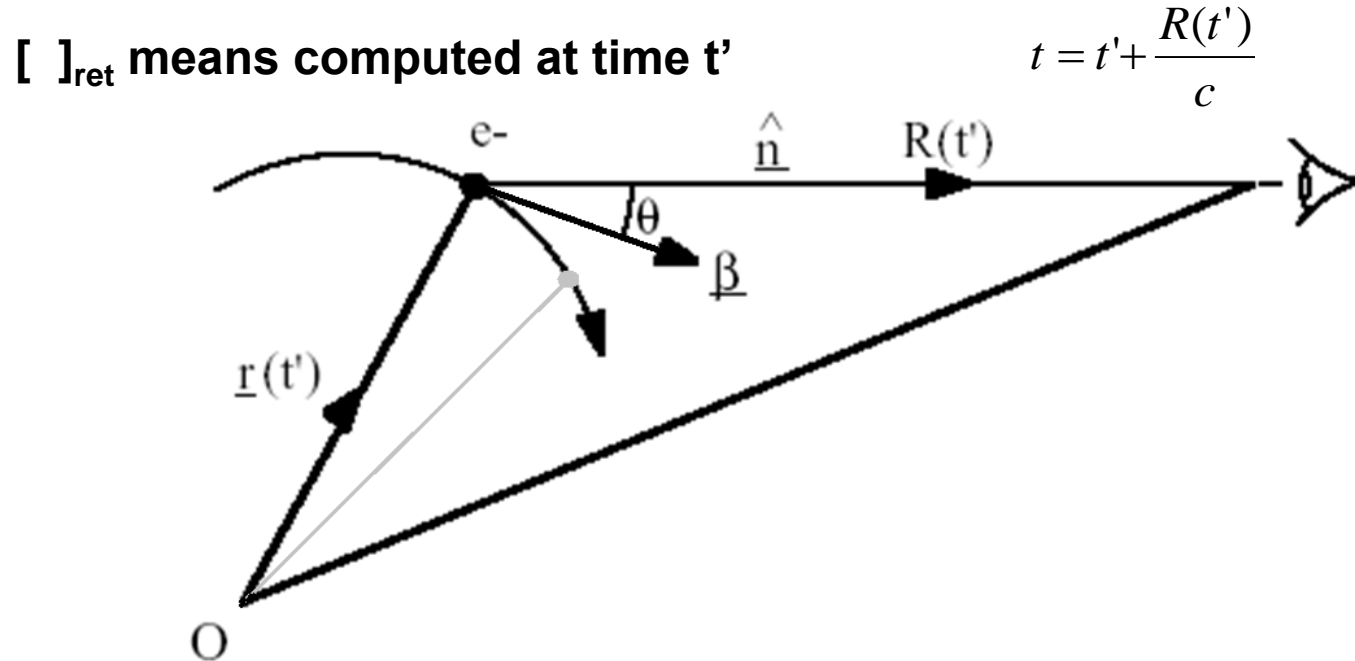
Using the properties of the Dirac deltas we can integrate and obtain the Lienard-Wiechert potentials

$$\Phi(\bar{x}, t) = \frac{1}{4\pi\epsilon_0} \left[ \frac{e}{(1 - \bar{\beta} \cdot \bar{n})R} \right]_{\text{ret}} \quad \bar{A}(\bar{x}, t) = \frac{1}{4\pi\epsilon_0 c} \left[ \frac{e\bar{\beta}}{(1 - \bar{\beta} \cdot \bar{n})R} \right]_{\text{ret}}$$

These are the potentials of the em fields generated by the charged particle in motion.

The trajectory itself is determined by external electric and magnetic fields

# Lienard-Wiechert Potentials (III)



Potentials and fields at position  $x$  at time  $t$  are determined by the characteristic of the electron motion at a time  $t'$

$t' - t$  is the time it takes for the em radiation to travel the distance  $R(t')$

i.e. grey is the position of the electron at time  $t$

# The Lienard-Wiechert fields

The electric and magnetic fields are computed from the potentials using

$$\vec{B} = \vec{\nabla} \wedge \vec{A} \quad E = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$$

and are called Lienard-Wiechert fields

The computation has to be done carefully since the potentials depend on  $t'$ , not  $t$ . The factor  $dt/dt'$  represents the Doppler factor. We get

$$\vec{E}(\vec{x}, t) = \frac{e}{4\pi\epsilon_0} \left[ \frac{\vec{n} - \vec{\beta}}{\gamma^2 (1 - \vec{\beta} \cdot \vec{n})^3 R^2} \right]_{ret} + \frac{e}{4\pi\epsilon_0 c} \left[ \frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{\beta}}{(1 - \vec{\beta} \cdot \vec{n})^3 R} \right]_{ret}$$

**velocity field**  $\propto \frac{1}{R^2}$      **acceleration field**  $\propto \frac{1}{R}$

$$\vec{B}(\vec{x}, t) = \frac{1}{c} [\vec{n} \times \vec{E}]_{ret}$$

If we consider the acceleration field we have  $\vec{E} \perp \vec{B} \perp \hat{n}$

and the correct dependence as  $1/R$  as for radiation field

# Power radiated

**Power radiated by a particle on a surface is the flux of the Poynting vector**

$$\bar{S} = \frac{1}{\mu_0} \bar{E} \times \bar{B}$$

$$\Phi_{\Sigma}(\bar{S})(t) = \iint_{\Sigma} \bar{S}(\bar{x}, t) \cdot \bar{n} d\Sigma$$

**Angular distribution of radiated power**

$$\frac{d^2P}{d\Omega} = (\bar{S} \cdot \bar{n})(1 - \bar{n} \cdot \bar{\beta}) R^2$$

**radiation emitted by the particle**

**We will analyse two cases:**

**acceleration orthogonal to the velocity → synchrotron radiation**

**acceleration parallel to the velocity → bremsstrahlung**

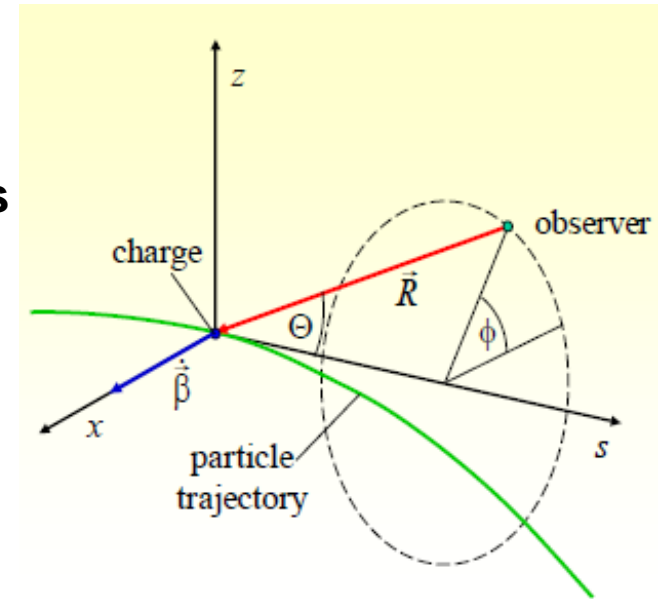
# Synchrotron radiation: non relativistic motion (I)

Assuming  $\vec{\beta} \approx \vec{0}$  and substituting the acceleration field

$$\vec{E}_{acc}(\vec{x}, t) = \frac{e}{4\pi\epsilon_0 c} \left[ \frac{\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})}{R} \right]_{ret}$$

The angular distribution of the power radiated is given by

$$\frac{d^2 P}{d\Omega} = \frac{1}{\mu_0 c} |R \vec{E}_{acc}|^2 = \frac{e^2}{(4\pi)^2 \epsilon_0 c} |\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})|^2$$



Working out the double cross product

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \text{ and } \vec{n} \cdot \vec{n} = n^2 = 1$$

We have

$$\begin{aligned} (\vec{n} \times [\vec{n} \times \dot{\vec{\beta}}])^2 &= (\vec{n} (\vec{n} \cdot \dot{\vec{\beta}}) - \dot{\vec{\beta}} (\vec{n} \cdot \vec{n}))^2 \\ &= n^2 (\vec{n} \cdot \dot{\vec{\beta}})^2 - 2\vec{n} (\vec{n} \cdot \dot{\vec{\beta}}) \dot{\vec{\beta}} + \dot{\vec{\beta}}^2 = \dot{\vec{\beta}}^2 - (\vec{n} \cdot \dot{\vec{\beta}})^2 \end{aligned}$$

# Synchrotron radiation: non relativistic motion (II)

Since

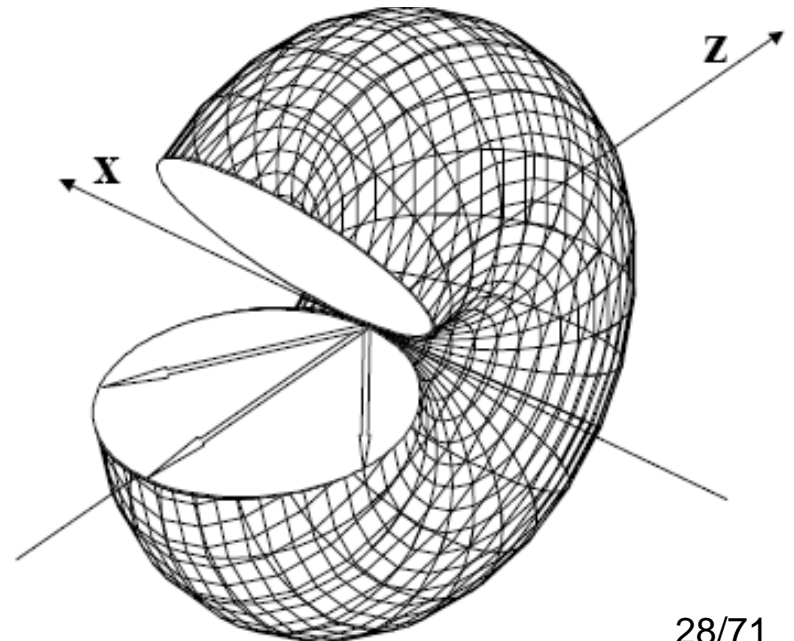
$$\vec{n} \cdot \dot{\vec{\beta}}^* = |\vec{n}| |\dot{\vec{\beta}}^*| \cos \Theta = |\dot{\vec{\beta}}^*| \cos \Theta$$

where  $\theta$  is the angle between the acceleration and the observation direction, we finally get

$$\left( \vec{n} \times \left[ \vec{n} \times \dot{\vec{\beta}}^* \right] \right)^2 = \dot{\vec{\beta}}^{*2} - \dot{\vec{\beta}}^{*2} \cos^2 \Theta = \dot{\vec{\beta}}^{*2} (1 - \cos^2 \Theta) = \dot{\vec{\beta}}^{*2} \sin^2 \Theta$$

The angular distribution of power reads

$$\frac{d^2 P}{d\Omega} = \frac{e^2}{(4\pi)^2 \varepsilon_0 c} \left| \dot{\vec{\beta}} \right|^2 \sin^2 \theta$$



# Synchrotron radiation: non relativistic motion (III)

Integrating over the angles gives the total radiated power

$$P = \frac{e^2}{(4\pi)^2 c \epsilon_0} \dot{\beta}^2 \int_0^{2\pi} \int_0^\pi \sin^3 \Theta d\Theta d\phi$$

This integral gives

$$P = \frac{e^2}{6\pi\epsilon_0 c} \left| \dot{\beta} \right|^2$$

**Larmor's formula**

It shows that radiation is emitted when the particle is accelerated.

Using

$$\bar{\beta} = \frac{\bar{p}}{mc}$$

we have (to be used later for the generalisation to the relativistic case)

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\bar{p}}{dt} \right|^2$$



# Synchrotron radiation: relativistic motion (I)

In the relativistic case the total radiated power is computed in the same way.  
Using only the acceleration field (large R)

$$\bar{\mathbf{E}}(\bar{\mathbf{x}}, t) \sim \frac{e}{4\pi\epsilon_0 c} \left[ \frac{\bar{\mathbf{n}} \times (\bar{\mathbf{n}} - \bar{\boldsymbol{\beta}}) \times \dot{\bar{\boldsymbol{\beta}}}}{(1 - \bar{\boldsymbol{\beta}} \cdot \bar{\mathbf{n}})^3 R} \right]_{\text{ret}}$$

The angular distribution of the power emitted is (use the retarded time!)

$$\frac{d^2 P}{d\Omega} = \frac{1}{\mu_0 c} \left| R \bar{\mathbf{E}}_{\text{acc}} \right|^2 (1 - \bar{\boldsymbol{\beta}} \cdot \bar{\mathbf{n}}) = \frac{e^2}{(4\pi)^2 \epsilon_0 c} \frac{\left| \bar{\mathbf{n}} \times [(\bar{\mathbf{n}} - \bar{\boldsymbol{\beta}}) \times \dot{\bar{\boldsymbol{\beta}}}] \right|^2}{(1 - \bar{\mathbf{n}} \cdot \bar{\boldsymbol{\beta}})^5}$$

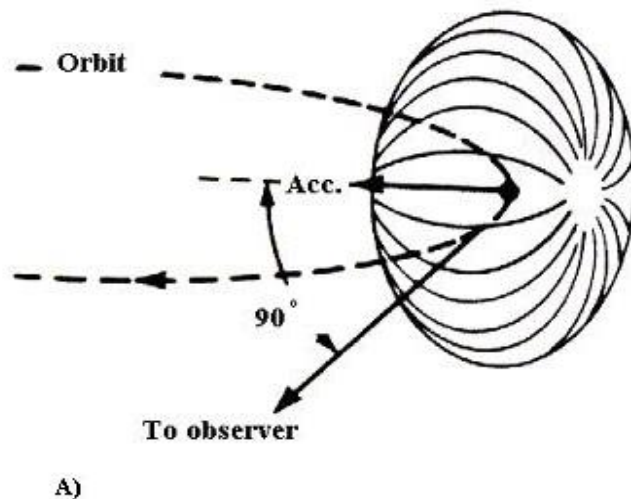
**The emission is peaked in the direction of the velocity**

The pattern depends on the details of velocity and acceleration but it is dominated by the denominator

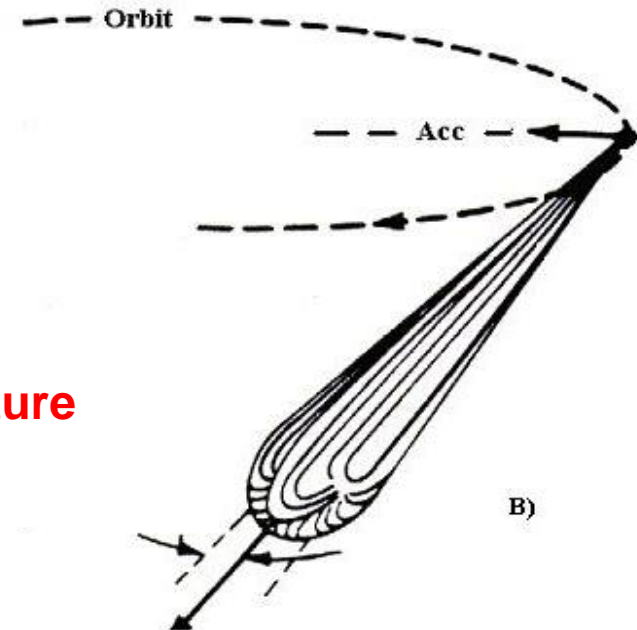
# velocity $\perp$ acceleration: synchrotron radiation (I)

Assuming  $\vec{\beta} \perp \dot{\vec{\beta}}$  and substituting the acceleration field

$$\frac{d^2 P}{d\Omega} = \frac{e^2}{(4\pi)^2 \epsilon_0 c^2} \frac{\left| \vec{n} \times \left[ (\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right] \right|^2}{(1 - \vec{n} \cdot \vec{\beta})^5} = \frac{e^2 |\dot{\vec{\beta}}|^2}{(4\pi)^2 \epsilon_0 c} \frac{1}{(1 - \beta \cos \theta)^3} \left[ 1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right]$$

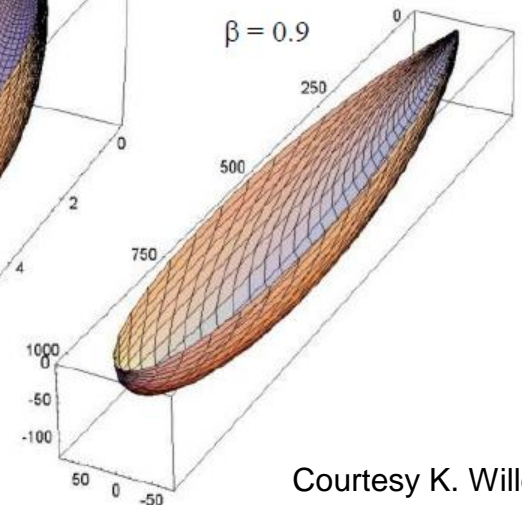
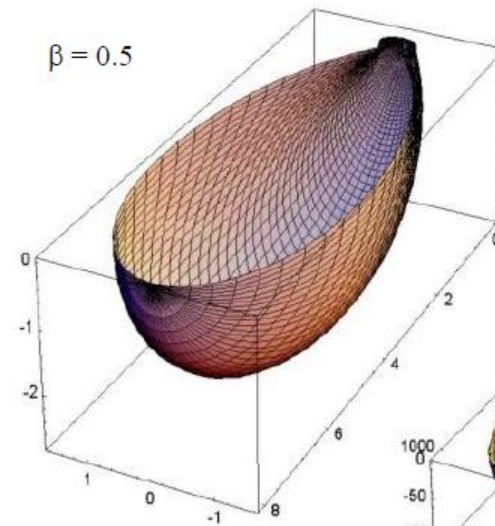
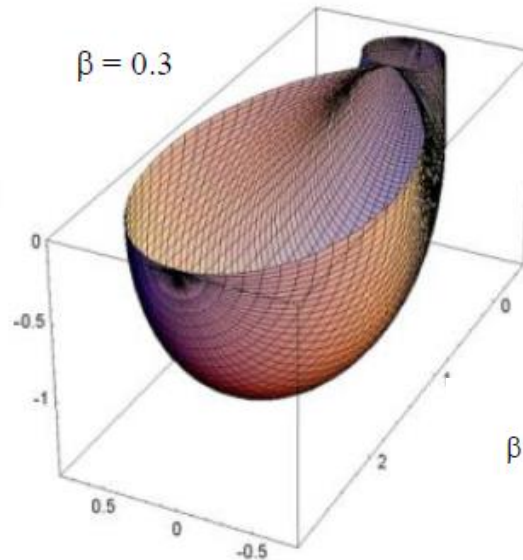
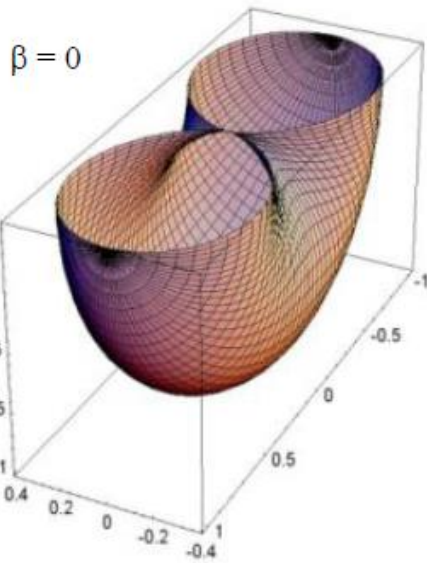


cone aperture  
 $\sim 1/\gamma$



When the electron velocity approaches the speed of light, the emission pattern is sharply collimated forward

# velocity $\perp$ acceleration: synchrotron radiation (II)



Courtesy K. Wille

# Total radiated power via synchrotron radiation

Integrating over the whole solid angle we obtain the total instantaneous power radiated by one electron

$$P = \frac{e^2}{6\pi\epsilon_0 c} \left| \dot{\vec{\beta}} \right|^2 \gamma^4 = \frac{e^2}{6\pi\epsilon_0 c} \left| \dot{\vec{\beta}} \right|^2 \frac{E^4}{E_o^4} = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\vec{p}}{dt} \right|^2 \gamma^2 = \frac{e^2 c}{6\pi\epsilon_0} \frac{\gamma^4}{\rho^2} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2$$

- Strong dependence  $1/m^4$  on the rest mass
- proportional to  $1/\rho^2$  ( $\rho$  is the bending radius)
- proportional to  $B^2$  ( $B$  is the magnetic field of the bending dipole)

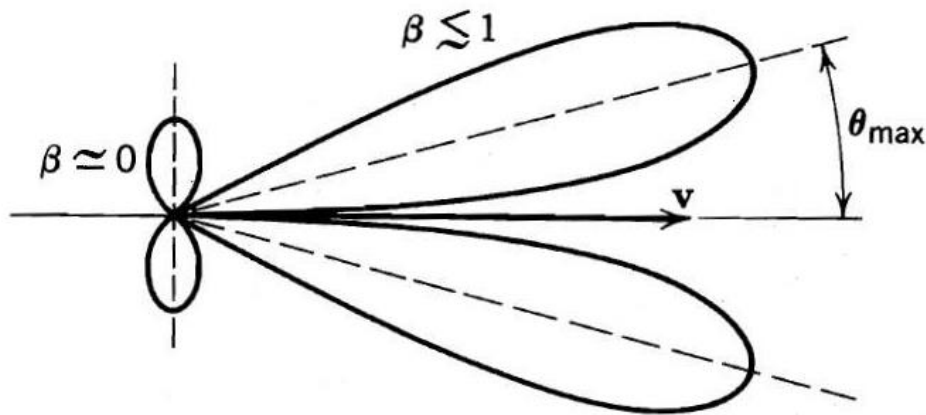
The radiation power emitted by an electron beam in a storage ring is very high.

The surface of the vacuum chamber hit by synchrotron radiation must be cooled.

# velocity || acceleration: bremsstrahlung

Assuming  $\vec{\beta} \parallel \dot{\vec{\beta}}$  and substituting the acceleration field

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 \epsilon_0 c} \frac{\left| \vec{n} \times \left[ (\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right] \right|^2}{(1 - \vec{n} \cdot \vec{\beta})^5} = \frac{e^2 |\dot{\vec{\beta}}|^2}{(4\pi)^2 \epsilon_0 c} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$



$$\theta_{\max} = \arccos \left[ \frac{1}{3\beta} \left( \sqrt{1 + 15\beta^2} - 1 \right) \right] \rightarrow \frac{1}{2\gamma}$$

Integrating over the angles as before gives the total radiated power

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\vec{p}}{dt} \right|^2$$

# Comparison of radiation from linear and circular trajectories

Back to the general expression for the acceleration field, integrating over the angles gives the total radiated power

$$P = \frac{e^2}{6\pi\epsilon_0 c} \gamma^6 \left[ (\dot{\vec{\beta}})^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right]$$

Relativistic generalization of Larmor's formula

The total radiated power can also be computed by relativistic transformation of the 4-acceleration in Larmor's formula

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\vec{p}}{dt} \right|^2 \quad \text{with} \quad t \rightarrow \tau \quad \text{and} \quad \frac{d\vec{p}}{dt} \rightarrow \frac{dp_\mu}{dt}$$

We build the relativistic invariant

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left[ \left| \frac{d\vec{p}}{d\tau} \right|^2 - \frac{1}{c^2} \left( \frac{dE}{d\tau} \right)^2 \right]$$

# Comparison of radiation from linear and circular trajectories

The particle energy is

$$E^2 = (mc^2)^2 + p^2 c^2$$

Therefore

$$E \frac{dE}{d\tau} = c^2 p \frac{dp}{d\tau} \quad \text{and} \quad \frac{dE}{d\tau} = v \frac{dp}{d\tau}$$

Inserting in the formula we get

$$P_s = \frac{e^2 c}{6\pi\epsilon_0 (m_0 c^2)^2} \left[ \left( \frac{dp}{d\tau} \right)^2 - \left( \frac{v}{c} \right)^2 \left( \frac{dp}{d\tau} \right)^2 \right] = \frac{e^2 c}{6\pi\epsilon_0 (m_0 c^2)^2} (1 - \beta^2) \left( \frac{dp}{d\tau} \right)^2$$

For a linear trajectory

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left| \frac{d\vec{p}}{dt} \right|^2$$



# Comparison of radiation from linear and circular trajectories

Repeating for a circular trajectory

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \left[ \left| \frac{d\vec{p}}{d\tau} \right|^2 - \frac{1}{c^2} \left( \frac{dE}{d\tau} \right)^2 \right]$$

The particle energy is now constant

$$E^2 = (mc^2)^2 + p^2 c^2$$

Therefore

$$\frac{dE}{d\tau} = 0$$

Inserting in the formula we get the total radiated power in a circular trajectory

$$P = \frac{e^2}{6\pi\epsilon_0 m^2 c^3} \gamma^2 \left| \frac{d\vec{p}}{dt} \right|^2$$

**This is  $\gamma^2$  larger than  
the linear case**

$$P(\mathbf{v} \parallel \mathbf{a}) \approx 1/\gamma^2 P(\mathbf{v} \perp \mathbf{a})$$

# Energy loss via synchrotron radiation emission in a storage ring

In the time  $T_b$  spent in the bendings  
the particle loses the energy  $U_0$

$$U_0 = \int P dt = P T_b = P \frac{2\pi\rho}{c} = \frac{e^2}{3\epsilon_0} \frac{\gamma^4}{\rho}$$

i.e. **Energy Loss per turn** (per electron)

$$U_0 (keV) = \frac{e^2 \gamma^4}{3\epsilon_0 \rho} = 88.46 \frac{E(GeV)^4}{\rho(m)}$$

**Power radiated by a beam of average  
current  $I_b$** : this power loss has to be  
compensated by the RF system

$$N_{tot} = \frac{I_b \cdot T_{rev}}{e}$$

$$P(kW) = \frac{e\gamma^4}{3\epsilon_0 \rho} I_b = 88.46 \frac{E(GeV)^4 I(A)}{\rho(m)}$$

**Power radiated by a beam of average  
current  $I_b$  in a dipole of length  $L$**   
(energy loss per second)

$$P(kW) = \frac{e\gamma^4}{6\pi\epsilon_0 \rho^2} L I_b = 14.08 \frac{L(m) I(A) E(GeV)^4}{\rho(m)^2}$$

# The radiation integral (I)

The energy received by an observer (per unit solid angle at the source) is

$$\frac{d^2W}{d\Omega} = \int_{-\infty}^{\infty} \frac{d^2P}{d\Omega} dt = c\epsilon_0 \int_{-\infty}^{\infty} |R\bar{E}(t)|^2 dt$$

Using the Fourier Transform we move to the frequency space

$$\frac{d^2W}{d\Omega} = 2c\epsilon_0 \int_0^{\infty} |R\bar{E}(\omega)|^2 d\omega$$

Angular and frequency distribution of the energy received by an observer

$$\frac{d^3W}{d\Omega d\omega} = 2\epsilon_0 c R^2 \left| \hat{\bar{E}}(\omega) \right|^2$$

Neglecting the velocity fields and assuming the observer in the far field:  
n constant, R constant

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\bar{n} \times [(\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}}]}{(1 - \bar{n} \cdot \bar{\beta})^2} e^{i\omega(t - \bar{n} \cdot \bar{r}(t)/c)} dt \right|^2$$

**Radiation Integral**

# The radiation integral (II)

The radiation integral can be simplified to [see Jackson]

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\infty}^{\infty} \bar{n} \times (\bar{n} \times \bar{\beta}) e^{i\omega(t - \bar{n} \cdot \bar{r}(t)/c)} dt \right|^2$$

**How to solve it?**

- ✓ **determine the particle motion**  $\bar{r}(t); \bar{\beta}(t); \dot{\bar{\beta}}(t)$
- ✓ **compute the cross products and the phase factor**
- ✓ **integrate each component and take the vector square modulus**

**Calculations are generally quite lengthy: even for simple cases as for the radiation emitted by an electron in a bending magnet they require Airy integrals or the modified Bessel functions (available in MATLAB)**

# Radiation integral for synchrotron radiation

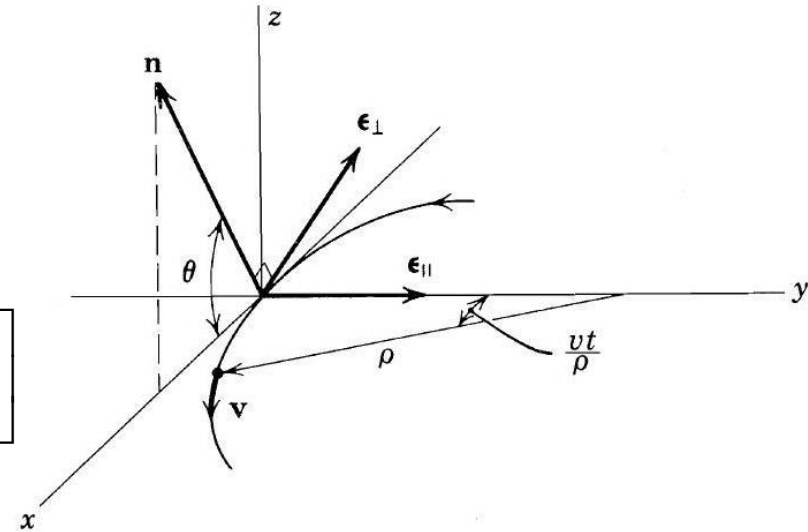
Trajectory of the arc of circumference [see Jackson]

$$\bar{r}(t) = \left( \rho \left( 1 - \cos \frac{\beta c}{\rho} t \right), \sin \frac{\beta c}{\rho} t, 0 \right)$$

In the limit of small angles we compute

$$\bar{n} \times (\bar{n} \times \bar{\beta}) = \beta \left[ -\bar{\varepsilon}_{\parallel} \sin \left( \frac{\beta c t}{\rho} \right) + \bar{\varepsilon}_{\perp} \cos \left( \frac{\beta c t}{\rho} \right) \sin \theta \right]$$

$$\omega \left( t - \frac{\bar{n} \cdot \bar{r}(t)}{c} \right) = \omega \left[ t - \frac{\rho}{c} \sin \left( \frac{\beta c t}{\rho} \right) \cos \theta \right]$$



Substituting into the radiation integral and introducing  $\xi = \frac{\rho \omega}{3c\gamma^3} (1 + \gamma^2 \theta^2)^{3/2}$

$$\frac{d^3 W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \varepsilon_0 c} \left( \frac{2\omega \rho}{3c\gamma^2} \right)^2 (1 + \gamma^2 \theta^2)^2 \left[ K_{2/3}^2(\xi) + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} K_{1/3}^2(\xi) \right]$$

# Critical frequency and critical angle

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \left( \frac{2\omega\rho}{3c\gamma^2} \right)^2 (1 + \gamma^2 \theta^2)^2 \left[ K_{2/3}^2(\xi) + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} K_{1/3}^2(\xi) \right]$$

Using the properties of the modified Bessel function we observe that the radiation intensity is negligible for  $\xi \gg 1$

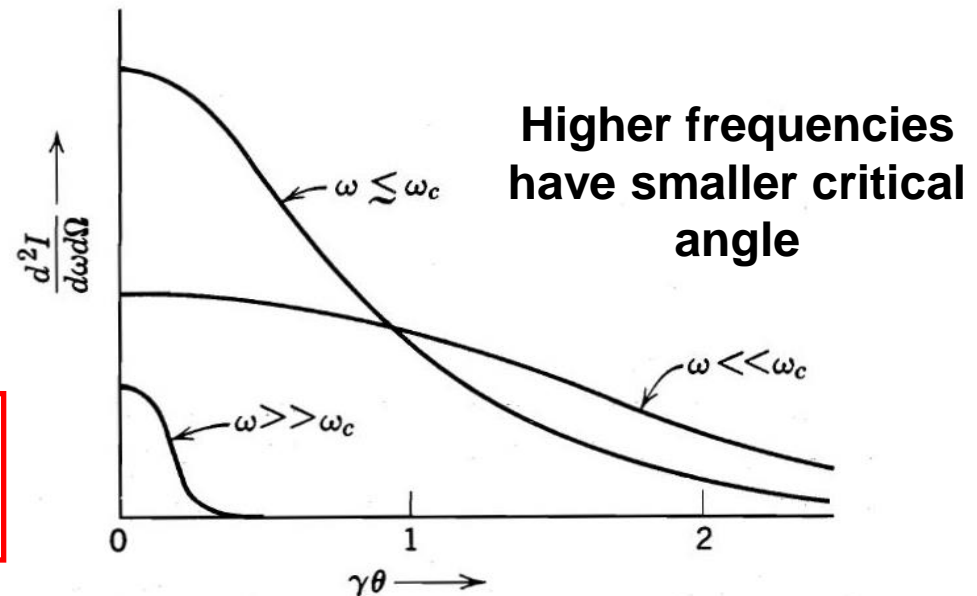
$$\xi = \frac{\omega\rho}{3c\gamma^3} (1 + \gamma^2 \theta^2)^{3/2} \gg 1$$

**Critical frequency**

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

**Critical angle**

$$\theta_c = \frac{1}{\gamma} \left( \frac{\omega_c}{\omega} \right)^{1/3}$$



For frequencies much larger than the critical frequency and angles much larger than the critical angle the synchrotron radiation emission is negligible

# Frequency distribution of radiated energy

Integrating on all angles we get the frequency distribution of the energy radiated

$$\frac{dW}{d\omega} = \iint_{4\pi} \frac{d^3 I}{d\omega d\Omega} d\Omega = \frac{\sqrt{3}e^2}{4\pi\epsilon_0 c} \gamma \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx$$

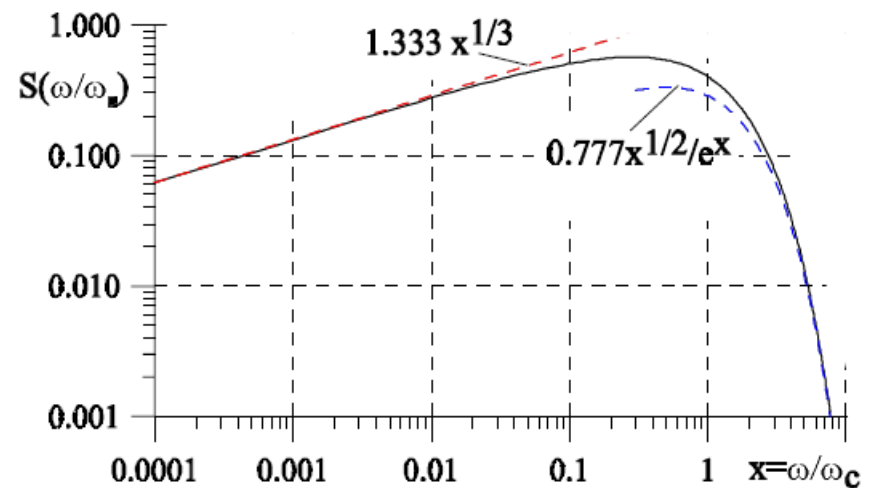
$$\frac{dW}{d\omega} \approx \frac{e^2}{4\pi\epsilon_0 c} \left( \frac{\omega\rho}{c} \right)^{1/3} \quad \omega \ll \omega_c$$

$$\frac{dW}{d\omega} \approx \sqrt{\frac{3\pi}{2}} \frac{e^2}{4\pi\epsilon_0 c} \gamma \left( \frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \quad \omega \gg \omega_c$$

often expressed in terms of the function  $S(\xi)$  with  $\xi = \omega/\omega_c$

$$S(\xi) = \frac{9\sqrt{3}}{8\pi} \xi \int_{\xi}^{\infty} K_{5/3}(x) dx \quad \int_0^{\infty} S(\xi) d\xi = 1$$

$$\frac{dW}{d\omega} = \frac{\sqrt{3}e^2 \gamma}{4\pi\epsilon_0 c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx = \frac{2e^2 \gamma}{9\epsilon_0 c} S(\xi)$$



# Frequency distribution of radiated energy

It is possible to verify that the integral over the frequencies agrees with the previous expression for the total power radiated [Hubner]

$$P = \frac{U_0}{T_b} = \frac{1}{T_b} \int_0^\omega \frac{dW}{d\omega} d\omega = \frac{1}{T_b} \frac{2e^2 \gamma}{9\epsilon_0 c} \omega_c \int_0^\omega \xi d\xi \int_\xi^\infty K_{5/3}(x) dx = \frac{e^2 c}{6\epsilon_0} \frac{\gamma^4}{\rho^2}$$

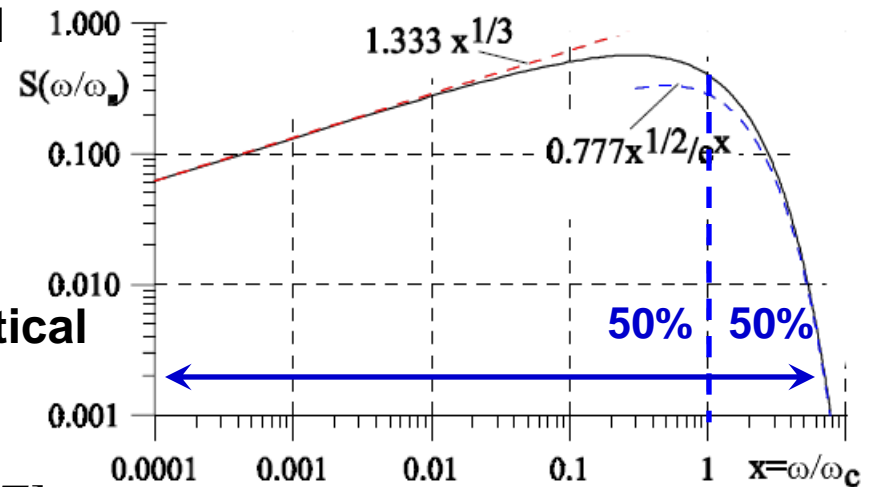
The frequency integral extended up to the critical frequency contains half of the total energy radiated, the peak occurs approximately at  $0.3\omega_c$

It is also convenient to define the critical photon energy as

$$\varepsilon_c = \hbar\omega_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

For electrons, the **critical energy** in practical units reads

$$\varepsilon_c [keV] = 2.218 \frac{E[GeV]^3}{\rho[m]} = 0.665 \cdot E[GeV]^2 \cdot B[T]$$





# Heuristic derivation of critical frequency

Synchrotron radiation is emitted in an arc of circumference with radius  $\rho$ , Angle of emission of radiation is  $1/\gamma$  (relativistic argument), therefore

$$\Delta T = \frac{\rho}{\beta c \gamma} \quad \text{transit time in the arc of dipole}$$

During this time the electron travels a distance

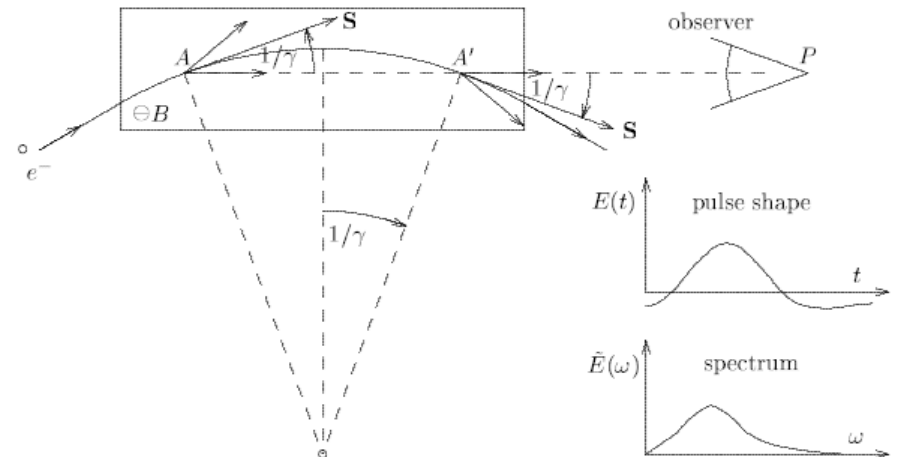
$$\Delta s = \beta c \Delta T = \frac{\rho}{\gamma}$$

The time duration of the radiation pulse seen by the observer is the difference between the time of emission of the photons and the time travelled by the electron in the arc

$$\tau = \Delta T - \frac{\Delta s}{\beta c} = \frac{\rho}{c \gamma} \left( \frac{1}{\beta} - 1 \right) = \frac{\rho}{c \gamma} \frac{1 - \beta^2}{\beta(1 + \beta)} \approx \frac{\rho}{2c \gamma^3}$$

The width of the Fourier transform of the pulse is

$$\Delta \nu \approx \frac{2c \gamma^3}{\rho}$$



# Polarisation of synchrotron radiation

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \left( \frac{2\omega\rho}{3c\gamma^2} \right)^2 (1 + \gamma^2 \theta^2)^2 \left[ K_{2/3}^2(\xi) + \frac{\gamma^2 \theta^2}{1 + \gamma^2 \theta^2} K_{1/3}^2(\xi) \right]$$

**Polarisation in  
the orbit plane**

**Polarisation orthogonal  
to the orbit plane**

In the orbit plane  $\theta = 0$ , the polarisation is purely horizontal

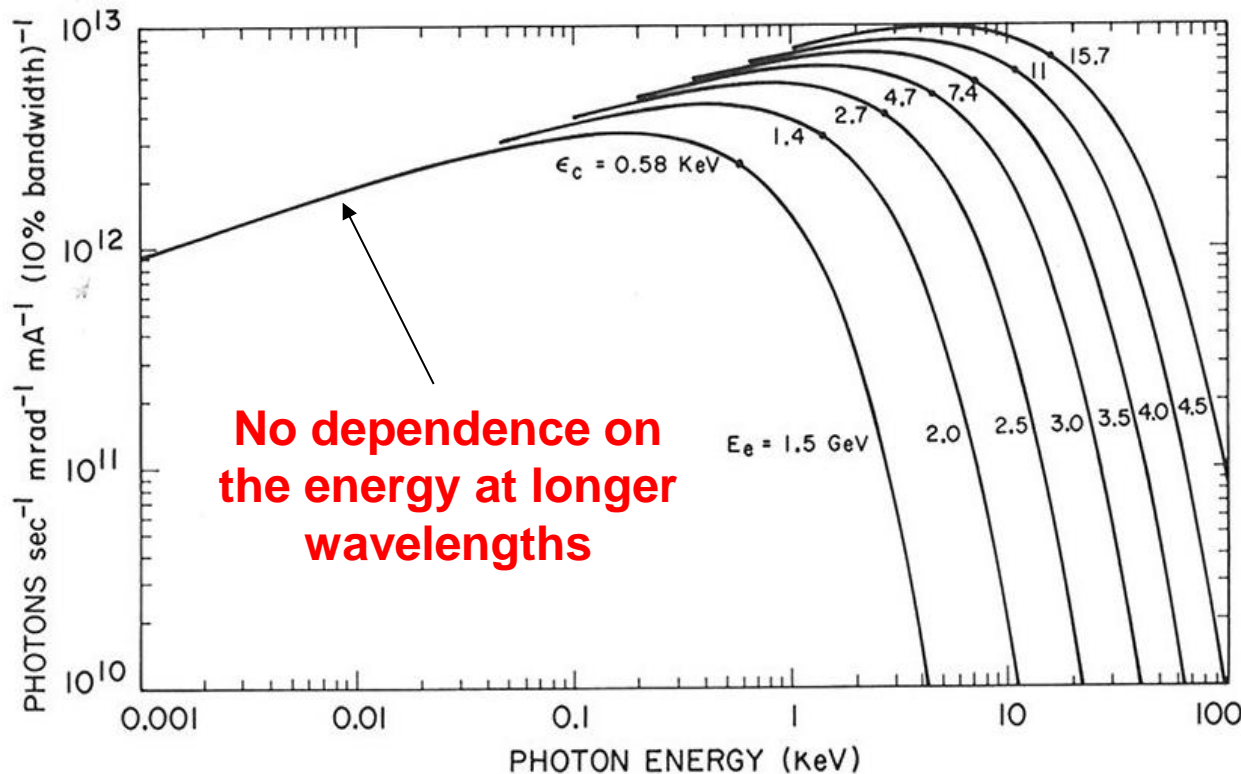
Integrating on all frequencies we get the angular distribution of the energy radiated

$$\frac{d^2W}{d\Omega} = \int_0^\infty \frac{d^3I}{d\omega d\Omega} d\omega = \frac{7e^2\gamma^5}{64\pi\epsilon_0\rho} \frac{1}{(1 + \gamma^2\theta^2)^{5/2}} \left[ 1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} \right]$$

Integrating on all the angles we get a polarization on the plan of the orbit 7 times larger than on the plan perpendicular to the orbit

# Synchrotron radiation emission as a function of beam the energy

Dependence of the frequency distribution of the energy radiated via synchrotron emission on the electron beam energy



**Critical frequency**

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

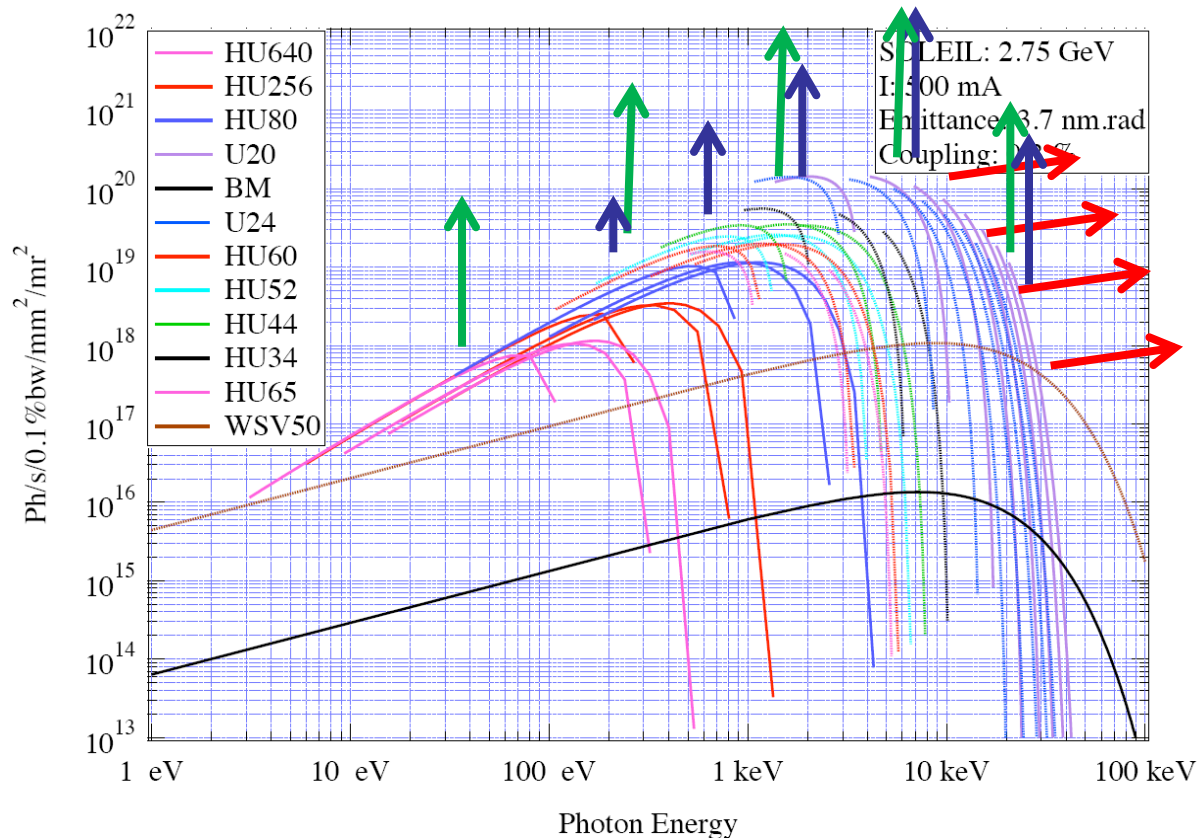
**Critical angle**

$$\theta_c = \frac{1}{\gamma} \left( \frac{\omega_c}{\omega} \right)^{1/3}$$

**Critical energy**

$$\epsilon_c = \hbar \omega_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

# Brilliance with IDs (medium energy light sources)



Brilliance dependence

with current

with energy

with emittance

**Medium energy** storage rings with **in-vacuum undulators** operated at low gaps (e.g. 5-7 mm) can reach 10 keV with a brilliance of  $10^{20}$  ph/s/0.1%BW/mm<sup>2</sup>/mrad<sup>2</sup>

# Radiation from undulators and wigglers

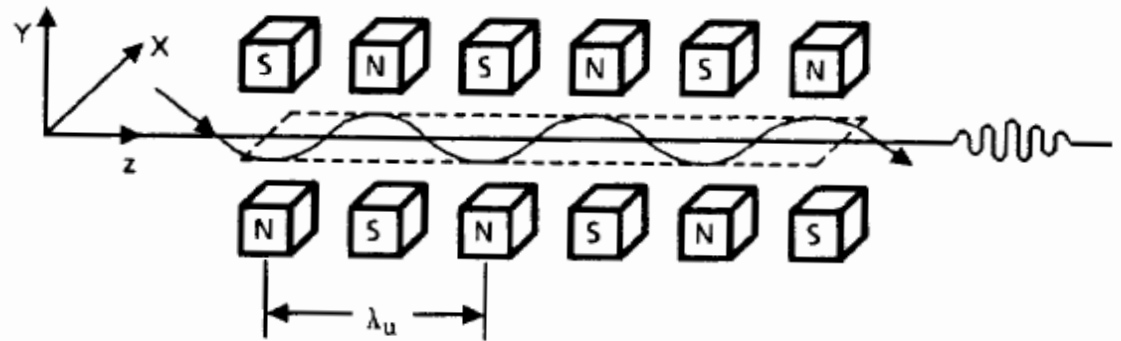
Radiation emitted by undulators and wigglers

Types of undulators and wigglers

# Undulators and wigglers

Periodic array of magnetic poles providing a sinusoidal magnetic field on axis:

$$B = (0, B_0 \sin(k_u z), 0)$$



Solution of equation of motions:

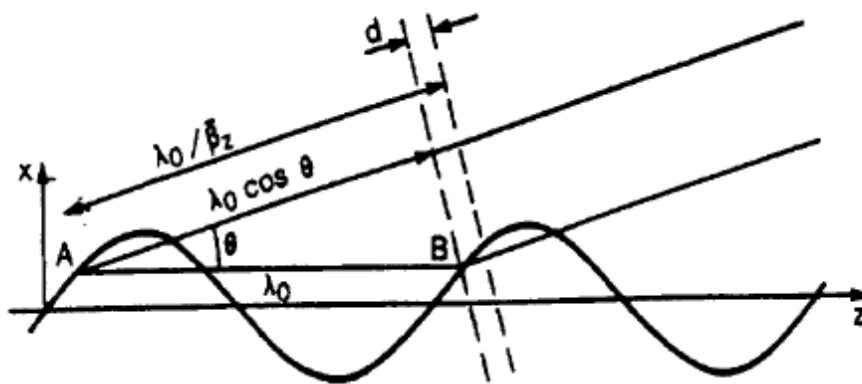
$$\vec{r}(t) = -\frac{\lambda_u K}{2\pi\gamma} \sin \omega_u t \cdot \hat{x} + \left( \bar{\beta}_z ct + \frac{\lambda_u K^2}{16\pi\gamma^2} \cos(2\omega_u t) \right) \cdot \hat{z}$$

$$K = \frac{eB_0\lambda_u}{2\pi mc}$$

Undulator  
parameter

$$\bar{\beta}_z = 1 - \frac{1}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$

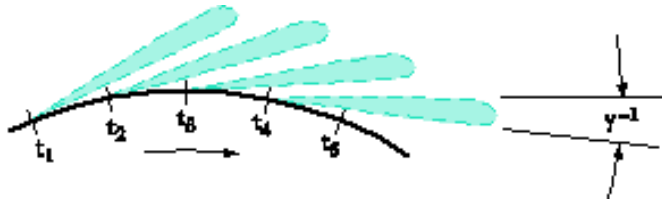
Constructive interference of radiation emitted at different poles



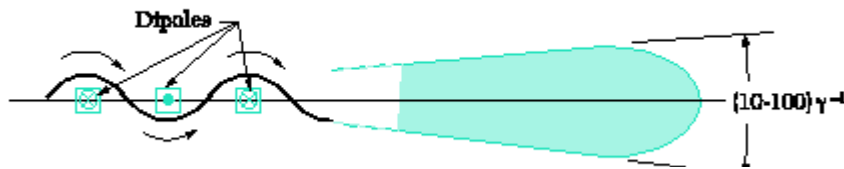
$$d = \frac{\lambda_u}{\bar{\beta}} - \lambda_u \cos \theta = n\lambda$$

$$\lambda_n = \frac{\lambda_u}{2\gamma^2 n} \left( 1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right)$$

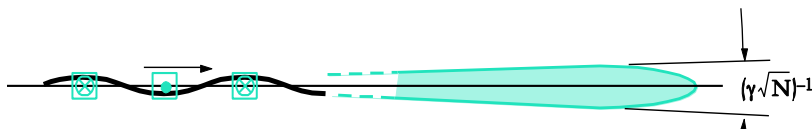
# From the lecture on synchrotron radiation



bending magnet - a “sweeping searchlight”



wiggler - incoherent superposition  $K > 1$   
Max. angle of trajectory  $> 1/\gamma$



undulator - coherent interference  $K < 1$   
Max. angle of trajectory  $< 1/\gamma$

Continuous spectrum characterized by  $\varepsilon_c$   
= critical energy

$$\varepsilon_c(\text{keV}) = 0.665 B(\text{T}) E^2(\text{GeV})$$

eg: for  $B = 1.4\text{T}$   $E = 3\text{GeV}$   $\varepsilon_c = 8.4\text{ keV}$

(bending magnet fields are usually lower  
 $\sim 1 - 1.5\text{T}$ )

Quasi-monochromatic spectrum with  
peaks at lower energy than a wiggler

$$\lambda_n = \frac{\lambda_u}{2n\gamma^2} \left( 1 + \frac{K^2}{2} \right) \approx \frac{\lambda_u}{n\gamma^2}$$

$$\varepsilon_n(\text{eV}) = 9.496 \frac{nE[\text{GeV}]^2}{\lambda_u[\text{m}] \left( 1 + \frac{K^2}{2} \right)}$$

# Radiation integral for a linear undulator (I)

The angular and frequency distribution of the energy emitted by a wiggler is computed again with the radiation integral:

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \bar{\beta}) e^{i\omega(t - \hat{n} \cdot \bar{r}/c)} dt \right|^2$$

Using the periodicity of the trajectory we can split the radiation integral into a sum over  $N_u$  terms

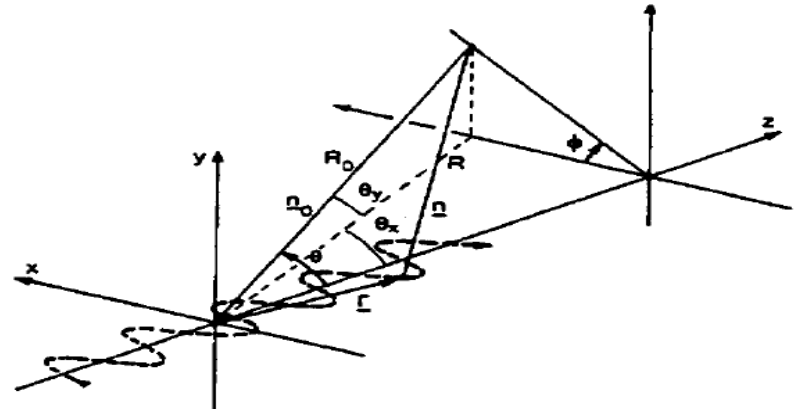
$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi\epsilon_0 4\pi^2 c} \left| \int_{-\lambda_u/2\bar{\beta}c}^{\lambda_u/2\bar{\beta}c} \hat{n} \times (\hat{n} \times \bar{\beta}) e^{i\omega(t - \hat{n} \cdot \bar{r}/c)} dt \right|^2 \left| 1 + e^{i\delta} + e^{i2\delta} + \dots + e^{i(N_u-1)\delta} \right|^2$$

where

$$\delta = \frac{2\pi\omega}{\omega_{res}(\theta)}$$

$$\omega_{res}(\theta) = \frac{2\pi c}{\lambda_{res}(\theta)}$$

$$\lambda_{res}(\theta) = \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right)$$





# Radiation integral for a linear undulator (II)

The radiation integral in an undulator or a wiggler can be written as

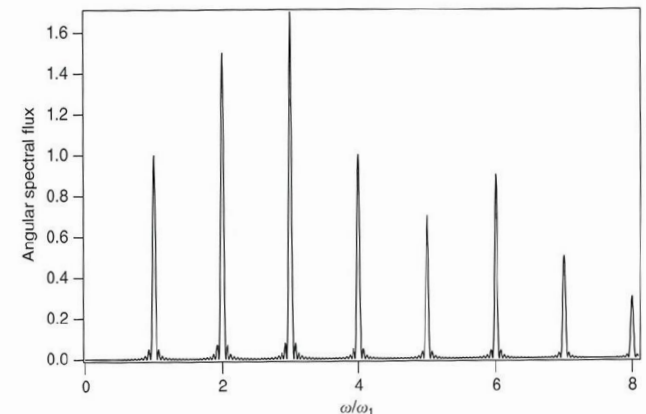
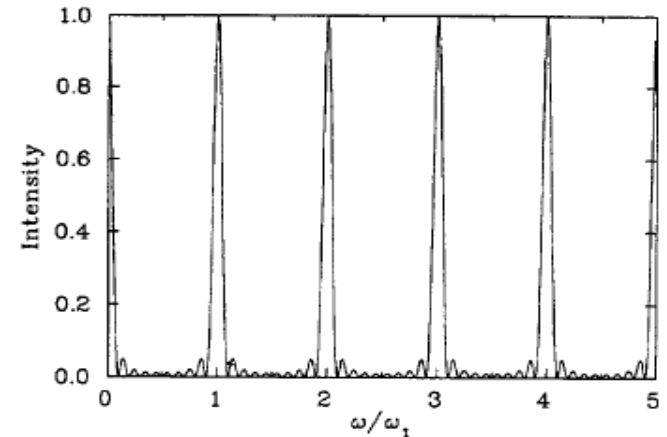
$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \gamma^2 N^2}{4\pi \epsilon_0 c} L \left( N \frac{\Delta\omega}{\omega_{res}(\theta)} \right) F_n(K, \theta, \phi) \quad \Delta\omega = \omega - n \omega_{res}(\theta)$$

The sum on  $\delta$  generates a series of sharp peaks in the frequency spectrum harmonics of the fundamental wavelength

$$L \left( N \frac{\Delta\omega}{\omega_{res}(\theta)} \right) = \frac{\sin^2(N\pi\Delta\omega / \omega_{res}(\theta))}{N^2 \sin^2(\pi\Delta\omega / \omega_{res}(\theta))}$$

The integral over one undulator period generates a modulation term  $F_n$  which depends on the angles of observations and  $K$

$$F_n(K, \theta, \phi) \propto \left| \int_{-\lambda_0/2\bar{\beta}c}^{\lambda_0/2\bar{\beta}c} \hat{n} \times (\hat{n} \times \bar{\beta}) e^{i\omega(t - \hat{n} \cdot \bar{r}/c)} dt \right|^2$$

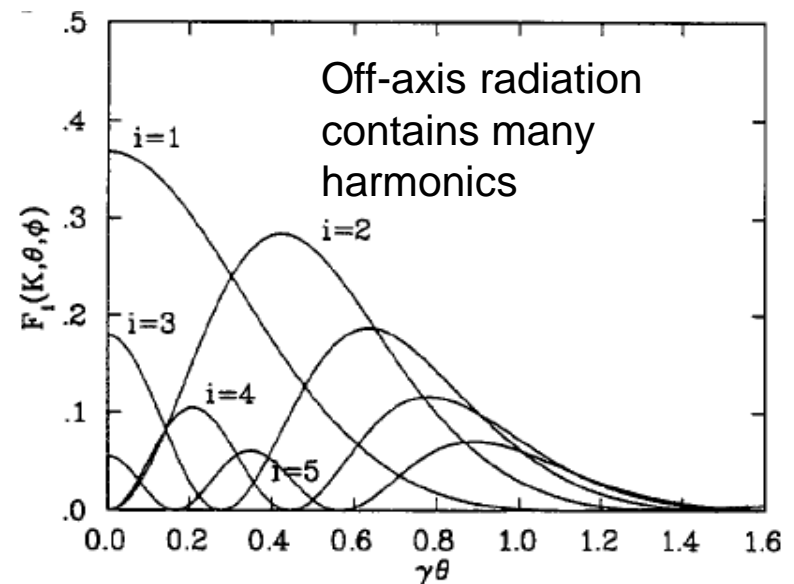
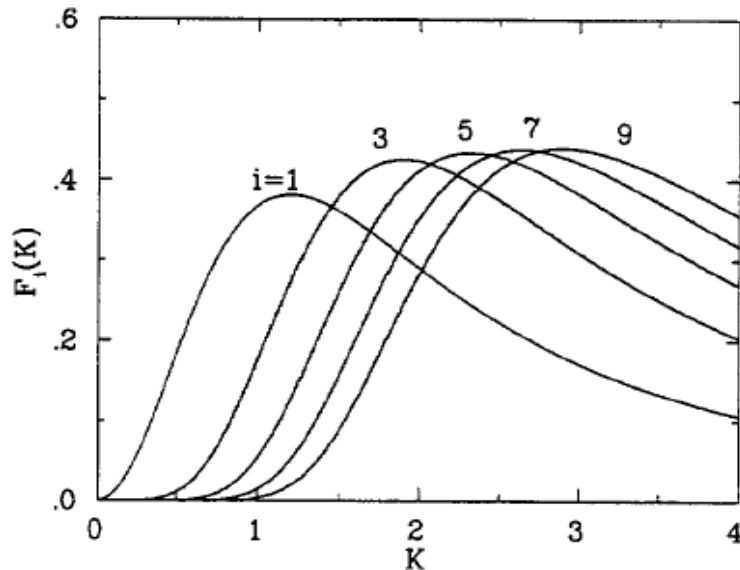


# Radiation integral for a linear undulator (II)

e.g. on axis ( $\theta = 0, \phi = 0$ ):

$$\frac{d^3W}{d\Omega d\omega} = \frac{e^2 \gamma^2 N^2}{4\pi \epsilon_0 c} L \left( N \frac{\Delta\omega}{\omega_{res}(0)} \right) F_n(K, 0, 0)$$

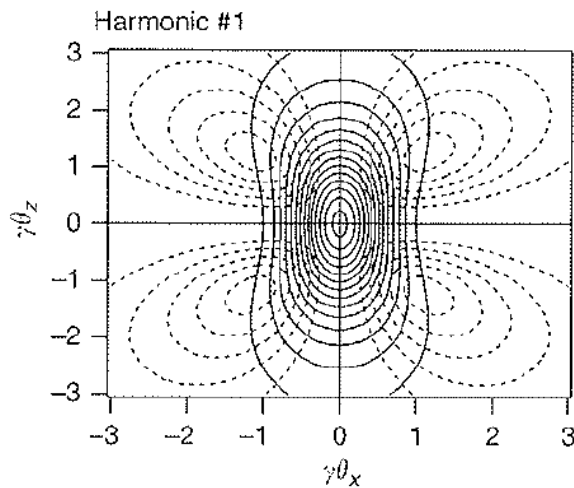
$$F_n(K, 0, 0) = \frac{n^2 K^2}{(1 + K^2/2)} \left[ J_{\frac{n+1}{2}}(Z) - J_{\frac{n-1}{2}}(Z) \right]^2 \quad Z = \frac{nK^2}{4(1 + K^2/2)}$$



Only odd harmonic are radiated on-axis;  
as  $K$  increases the higher harmonic becomes stronger

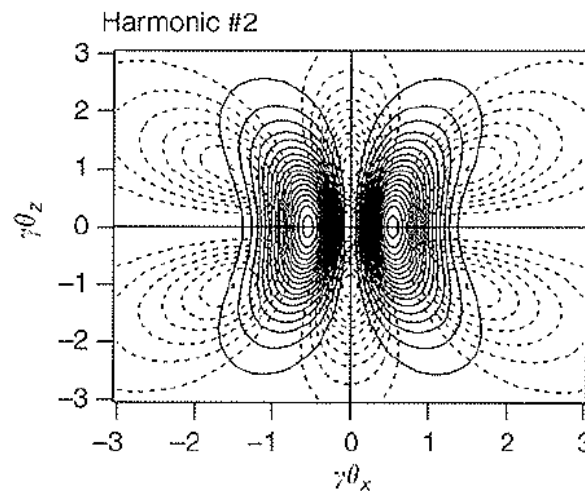
# Angular patterns of the radiation emitted on harmonics

Angular spectral flux as a function of frequency for a linear undulator; linear polarisation solid, vertical polarisation dashed ( $K = 2$ )



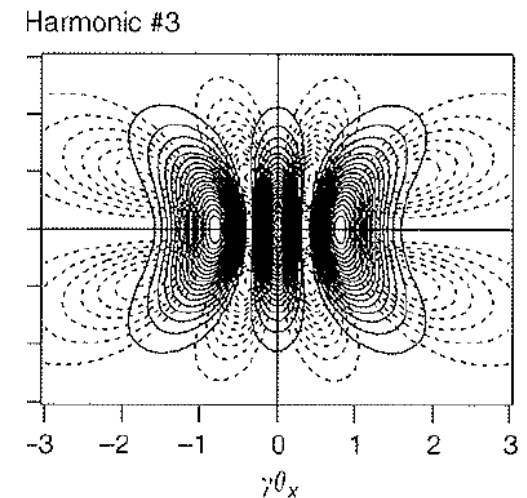
$$\lambda_1 = \frac{\lambda_u}{2\gamma^2} \left( 1 + \frac{K^2}{2} \right)$$

Fundamental wavelength emitted by the undulator



$$\lambda_2 = \frac{\lambda_1}{2}$$

2<sup>nd</sup> harmonic, not emitted on-axis !

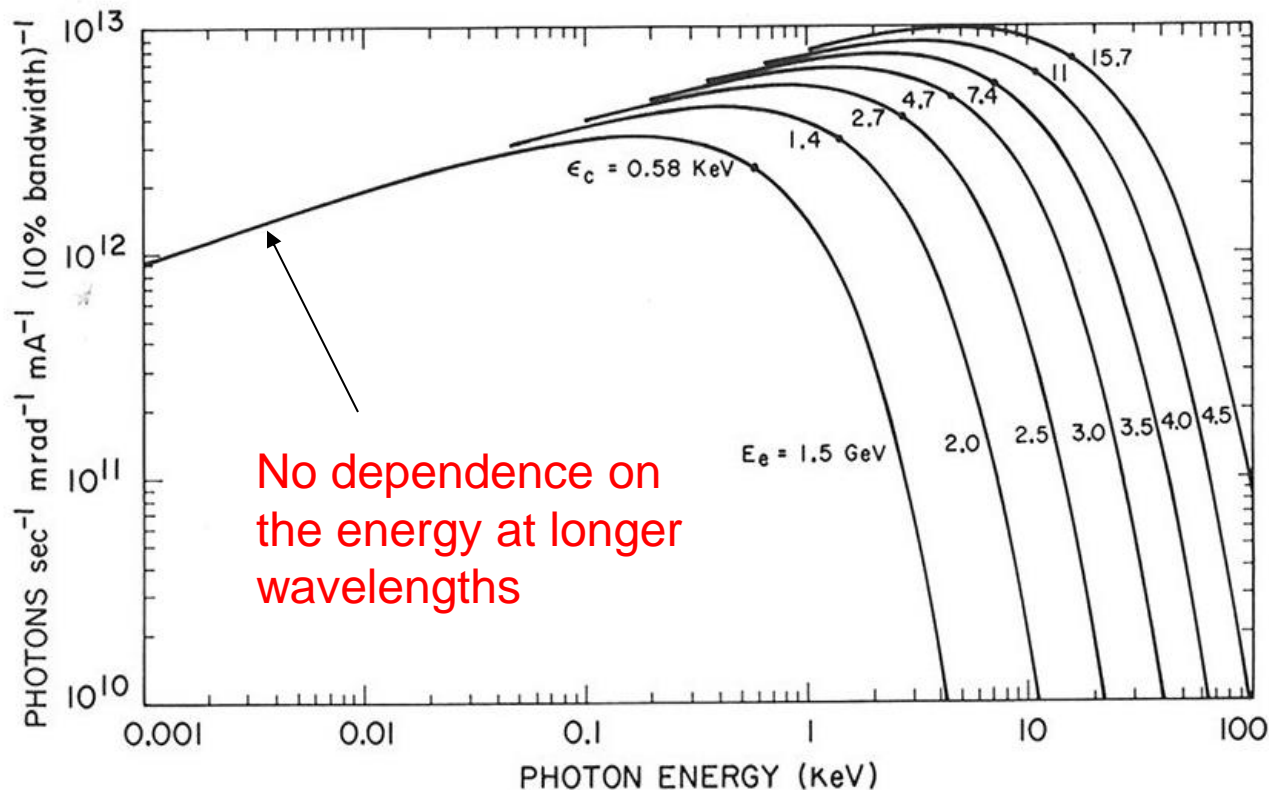


$$\lambda_2 = \frac{\lambda_1}{3}$$

3<sup>rd</sup> harmonic, emitted on-axis !

# Synchrotron radiation emission from a bending magnet

Dependence of the frequency distribution of the energy radiated via synchrotron emission on the electron beam energy



Critical frequency

$$\omega_c = \frac{3}{2} \frac{c}{\rho} \gamma^3$$

Critical angle

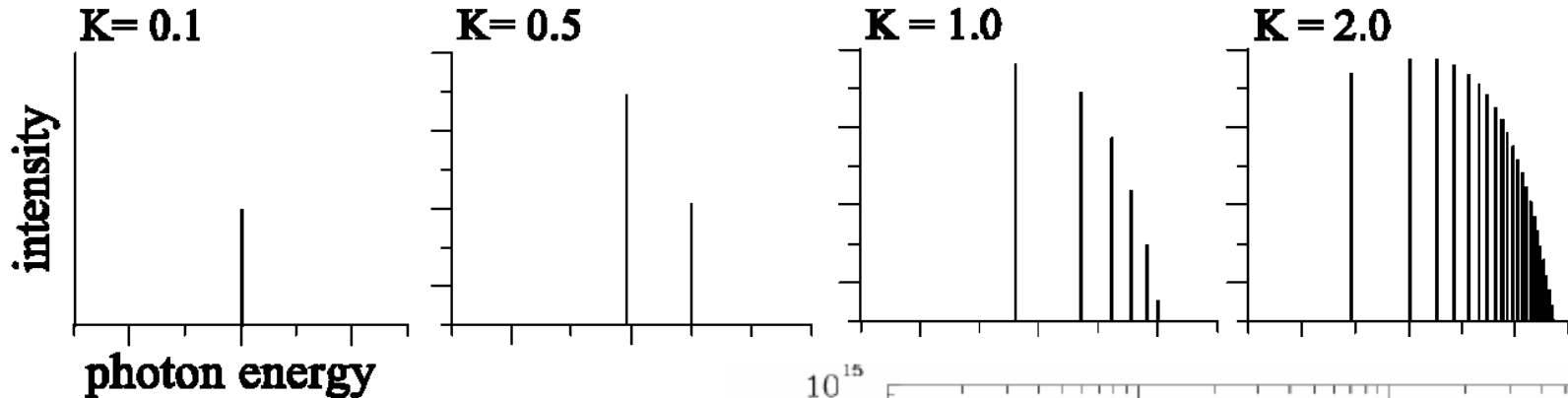
$$\theta_c = \frac{1}{\gamma} \left( \frac{\omega_c}{\omega} \right)^{1/3}$$

Critical energy

$$\varepsilon_c = \hbar \omega_c = \frac{3}{2} \frac{\hbar c}{\rho} \gamma^3$$

# Undulators and wigglers (large K)

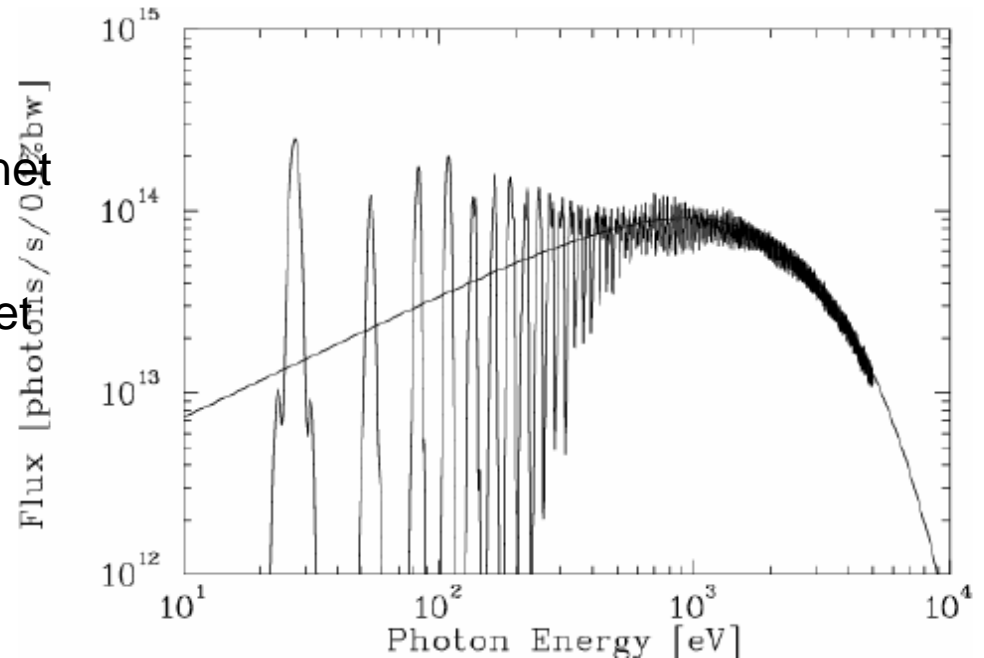
Radiated intensity emitted vs K



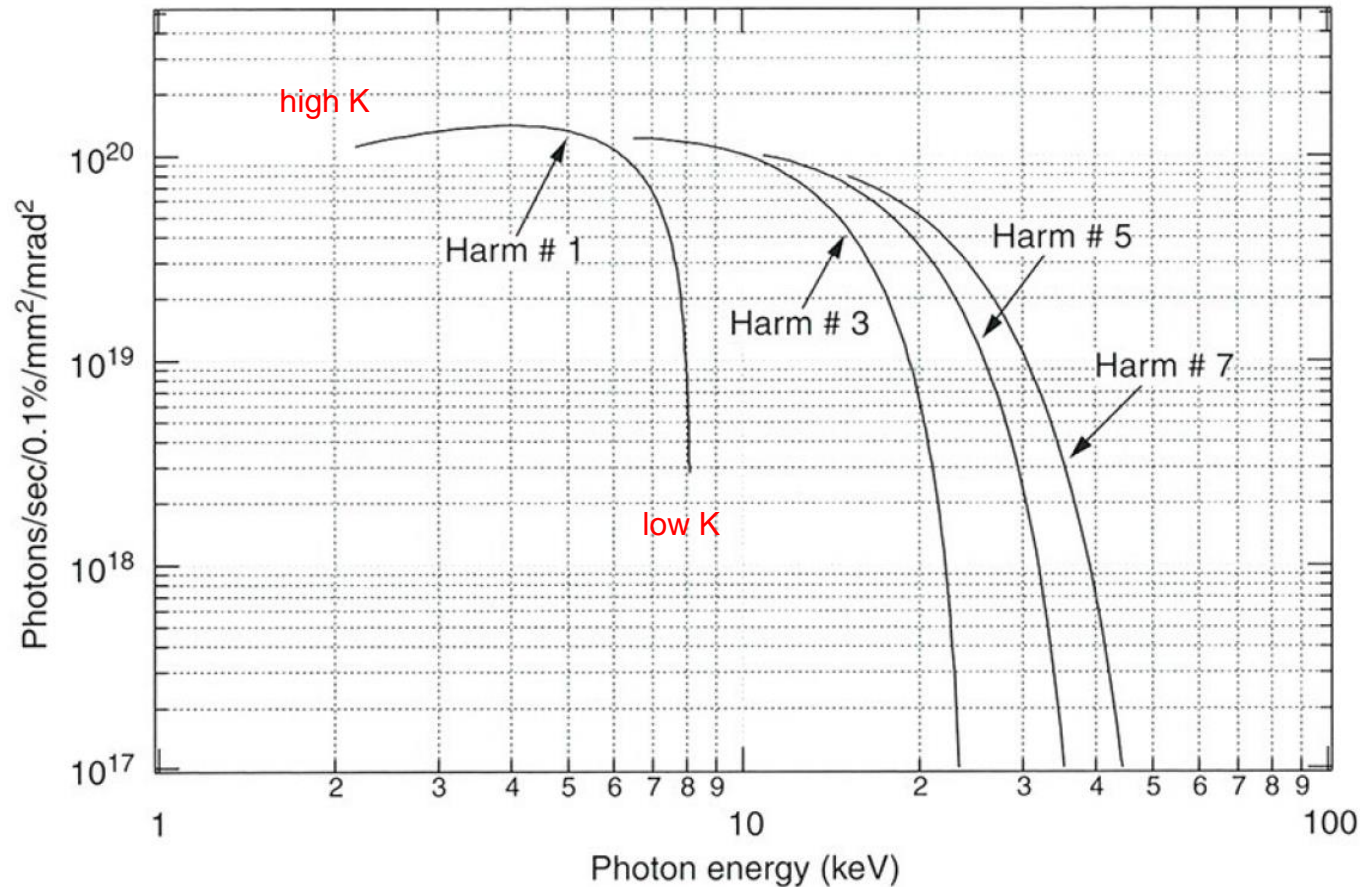
For large  $K$  the wiggler spectrum becomes similar to the bending magnet spectrum,  $2N_u$  times larger.

Fixed  $B_0$ , to reach the bending magnet critical wavelength we need:

K	1	2	10	20
n	1	5	383	3015



# Undulator tuning curve (with K)



$$K = \frac{eB_0\lambda_u}{2\pi mc}$$

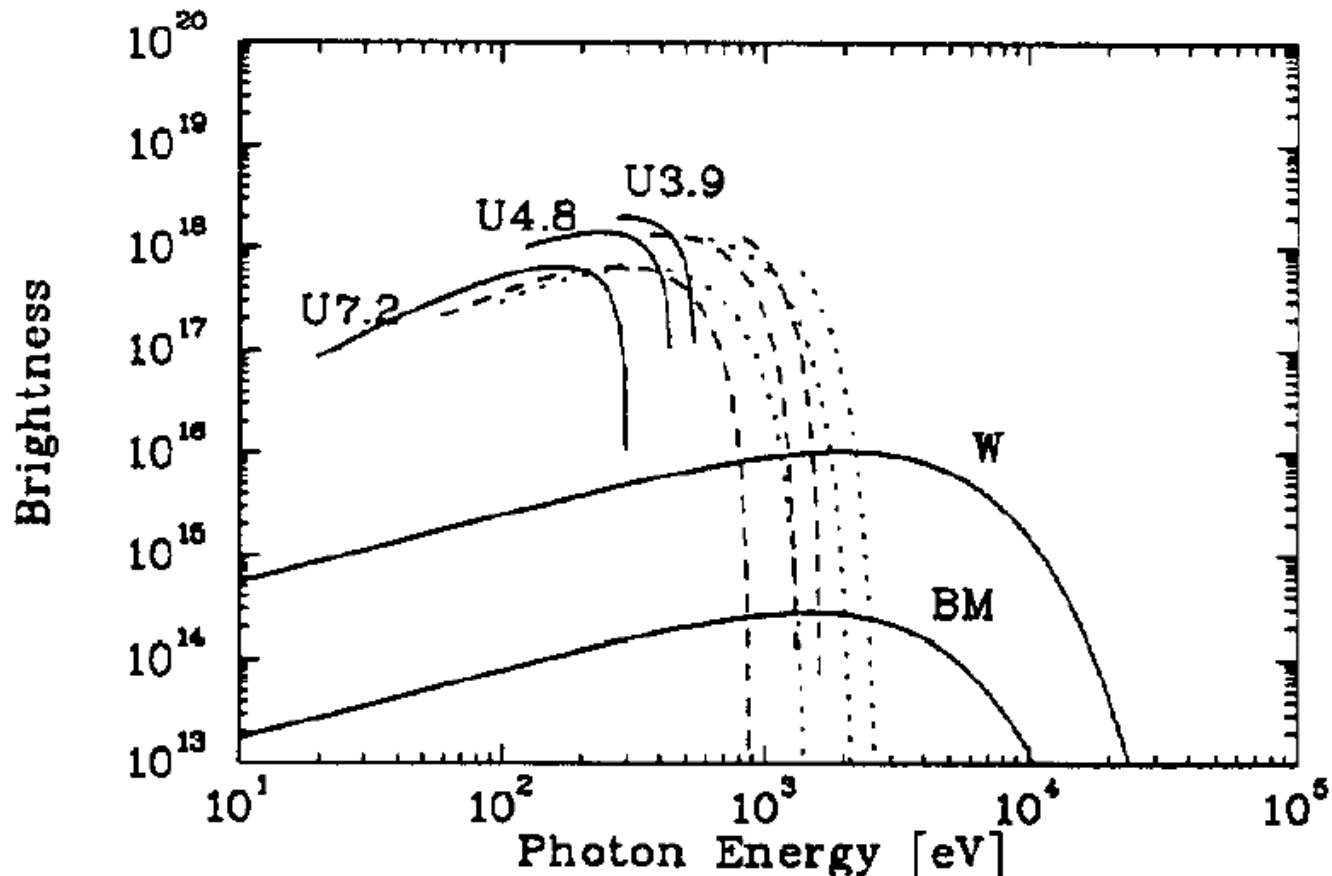
Undulator  
parameter

K decreases by  
opening the gap  
of the undulator  
(reducing B)

Brightness of a 5 m undulator 42 mm period with maximum  $K = 2.42$  (ESRF)  
Varying  $K$  one varies the wavelength emitted at various harmonics (not all wavelengths of this graph are emitted at a single time)

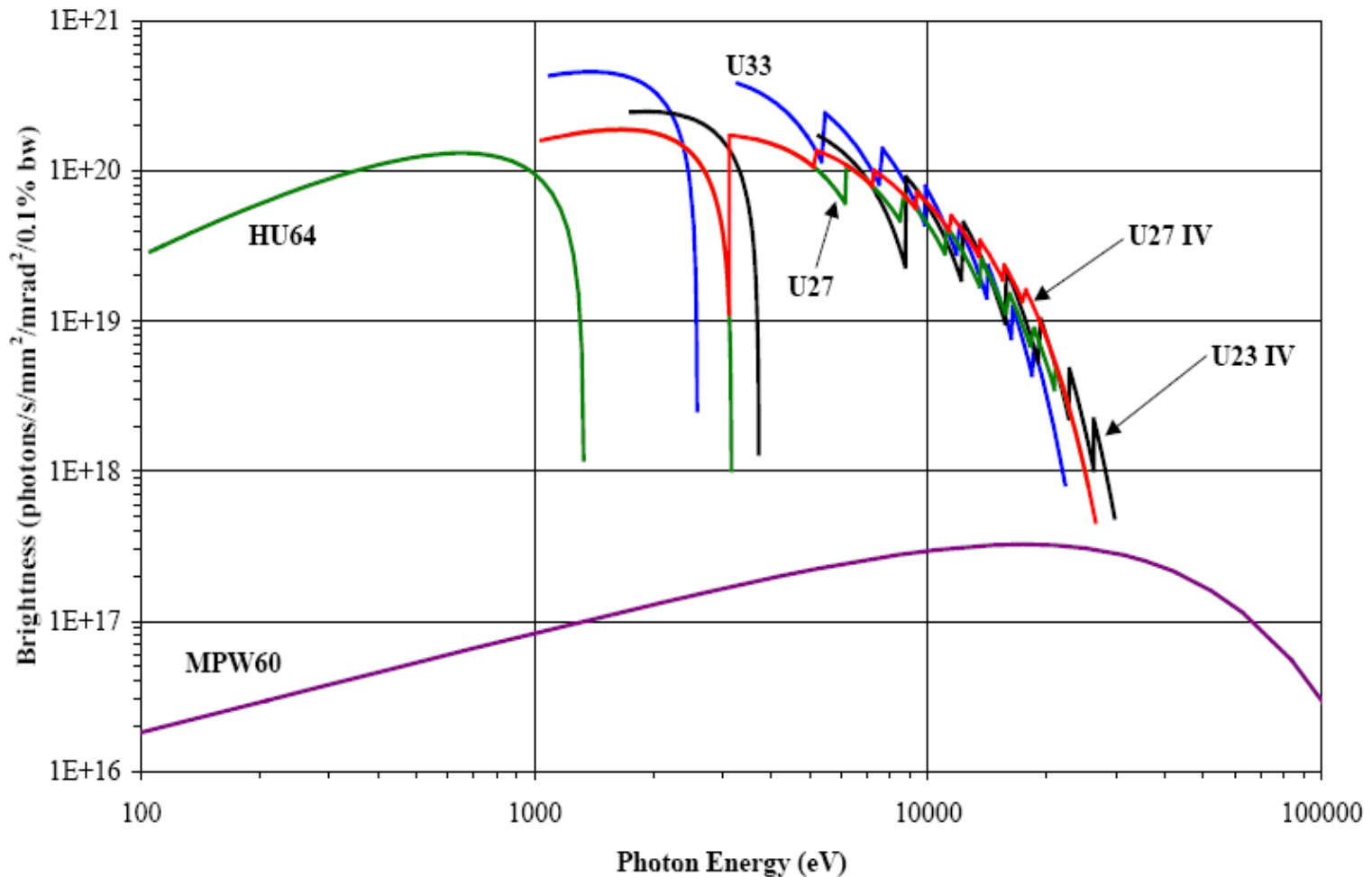
# Spectral brightness of undulators of wiggler

Comparison of undulators for a 1.5 GeV ring for three harmonics (solid dashed and dotted) compared with a wiggler and a bending magnet (ALS)



# Diamond undulators and wigglers

Spectral brightness for undulators and wigglers in state-of-the-art 3<sup>rd</sup> generation light sources





# Summary of radiation characteristics of undulators or wiggler

Undulators have weaker field or shorter periods ( $K < 1$ )

Produce narrow band radiation and harmonics

Intensity is proportional to  $N_u^2$

Wigglers have higher magnetic field ( $K > 1$ )

Produce a broadband radiation

Intensity is proportional to  $N_u$

# Type of undulators and wigglers

**Electromagnetic undulators:** the field is generated by current carrying coils; they may have iron poles;

**Permanent magnet undulators:** the field is generated by permanent magnets Samarium Cobalt (SmCo; 1T) and Neodymium Iron Boron (NdFeB; 1.4T); they may have iron poles (hybrid undulators);

**APPLE-II:** permanent magnets arrays which can slide allowing the polarisation of the magnetic field to be changed from linear to circular

**In-vacuum:** permanent magnets arrays which are located in-vacuum and whose gap can be closed to very small values (< 5 mm gap!)

**Superconducting wigglers:** the field is generated by superconducting coils and can reach very high peak fields (several T, 3.5 T at Diamond)

# Electromagnetic undulators (I)



HU64 at SOLEIL:  
variable polarisation electromagnetic  
undulator

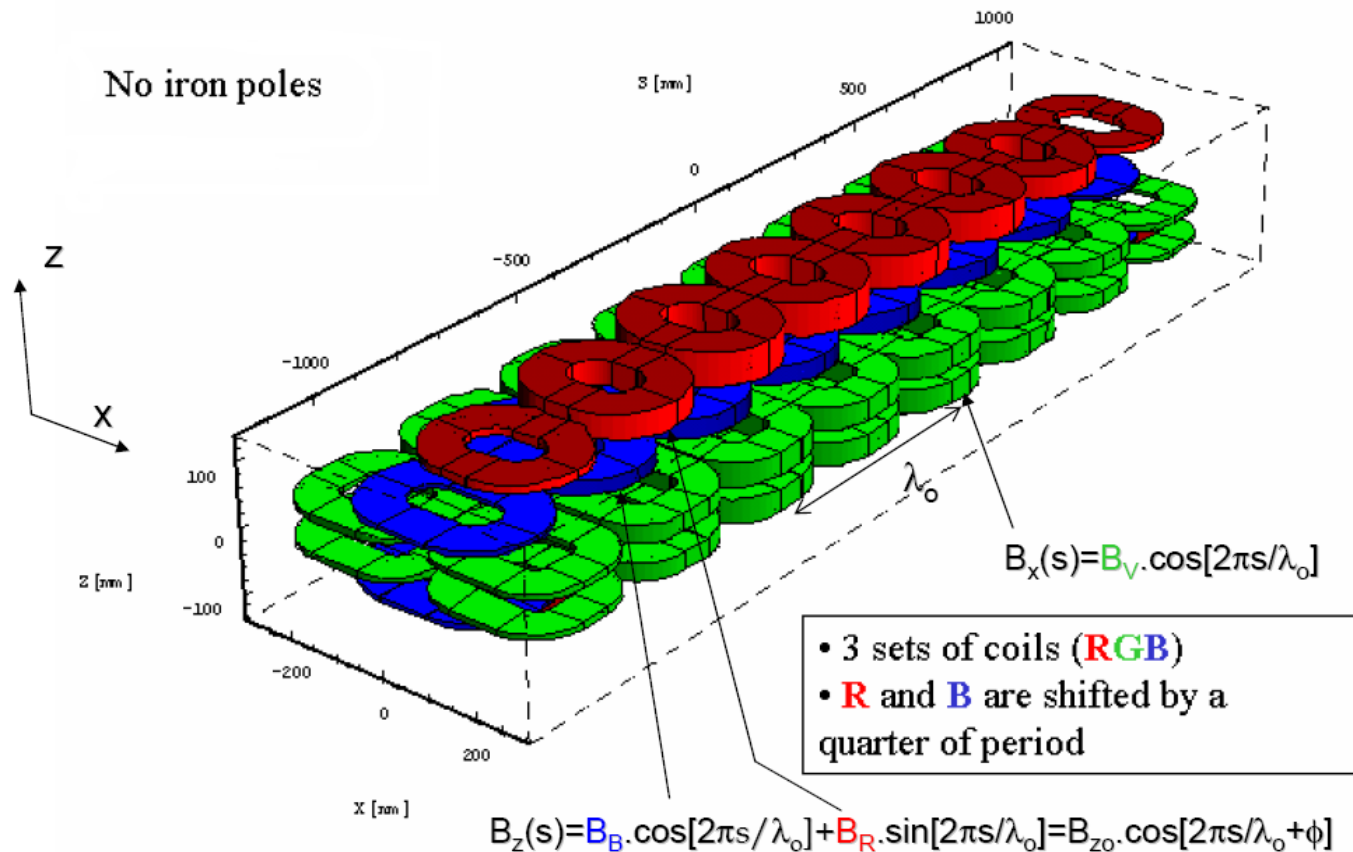
Period 64 mm

14 periods

Min gap 19 mm

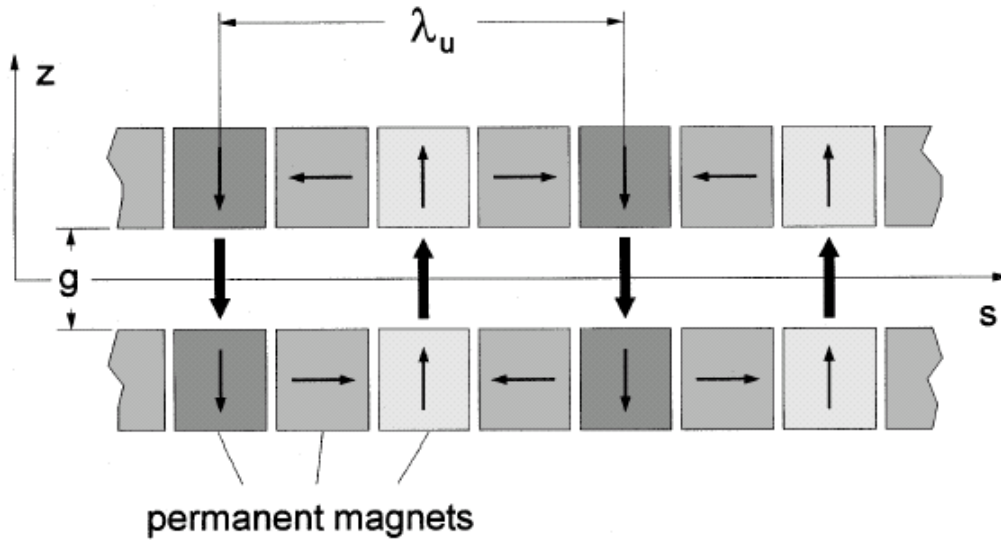
Photon energy < 40 eV (1 keV with EM undulators)

## Electromagnetic undulators (II)

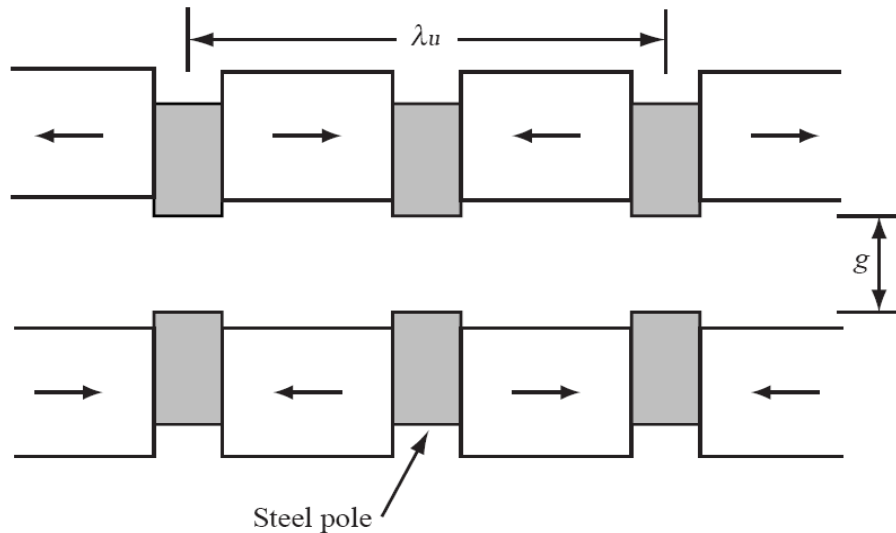


Depending on the way the coil power supplies are powered it can generate linear H, linear V or circular polarisations

# Permanent magnet undulators



Halback  
configuration



hybrid  
configuration with  
steel poles

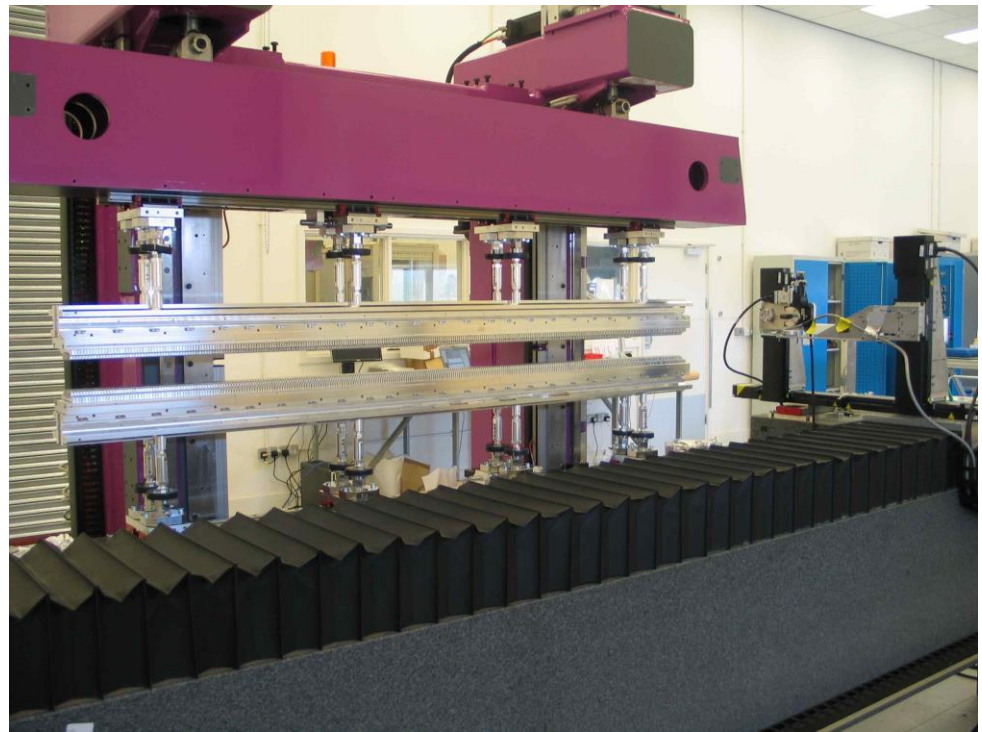


# In-vacuum undulators



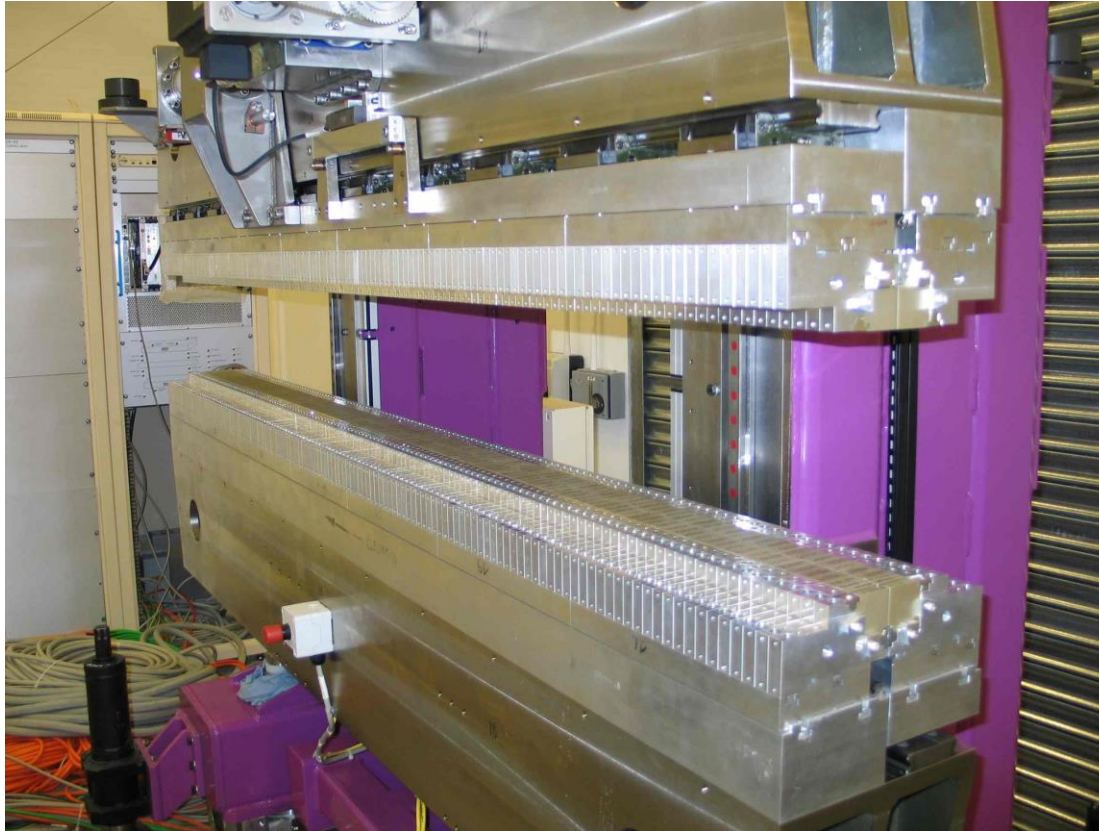
U27 at Diamond

27 mm, 73 periods 7 mm gap,  $B = 0.79 \text{ T}$ ;  $K = 2$



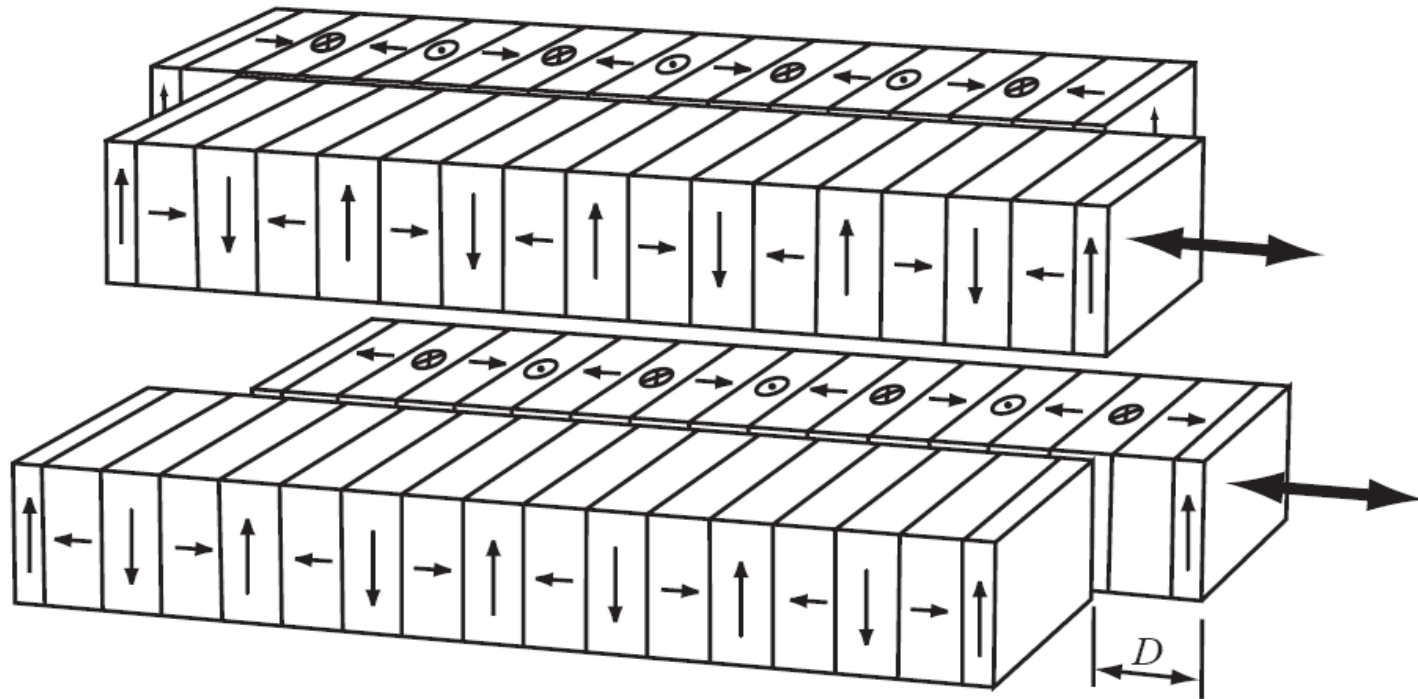


# Apple-II type undulators (I)



HU64 at Diamond; 33 period of 64 mm;  $B = 0.96$  T;  
gap 15 mm;  $K_{\text{max}} = 5.3$

## Apple-II type undulators (II)



Four independent arrays of permanent magnets

Diagonally opposite arrays move longitudinal, all arrays move vertically

Sliding the arrays of magnetic pole it is possible to control the polarisation of the radiation emitted

# Superconducting Wigglers



Superconducting wigglers are used when a high magnetic field is required

3 - 10 T

They need a cryogenic system to keep the coil superconductive

Nb<sub>3</sub>Sn and NbTi wires

SCMPW60 at Diamond

3.5 T coils cooled at 4 K

24 period of 64 mm

gap 10 mm

Undulator  $K = 21$

# Summary and bibliography

**Accelerated charged particles emit electromagnetic radiation**

**Synchrotron radiation is stronger for light particles and is emitted by bending magnets in a narrow cone within a critical frequency**

**Undulators and wigglers enhance the synchrotron radiation emission**

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