Wake Fields and Instabilities

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- **Introduction to wake fields/potentials**
- **Instability mechanism**
- **Instability in Linacs**
- **Instability in Circular Accelerators**

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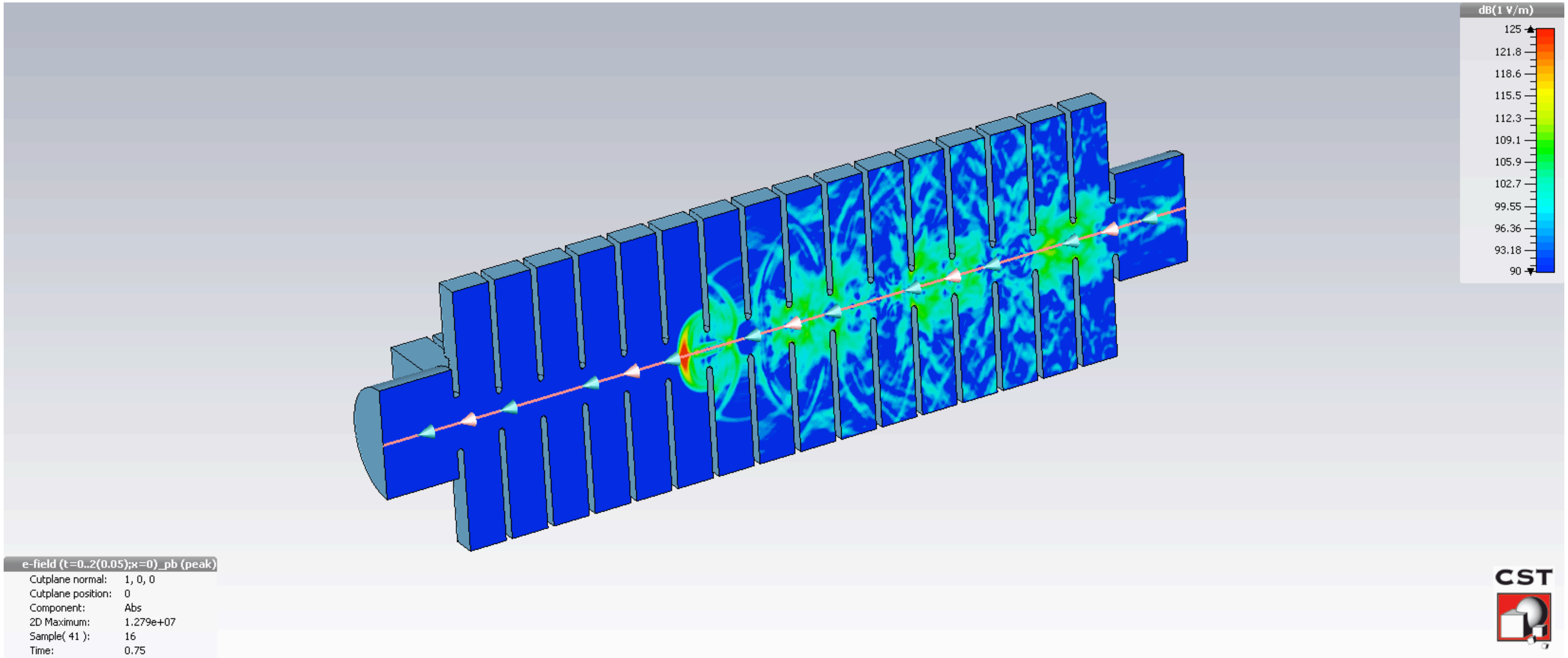
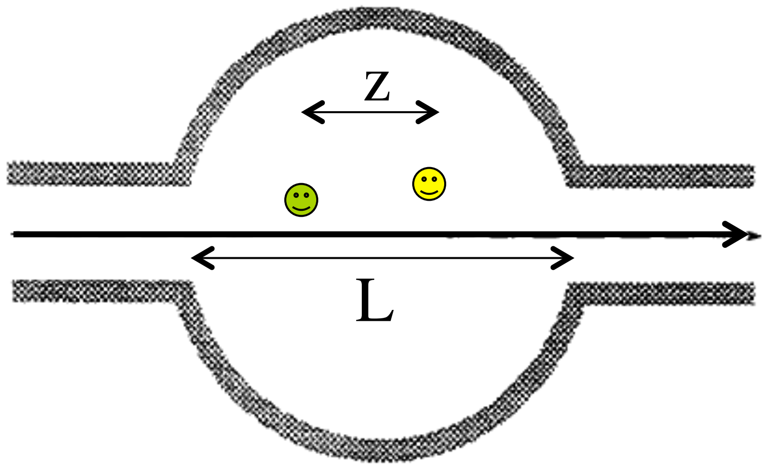

dB(1 V/m)
125
121.8
118.6
115.5
112.3
109.1
105.9
102.7
99.55
96.36
93.18
90
e-field (t=0..2(0.05);x=0)_pb (peak)
Cutplane normal: 1, 0, 0
Cutplane position: 0
Component: Abs
2D Maximum: 1.279e+07
Sample(41): 16
Time: 0.75
CST

Wake Fields and Wake Potentials



$$\boldsymbol{F} = q\left[E_z\hat{z} + \left(E_x - cB_y\right)\hat{x} + \left(E_y + cB_x\right)\hat{y}\right] \equiv \boldsymbol{F}_{//} + \boldsymbol{F}_{\perp}$$

This force depends on the longitudinal and transverse position of the two particles. It is useful to distinguish two effects on the **test charge** :

1) a longitudinal force which changes its energy,

2) a transverse force which deflects its trajectory.

If we consider a device of length L, we can perform the integral of the force acting on the test charge along the longitudinal path and get:

the Energy Gain (J):

$$\boxed{U(z) = \int_0^L F_{//} ds}$$

the Transverse Deflecting Kick (N · m) is:

$$\boxed{\boldsymbol{M}\left(r_0, z\right) = \int_0^L \mathbf{F}_{\perp} ds}$$

These quantities are both function of the distance z between the two particles. The transverse wake potential depends also on r_0, the transverse position of the source charge.

Note that the integration is performed over a given path of the trajectory.

These quantities, normalised to the charges, are called ***wake fields***

Longitudinal wake field
(Volt/Coulomb)

$$\boxed{w_{//}(z) = -\frac{U(z)}{q^2}}$$

Transverse dipole wake field
(Volt/Coulomb/meter)

$$\boxed{\boldsymbol{w}_{\perp}(z) = \frac{1}{r_0}\frac{\boldsymbol{M}(r_0, z)}{q^2}}$$

The minus sign in the longitudinal wake field means that the test charge loses energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

Coupling Impedance

The wake fields are generally useful to study the beam dynamics in the time domain (for example instabilities in a LINAC). If we take the equation of motion in the frequency domain (a trick generally used to study instabilities in circular accelerators), we need the Fourier transforms of the wake fields. Since these quantities have ohms units they are called ***coupling impedances***:

Longitudinal impedance (Ω)

$$Z_{//}(\omega) = \frac{1}{c}\int_{-\infty}^{\infty} w_{//}(z) e^{i\frac{\omega z}{c}} dz$$

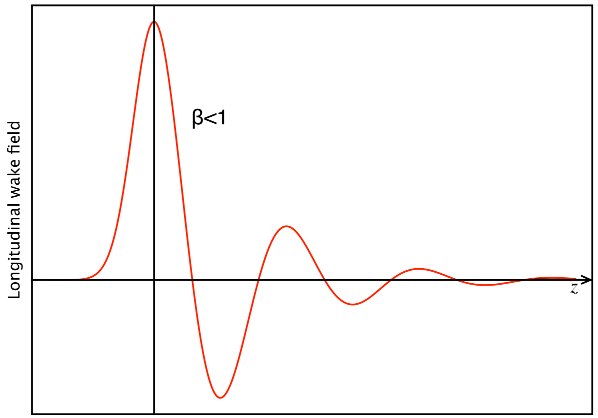
Transverse dipole impedance (Ω/m)

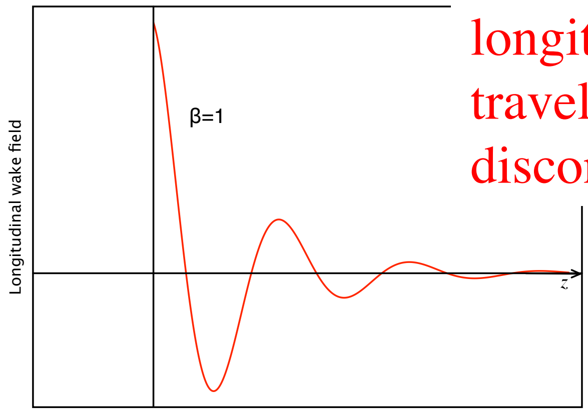
$$\boldsymbol{Z}_{\perp}(\omega) = -\frac{i}{c}\int_{-\infty}^{\infty} \boldsymbol{w}_{\perp}(z) e^{i\frac{\omega z}{c}} dz$$

It is also useful to define the ***loss factor*** as the normalised energy lost by the **source charge q**

$$\boxed{k = -\frac{U(z=0)}{q^2} \overset{??}{=} w_{//}(z=0)}$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the light velocity the longitudinal wake field is discontinuous at z=0





Causality requires that the longitudinal wake field of a charge travelling with the speed of light is discontinuous in the origin.

The exact relationship between k and $w(z\rightarrow 0)$ is given by the ***beam loading theorem*:**

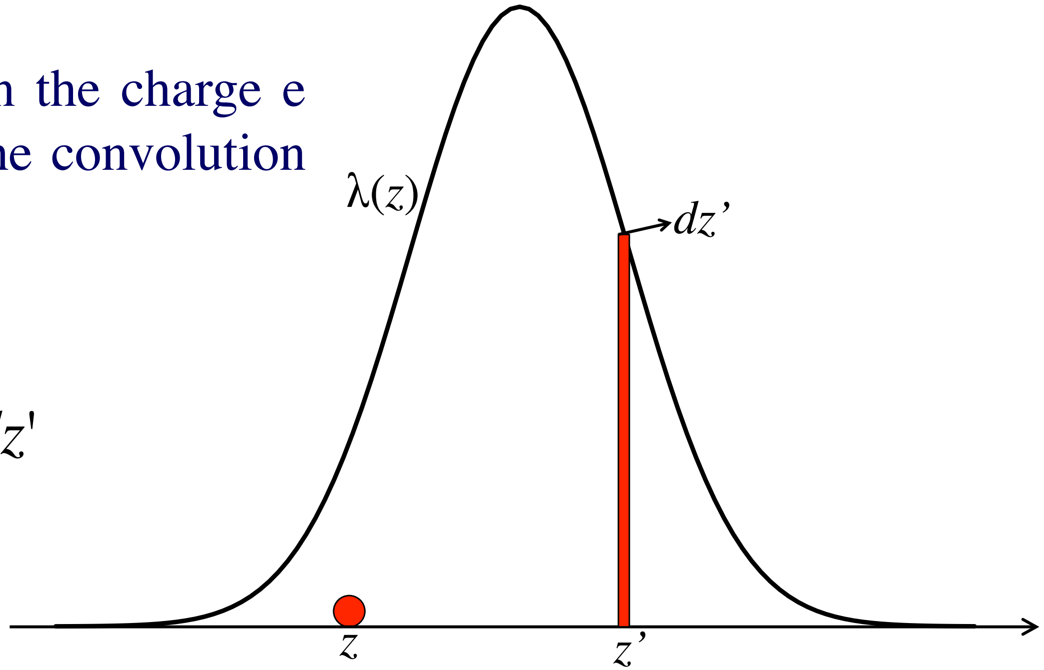
$$\boxed{k = \frac{w_{//}(z \rightarrow 0)}{2}}$$

Wake potentials and energy loss of a bunched distribution

When we have a bunch with longitudinal density λ(z), we may ask ourselves what is the amount of energy lost or gained by a single charge *e* in the beam

To this end we calculate the effect on the charge e from the whole bunch by means of the convolution integral:

$$U(z) = -e\int_{-\infty}^{\infty} w_{//}(z'-z)\lambda(z')dz'$$



Which allows to define the ***longitudinal wake potential of a distribution***

$$W_{//}(z) = -\frac{U(z)}{qe} = \frac{1}{q}\int_{-\infty}^{\infty} w_{//}(z'-z)\lambda(z')dz'$$

The total energy lost by the bunch is computed summing up the losses of all the particles:

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{\infty} U(z)\lambda(z)dz = -q\int_{-\infty}^{\infty} W_{//}(z)\lambda(z)dz$$

Numerical Analysis

The study of the fields requires to solve the Maxwell's equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the em design of accelerator devices, and new ones are developed. Examples of codes: **CST STUDIO SUITE, GDFIDL, ACE3P, ABCI, …**

Theoretical Analysis

The wake potentials depend on the particular charge distribution of the beam. It is therefore desirable to know what is the effect produced by a single charge, i.e. **find the Green function** (wake field), in order to reconstruct the fields produced by any charge distribution.

Example of longitudinal wake field and coupling impedance: space charge

Even if in the ultra-relativistic limit with $\gamma \longrightarrow \infty$, we have seen that there is no space charge effect, we can still define a wake field by considering a moderately relativistic beam with $\gamma>>1$ but not infinite. It turns out that the space charge forces can fit into the definition of wake field, and when that is done, we find that the wake depends on beam properties such as the transverse beam radius a and the beam energy γ. Let us consider a relativistic beam with cylindrical symmetry and uniform transverse distribution. We have already obtained the longitudinal force acting on a charge of the beam travelling inside a cylindrical pipe of radius b:

$$F_{//}(r,z) = \frac{-q}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\ \lambda(z)}{\partial\ z}$$

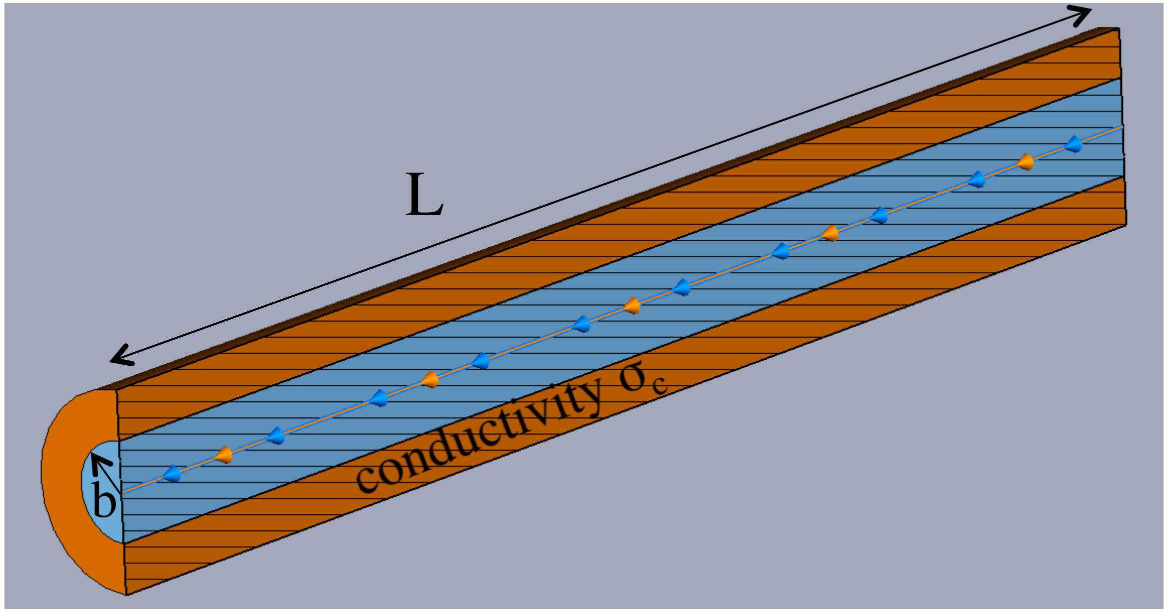
Example of longitudinal wake field and coupling impedance: space charge

Since the space charge forces move together with the beam, they are constant along the accelerator if the beam pipe remains constant. We can therefore define the longitudinal wake field per unit length (V/Cm). To get the longitudinal wake field of a piece of pipe, we just multiply by the pipe length. Assuming r→0 (particle on axis), and a charge line density given by $\lambda(z) = q_0\delta(z)$ we obtain

$$\frac{dw_{//}(z)}{ds} = \frac{1}{4\pi\varepsilon_0\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)\frac{\partial}{\partial z}\delta(z)$$

$$\frac{\partial Z_{//}(\omega)}{\partial s} = \frac{1}{v}\int_{-\infty}^{\infty}\frac{\partial w_{//}(z)}{\partial s}e^{i\frac{\omega z}{v}}dz = \frac{1 + 2\ln(b/a)}{v4\pi\varepsilon_0\gamma^2}\int_{-\infty}^{\infty}\frac{d}{dz}\delta(z)e^{i\frac{\omega z}{v}}dz = \frac{i\omega Z_0}{4\pi c\beta^2\gamma^2}\left(1 + 2\ln\frac{b}{a}\right)$$

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)



Hp: high conductivity such that

$$\delta_w << \frac{c^2}{\omega^2 b}$$

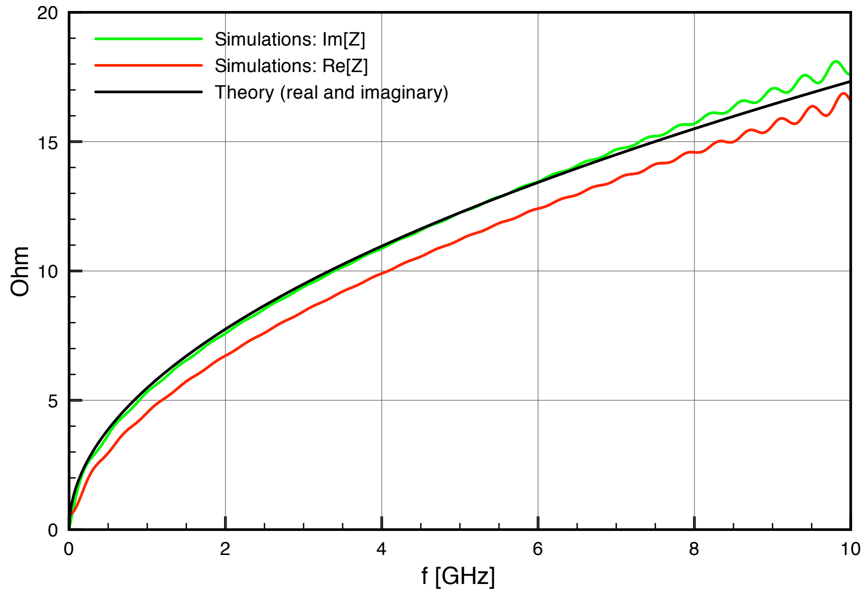
$$\delta_w << b$$

$$Z_{//}(\omega) = \left[1 - i\,\mathrm{sgn}(\omega)\right]\frac{L}{2\pi b}\sqrt{\frac{Z_0|\omega|}{2c\sigma_c}}$$

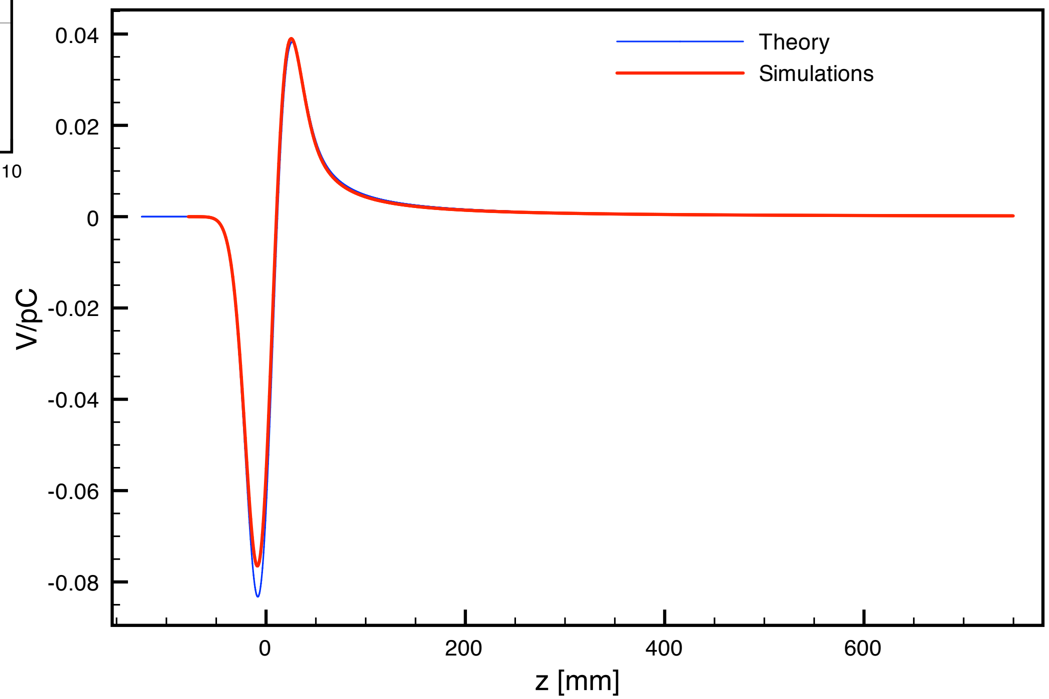
$$w_{//}(z) = \frac{Lc}{4\pi b}\sqrt{\frac{Z_0}{\sigma_c}}\frac{1}{|z|^{3/2}}$$

not valid for small z and when ω -> 0

Example of longitudinal wake field and coupling impedance: finite conductivity of a circular pipe wall (resistive wall)

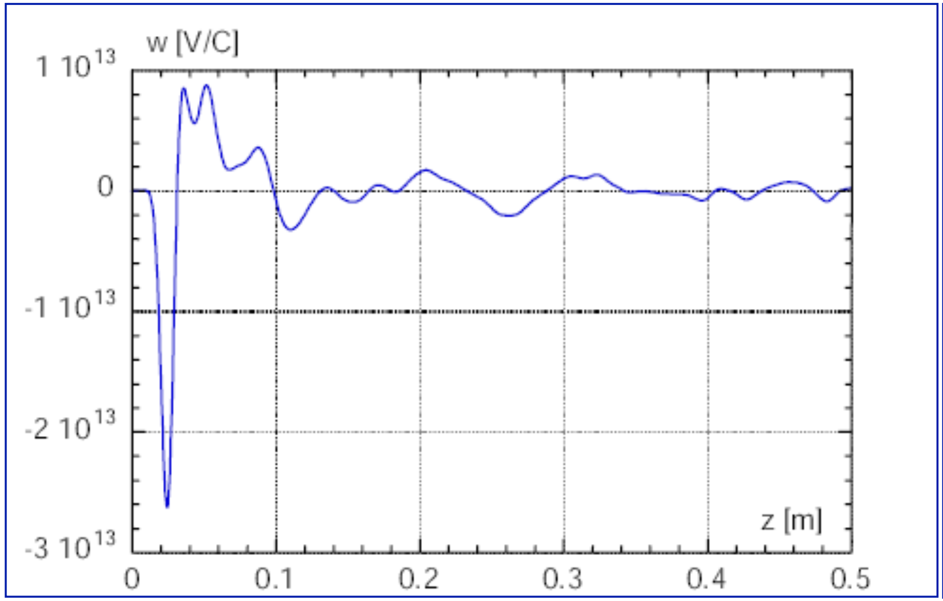


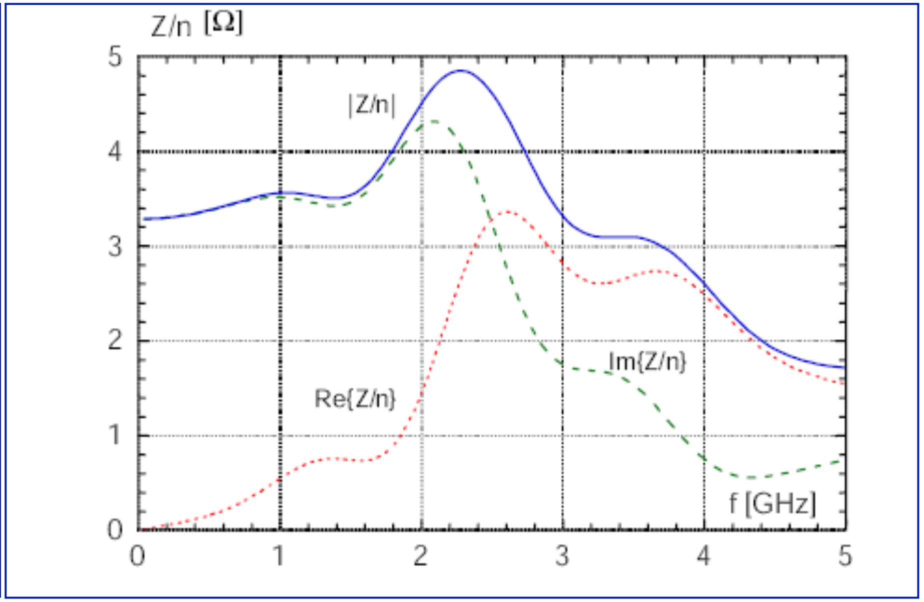
Impedance comparison



Wake potential comparison

Example of wake potential and longitudinal coupling impedance for an entire machine: DAΦNE accumulator

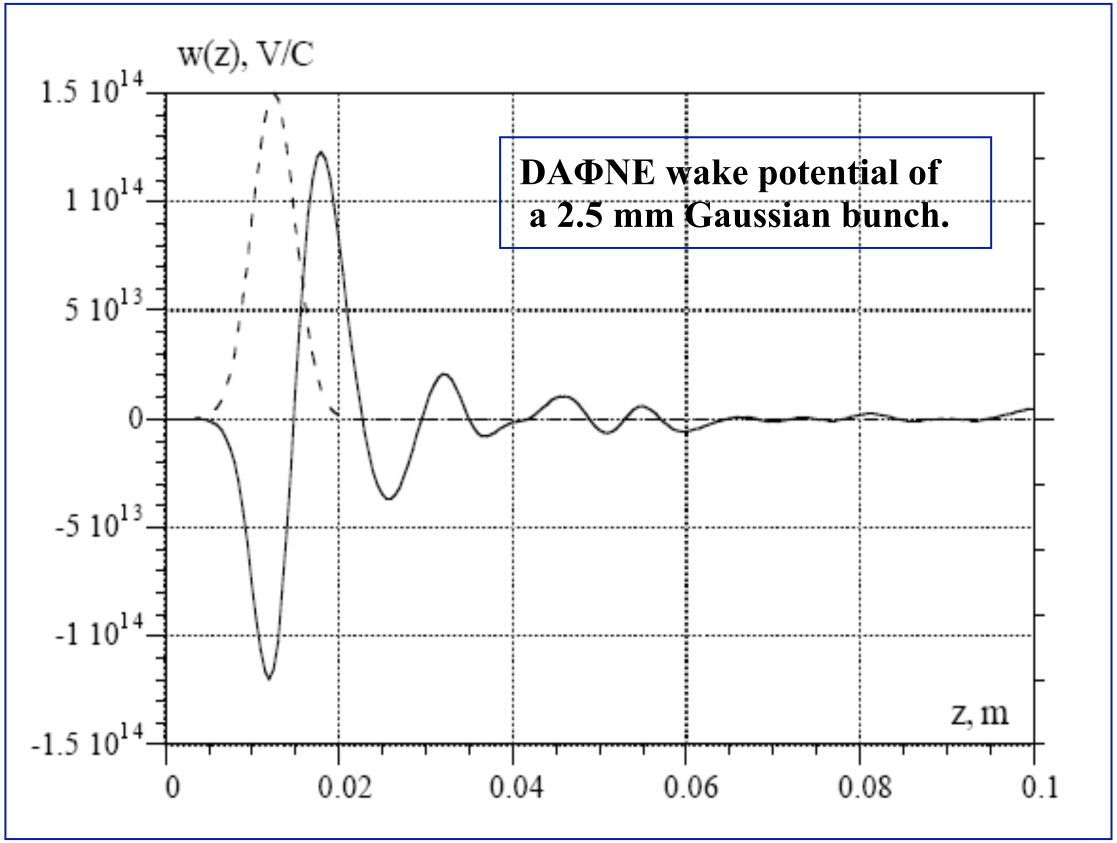




DAΦNE accumulator wake potential of a 2.5 mm Gaussian bunch.

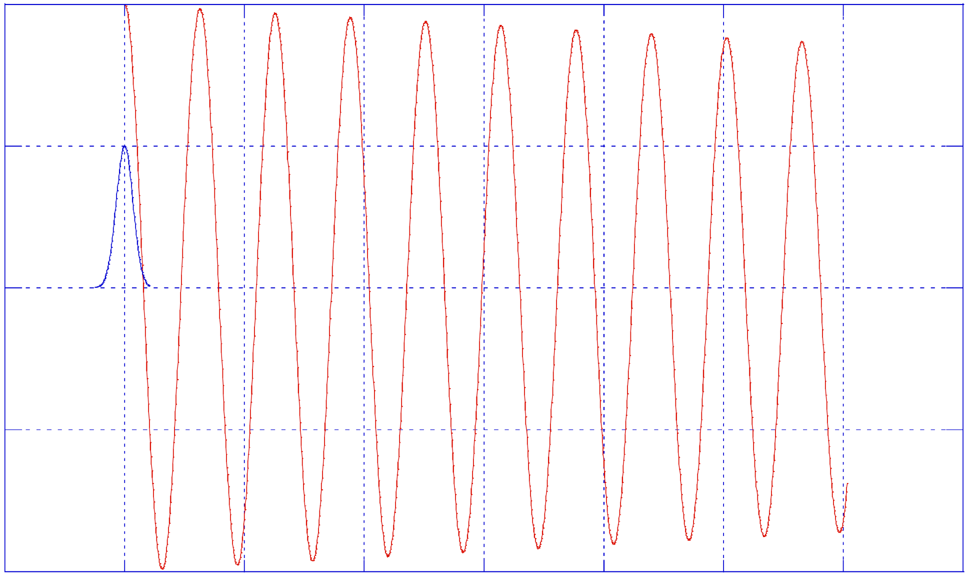
$$\frac{Z_{\|}(\omega)}{n} = \frac{Z_{\|}(\omega)}{\omega/\omega_o}$$

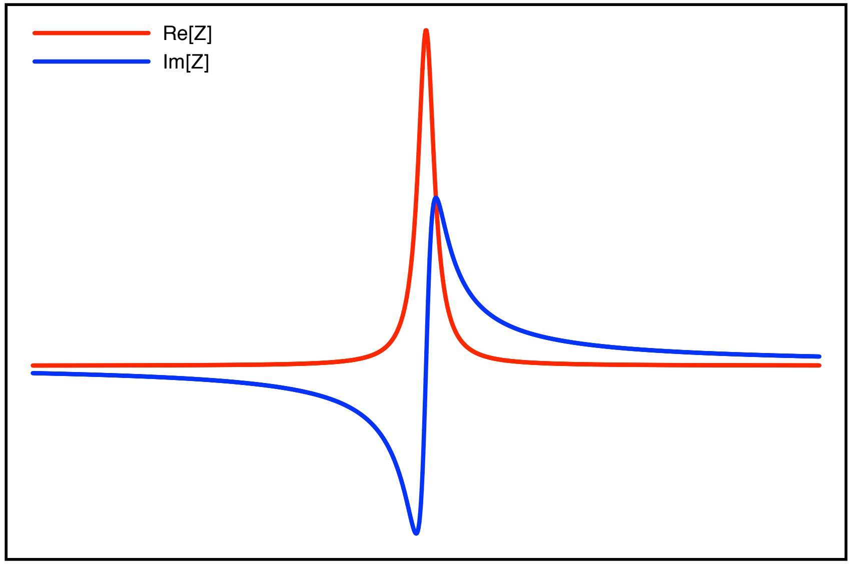
Short range wake field/potential acts over the bunch length



- Vanishes after a distance of few bunch lengths
- Poor frequency resolution of Fourier transform of coupling impedance => broad band impedance

Long range wake field/potential acts on many bunches/multi-turn

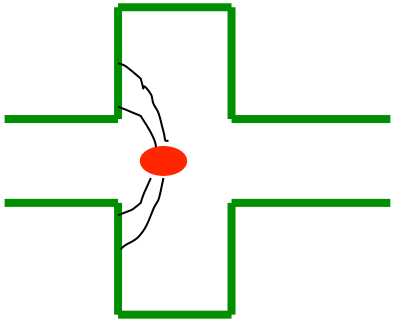


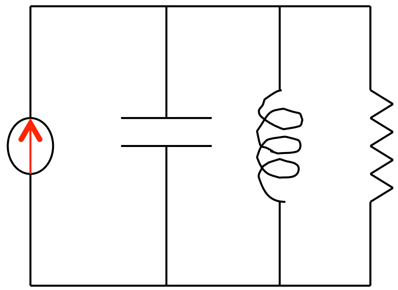


- Field oscillates over long distances
- Produced by high Q resonant modes
- Described by only 3 parameters: Q, ω_r and R_s
- High peak impedance

Longitudinal wake field of a resonant mode

When a charge crosses a resonant structure, it excites resonant modes (fundamental and HOMs). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.





Just after the charge passage, the capacitor is charged with a voltage $V_o = q_o / C$ and the electric field is $E_{so} = V_o / l_o$.

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\boxed{\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0}$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount V_0. This potential will oscillate and decay producing a current flow in the resistor and inductance.

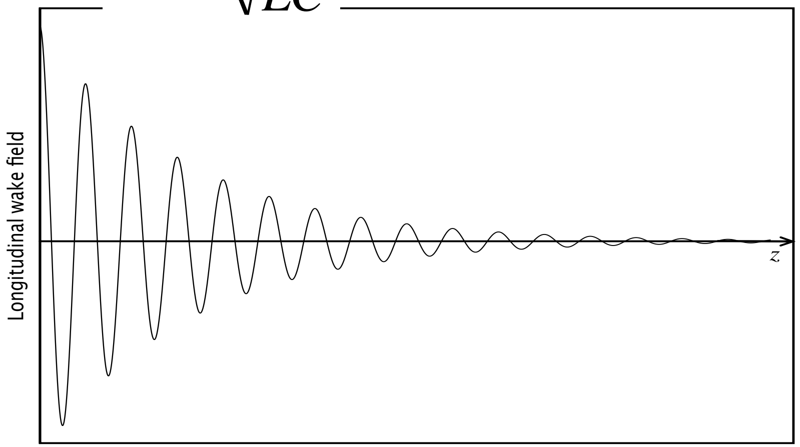
For t>0 the potential satisfies the following equations and initial conditions:

$$\boxed{\begin{aligned} &\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0 \\ &V(t=0^+) = \frac{q_0}{C} \equiv V_0 \\ &\dot{V}(t=0^+) = \frac{\dot{q}}{C} = -\frac{I(0^+)}{C} = -\frac{V_0}{RC} \end{aligned}}$$

$$V(t) = V_0 e^{-\gamma t}\left[\cos(\omega_n t) - \frac{\gamma}{\omega_n}\sin(\omega_n t)\right]$$

$$\omega_n^2 = \omega_r^2 - \gamma^2 \qquad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\gamma = \frac{1}{2RC}$$



putting z = -ct (z is negative behind the source charge),

$$\left(w_0 = \frac{1}{C}\right)$$

$$\boxed{w_{//}(z) = \frac{V(z)}{q_0} = w_0 e^{\gamma z/c}\left[\cos(\omega_n z/c) + \frac{\gamma}{\omega_n}\sin(\omega_n z/c)\right] H(-z)}$$

Coupling impedances of a resonant mode

Longitudinal Impedance:

$$\boxed{Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)}}$$

The parameters R_s, Q and ω_r, that can be evaluated by computer codes, can be related to the parameters *RLC* of the parallel circuit

shunt impedance: $R_s = R = \dfrac{w_o}{2\gamma}$ quality factor: $Q = \dfrac{\omega_r}{2\gamma}$

Transverse Wakefield and Impedance:

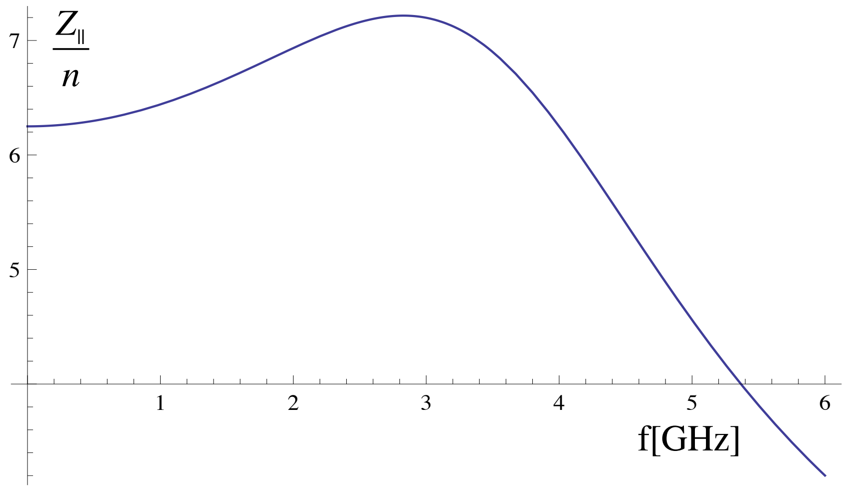
$$w_{\perp}(z) = \frac{R_{\perp}\omega_r}{Q} e^{\Gamma\, z/c} \sin(\bar{\omega} z / c) \qquad Z_{\perp}(\omega) = \frac{\bar{\omega}}{\omega} \frac{R_{\perp}}{1 + iQ_r\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)}$$

Some remarks on the longitudinal impedance of a resonant mode

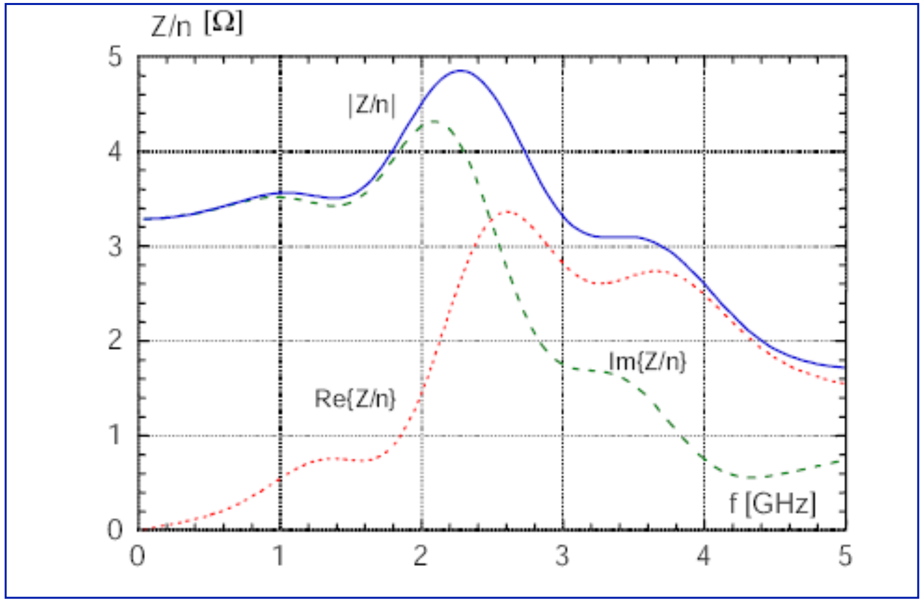
$$Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

This impedance can be also used as a simplified impedance model of a whole machine for the short range wake fields assuming Q ~ 1 (it is called **Broad Band Impedance Model**)

Broad Band Resonator Model



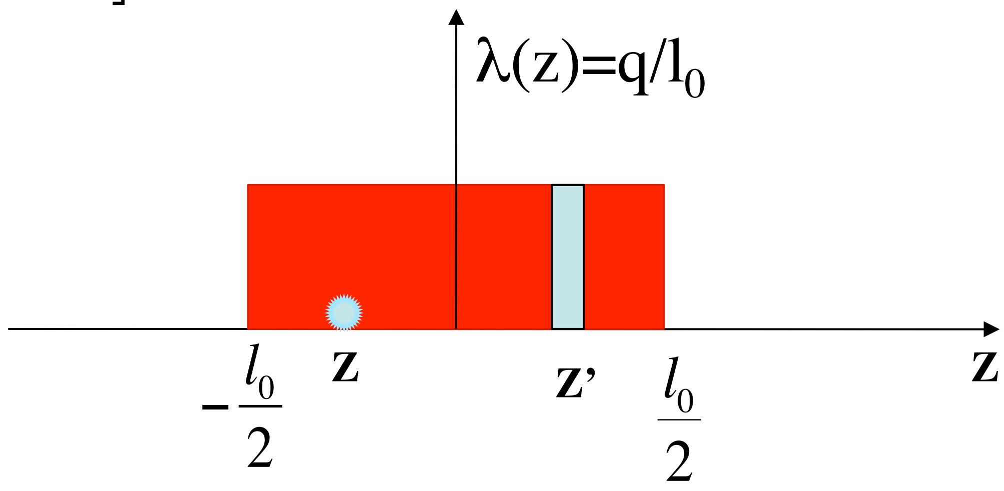
DAΦNE Accumulator Impedance



Wake Fields Effects In Linear Accelerators

Example: Energy lost by a finite uniform beam due to a resonant mode

$$w_{//}(z) = w_o e^{\gamma\, z/c}\left[\cos(\omega_n z/c) + \frac{\gamma}{\omega_n}\sin(\omega_n z/c)\right] H(-z) \approx w_o \cos(\omega_r z/c) H(-z)$$



$$U(z) = -e\int_{-\infty}^{+\infty} w_{//}(z'-z)\lambda(z')dz'$$

$$U(z) = -\frac{eqw_0}{l_0}\int_{z}^{l_0/2}\cos\left[\frac{\omega_r}{c}(z'-z)\right]dz'$$

$$z'-z = x$$

$$U(z) = -\frac{eqw_0}{l_0}\int_{0}^{(l_0/2-z)}\cos\left(\frac{\omega_r}{c}x\right)dx = -\frac{eqw_0}{l_0}\left[\frac{\sin\left(\frac{\omega_r}{c}x\right)}{\left(\frac{\omega_r}{c}\right)}\right]_0^{(l_0/2-z)}$$

$$U(z) = -\frac{eqw_0}{2}\left[\frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2}-z\right)\right]}{\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)}\right]$$

Wake potential?

Energy spread (U_{max}-U_{min})?

Energy loss

$$U_{bunch} = \frac{1}{e}\int_{-\infty}^{+\infty} U(z)\lambda(z)dz \approx \frac{-q^2 w_0}{2l_0\left(\frac{\omega_r}{c}\frac{l_0}{2}\right)} \int_{-\frac{l_0}{2}}^{\frac{l_0}{2}} \sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right] dz$$

$$U_{bunch} = \frac{-q^2 w_0 c}{\omega_r l_0^2}\left|\frac{-\cos\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{-\frac{\omega_r}{c}}\right|_{-\frac{l_0}{2}}^{\frac{l_0}{2}}$$

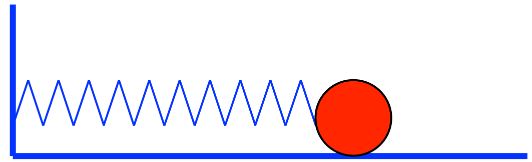
$$U_{bunch} = -\frac{q^2 w_0 c^2}{\omega_r^2 l_0^2}\left[1 - \cos\left(\frac{\omega_r l_0}{c}\right)\right] = -\frac{2q^2 w_0 c^2}{\omega_r^2 l_0^2}\sin^2\left(\frac{\omega_r l_0}{2c}\right)$$

$$U_{bunch} = -\frac{q^2 w_0}{2}\frac{\sin^2\left(\frac{\omega_r l_0}{2c}\right)}{\left(\frac{\omega_r l_0}{2c}\right)^2}$$

$$\lim_{l_0 \to 0}\left(U_{bunch}\right) = -\frac{q^2 w_0}{2}$$

Instabilities: driven oscillators

Consider an harmonic oscillator with natural frequency ω, with an external excitation at frequency Ω:



$$\ddot{x} + \omega^2 x = A\cos(\Omega t)$$

General solution:

$$x(t) = x^{free}(t) + x^{driven}(t)$$

$$\cos(\Omega \mathrm{t}) \Rightarrow e^{i\Omega t}$$

$$x^{free}(t) = \tilde{x}_m^f e^{i\omega t}$$

$$x^{driven}(t) = \tilde{x}_m^d e^{i\Omega t}$$

substitution in the diff. equation:

$$(\omega^2 - \Omega^2)\tilde{x}_m^d e^{i\Omega t} = Ae^{i\Omega t}$$

$$x^{driven}(t) = \frac{A}{(\omega^2 - \Omega^2)} e^{i\Omega t}$$

The general solution has to satisfy the initial conditions at t=0. In our case we assume that the oscillator is at rest for t=0:

$$x^{free}(t=0) = -x^{driven}(t=0)$$

$$\tilde{x}_m^f = -\frac{A}{\omega^2 - \Omega^2}$$

thus we get:

$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[e^{i\Omega t} - e^{i\omega t}\right]$$

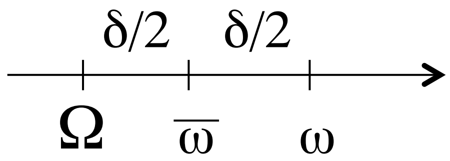
taking only the real part:

$$x(t) = \frac{A}{\omega^2 - \Omega^2}\left[\cos(\Omega t) - \cos(\omega t)\right]$$

This expression is suitable for deriving the response of the oscillator driven at resonance or at frequency very close:

$$\omega = \Omega + \delta, \quad \delta \to 0$$

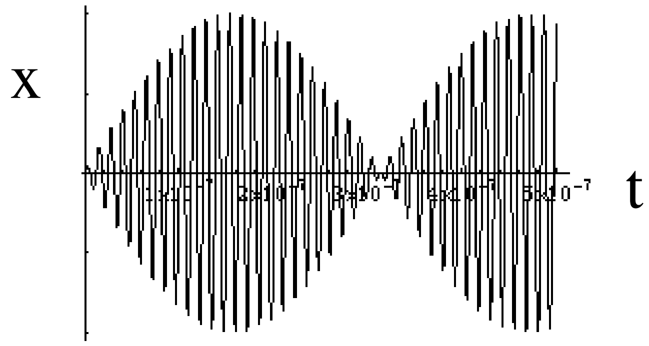
$$\bar{\omega} = (\omega + \Omega)/2; \ \omega = \bar{\omega} + \delta/2, \ \Omega = \bar{\omega} - \delta/2$$



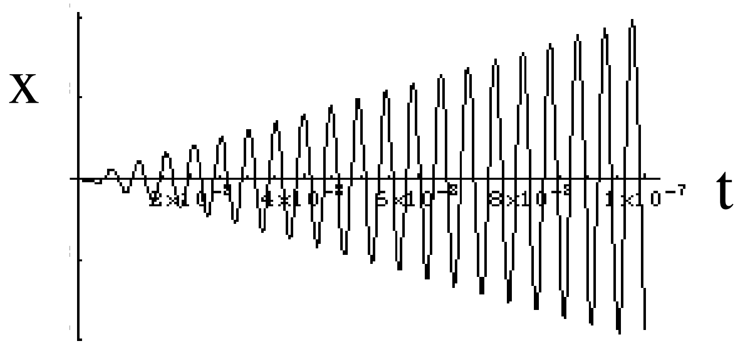
$$x(t) \approx \frac{A}{2\bar{\omega}\delta} \ \Big\{ \ \big[\cos(\bar{\omega}t)\cos(\delta t/2) + \sin(\bar{\omega}t)\sin(\delta t/2)\big] +$$

$$- \ \big[\cos(\bar{\omega}t)\cos(\delta t/2) - \sin(\bar{\omega}t)\sin(\delta t/2)\big] \ \Big\}$$

$$x(t) = \overbrace{\frac{A}{\bar{\omega}\delta} \ \sin\left(\frac{\delta t}{2}\right)}^{\text{amplitude modulation}} \sin(\bar{\omega}t) \equiv \frac{At}{2\bar{\omega}}\sin(\bar{\omega}t)\frac{\sin\left(\frac{\delta t}{2}\right)}{\frac{\delta t}{2}}$$

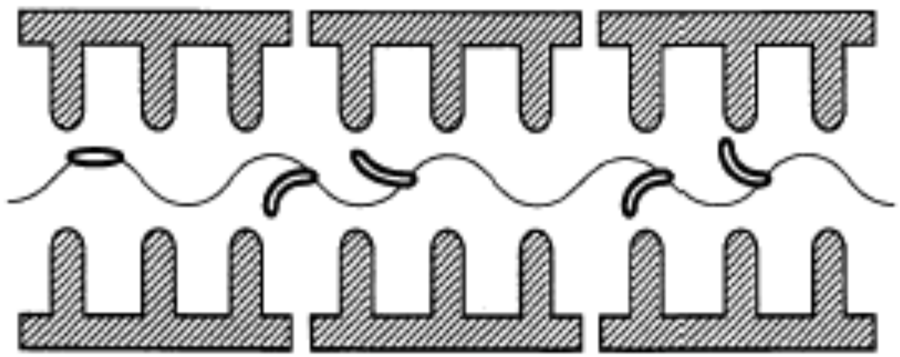


$$\lim_{\delta \to 0} x(t) = \frac{At}{2\bar{\omega}}\sin(\bar{\omega}t)$$

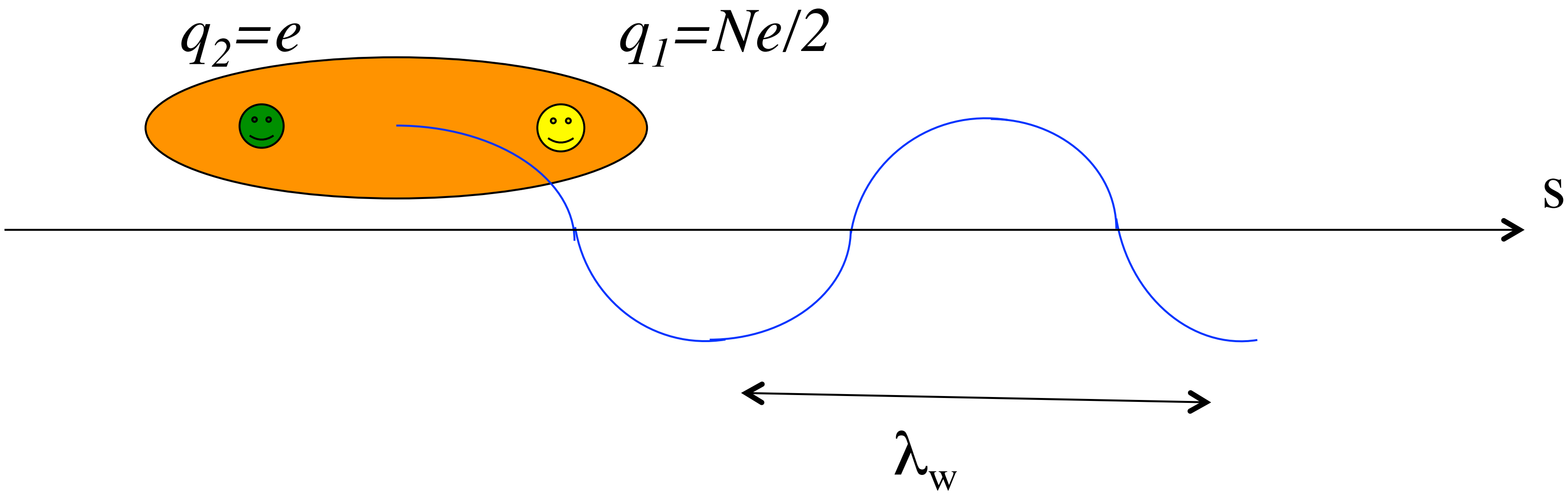


Single Bunch Beam Break Up in Linacs

A beam injected off-centre in a LINAC, because of the focusing quadrupoles, executes betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges.



In order to understand the effect, we consider a simple model with only two charges $q_1=Ne/2$ (source charge = half bunch) and $q_2=e$ (test charge = single charge).



the source charge executes free betatron oscillations:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right); \quad \frac{\omega_y}{c} = \frac{2\pi}{\lambda_w}$$

the test charge, at a distance z behind, over a length L_w experiences a deflecting force proportional to the displacement y_1, and dependent on the distance z:

$$\boxed{w_\perp(z) = \frac{1}{r_0}\frac{M(r_0,z)}{q^2}}$$

$$e \qquad Ne/2$$

$$M(r_0,z) = \int_0^{L_w} F_\perp ds = \langle F_\perp(r_0,z)\rangle L_w \quad \Longrightarrow \quad \langle F_\perp(z,y_1)\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

This force drives the motion of the test charge:

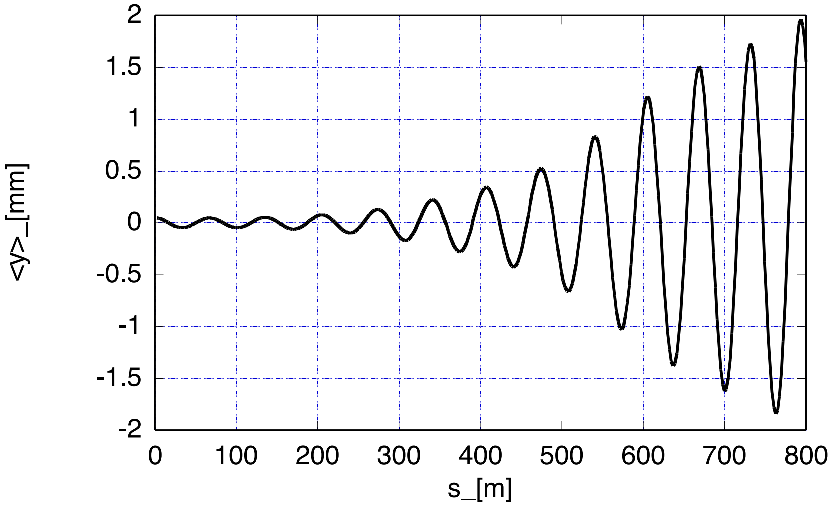
$$\underbrace{y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{1}{\beta^2 E_o}\langle F_\perp(z,y_1)\rangle}_{\text{betatron motion equation with coherent force}} = \frac{Ne^2 w_\perp(z)}{2\beta^2 E_o L_w}\hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)$$

This is the typical equation of a resonator driven at the resonant frequency. The solution is given by the superposition of the "free" oscillation and a "driven" oscillation which, being driven at the resonant frequency, grows linearly with s.

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c} s\right) + y_2^{driven}$$

$$y_2^{driven} = \frac{cNe^2 w_\perp(z) s}{4\omega_y E_o L_w} \hat{y}_1 \sin\left(\frac{\omega_y}{c} s\right)$$

$$(\beta = 1)$$



At the end of the LINAC of length L_L, the oscillation amplitude is grown by $\left(y_1(0) = \hat{y}_1 = y_2(0) = \hat{y}_2\right)$

$$\boxed{\left(\frac{y_2(L_L) - \hat{y}_2}{\hat{y}_2}\right)_{\max} = \left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_0/e)L_w}}$$

Balakin-Novokhatsky-Smirnov Damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the **"resonant" driving force**.

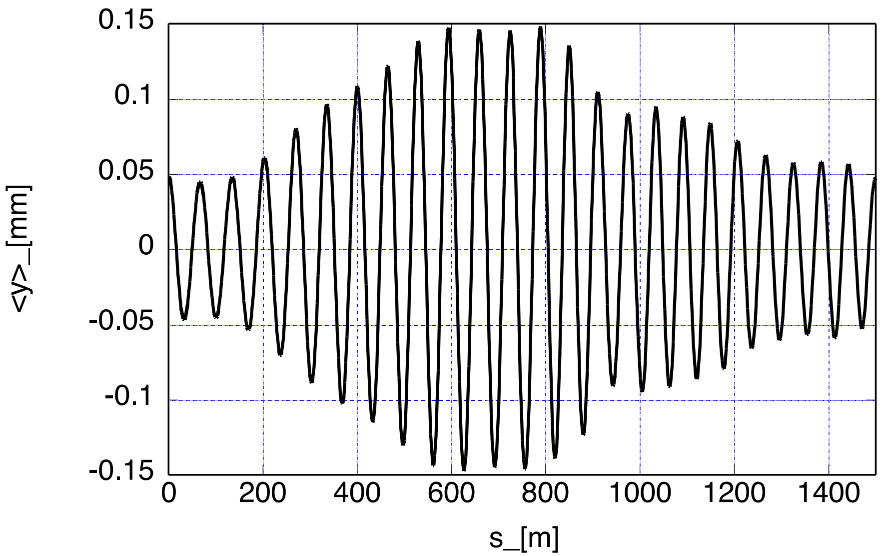
If the tail and the head of the bunch oscillate with different frequencies, this effect can be significantly removed.

Let us assume that the tail oscillates with a frequency $\omega_y + \Delta\omega_y$, the equation of motion becomes:

$$\boxed{y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2\beta^2 E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)}$$

the solution of which is:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + \frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_o L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y}{c} s\right) - \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right)\right]$$



by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail.

$$\boxed{\hat{y}_2 = \hat{y}_1} \qquad \frac{c^2 N e^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_o L_w} = 1 \quad \Rightarrow \quad \boxed{y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) = y_1(s)}$$

$$\boxed{\Delta\omega_y = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y E_o L_w}}$$

The extra focusing at the tail can be obtained by:

- Using an RFQ, where head and tail see a different focusing strength,
- Exploit the energy distribution along the bunch which, because of the chromaticity, induces a spread in the betatron frequencies. An energy spread correlated with the longitudinal position is attainable with the external accelerating voltage, or with the wake fields.

Instabilities in Circular Accelerators

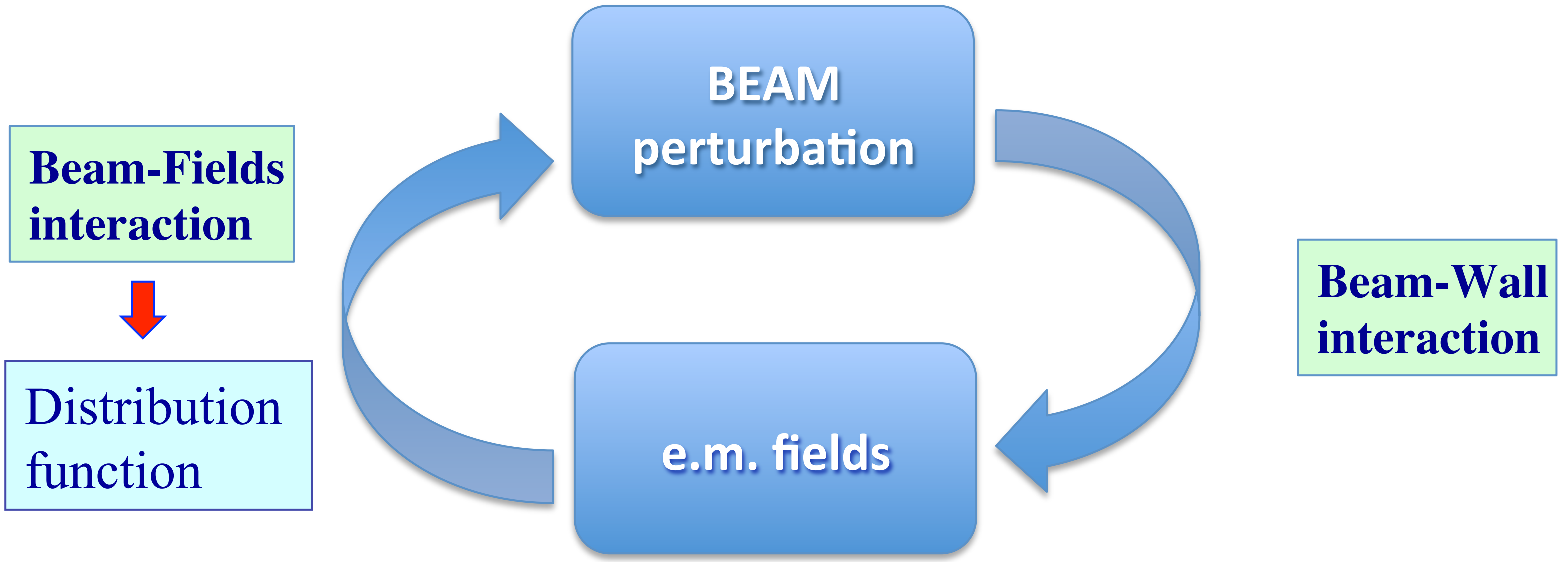
Longitudinal effects on beam dynamics

Short range wake fields:

- Potential well distortion ➔ deformation of the longitudinal distribution
- Longitudinal emittance growth, microwave instability

Long range wake fields:

- Robinson instabilities (RF fundamental mode)
- Coupled bunch instability (HOMs)



This can be considered a feedback system where the gain depends on the current

- At low current the feedback is stable and we find a stationary distribution function
- At high current the gain is so high that the system becomes unstable

$$\lambda(z;t) = \lambda_0(z) + \lambda_1(z;t)$$

stationary distribution function

perturbation (responsible of the instability)

Short range wake fields at low current: Potential well distortion

The longitudinal motion of a particle in the bunch is confined by the potential energy due to the RF voltage and to the wake fields

$$\Psi(z) = \frac{1}{L_0}\int_0^z \left[eV_{RF}(z') - U_0\right]dz' - \frac{e^2 N_p}{L_0}\int_0^z dz' \int_{-\infty}^{\infty} \lambda_0(z'') w_{\parallel}(z'' - z')\,dz''$$

The energy distribution is Gaussian with an RMS energy spread $\sigma_{\varepsilon 0}$ not modified by the wake fields. The longitudinal distribution is described by an integral equation known as the Haissinski equation

$$\lambda_0(z) = \bar{\lambda}\exp\left[-\frac{1}{E_0 \alpha_c \sigma_{\varepsilon 0}^2}\Psi(z)\right]$$

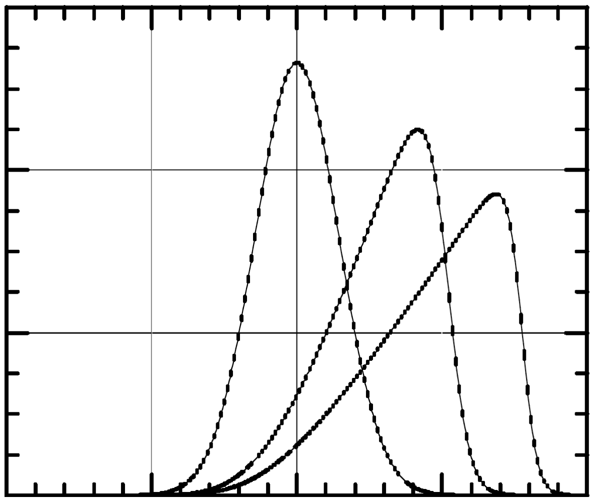
Particular solution of Haissinski equation

No wake field contribution: a linear expansion of V_{RF} around z=0 gives

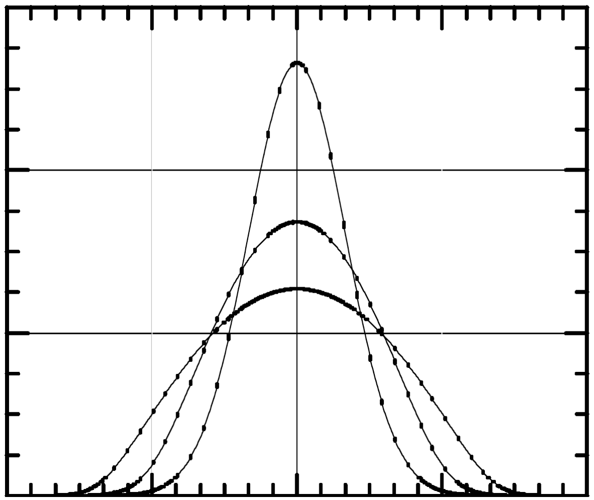
$$\Psi(z) = \frac{2\pi h e \hat{V}\sin(\phi_s)}{L_0^2}\int_0^z z'\,dz' = \frac{\omega_{s0}^2 E_0}{2\alpha_c c^2} z^2$$

$$\lambda_0(z) = \bar{\lambda}\exp\left[-\frac{z^2}{2\sigma_{z0}^2}\right] \qquad \bar{\lambda} = \frac{eN}{\sqrt{2\pi}\sigma_{z0}} \qquad \sigma_{z0} = \frac{\alpha_c c \sigma_{\varepsilon 0}}{\omega_{s0}}$$

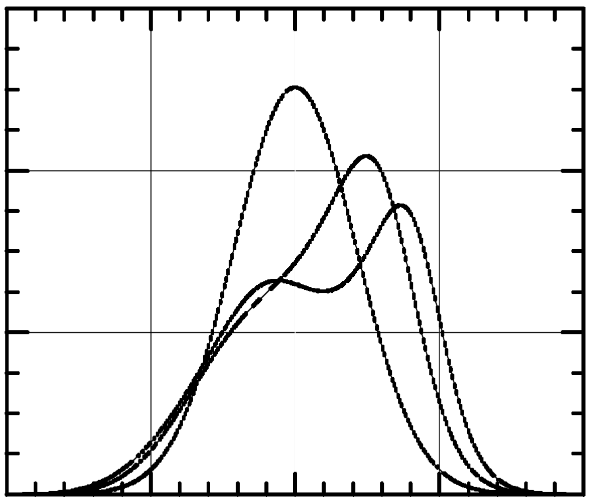
Pure resistive impedance



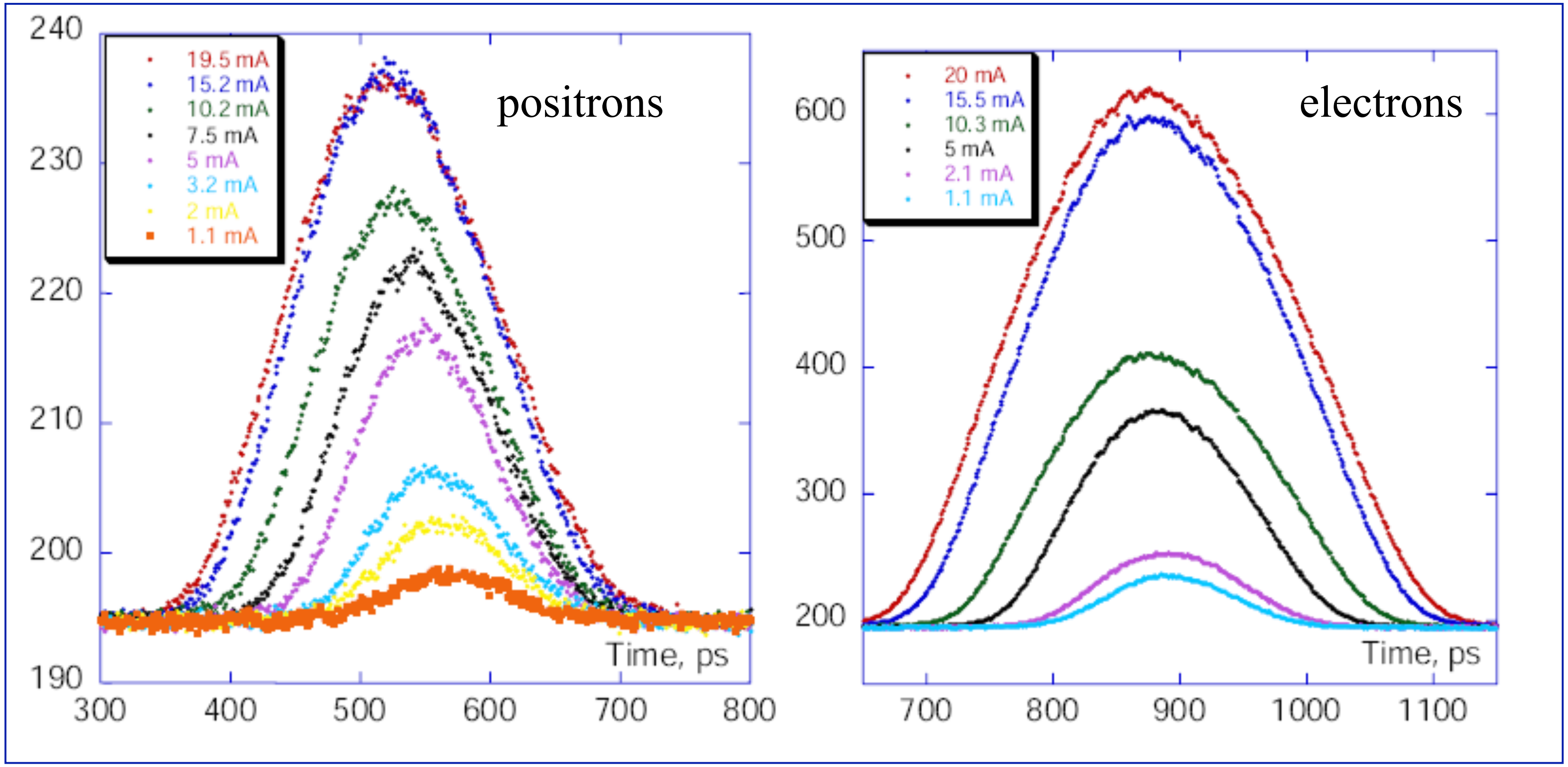
Pure inductive impedance



Broad band resonator

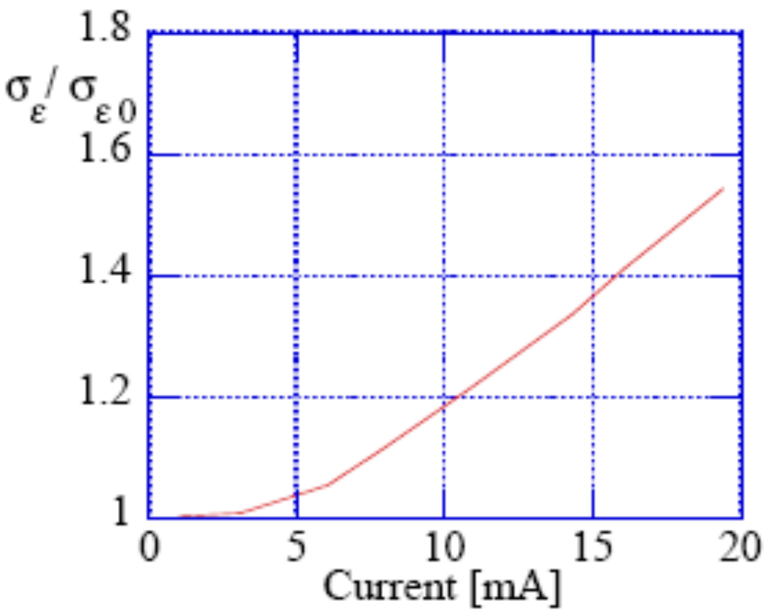


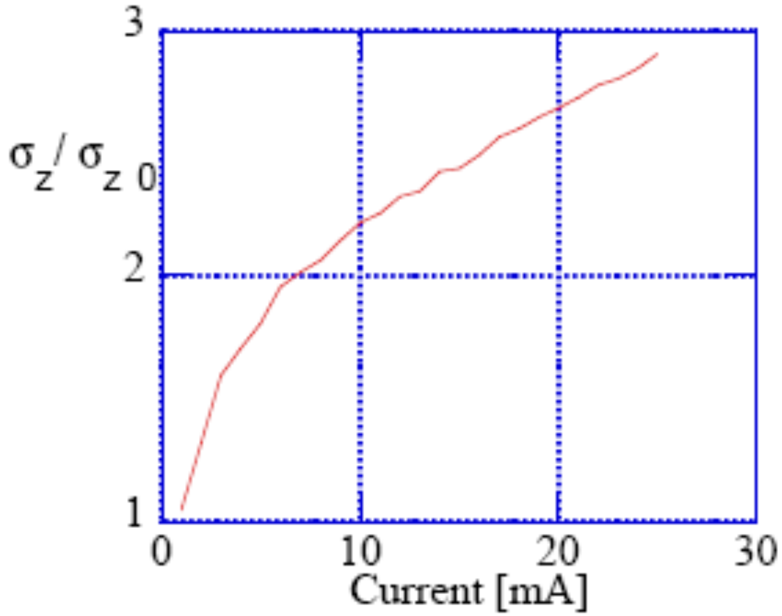
Typical measured bunch distributions in the DAΦNE Rings. The head is to the left



High current: longitudinal emittance growth, microwave instability

• Observe energy spread and bunch length as a function of the current.

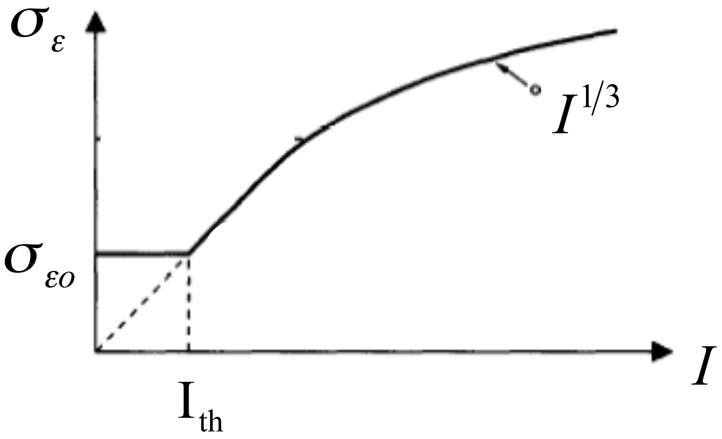




• σ_ε is almost constant up to a threshold current after which it starts to increase with the current according to a given power law (in most cases 1/3 power).

• σ_z starts to increase from the very beginning (potential well distortion), and, after the same threshold current, it grows with the same power law.

Longitudinal emittance growth & microwave instability



Threshold current:

$$\boxed{\frac{\hat{I}\left|Z_{\parallel}/n\right|}{2\pi\alpha_c\left(E_0/e\right)\sigma_\varepsilon^2} \leq 1}$$

$$\hat{I} = \frac{ceN_p}{\sqrt{2\pi}\sigma_z}$$

$$n = \frac{\omega}{\omega_o}$$

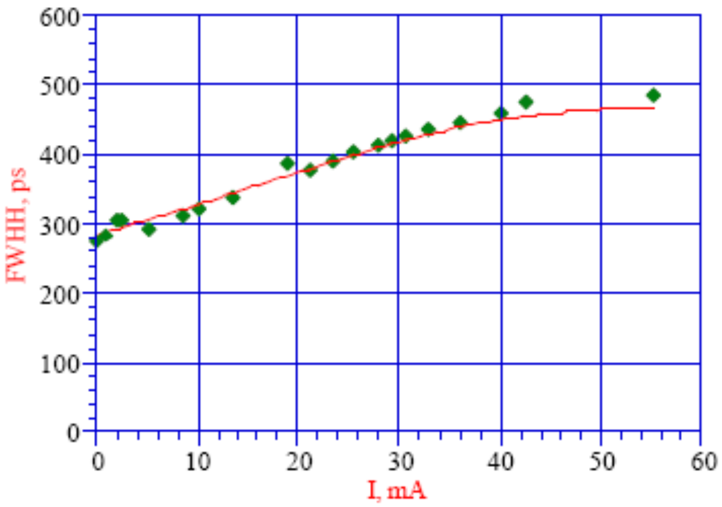
remember:

$$\sigma_z = \frac{\alpha_c c\sigma_\varepsilon}{\omega_s}$$

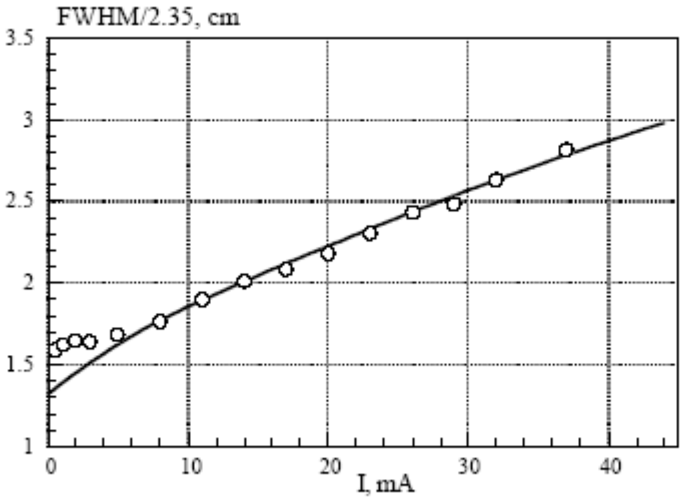
Above threshold: Boussard criterion

$$\boxed{\sigma_z = \left(\frac{R^3\left|Z/n\right|\xi}{\sqrt{2\pi}}\right)^{1/3}} \qquad \xi = \frac{I\alpha_c}{\nu_s^2 E_0/e}$$

Bunch lengthening in DAFNE



DAFNE Accumulator.
Dots: measurement results
Solid line: numerical simulations.



DAFNE main rings
Circles: measurement results.
Solid line: numerical simulations

NOTICE
Numerical simulations performed in the design phase, before measurements: good impedance model of the machine

Longitudinal microwave instability is fast but not destructive

Design strategy: proper design of vacuum chamber

- Single bunch: low broad band impedance Z/n

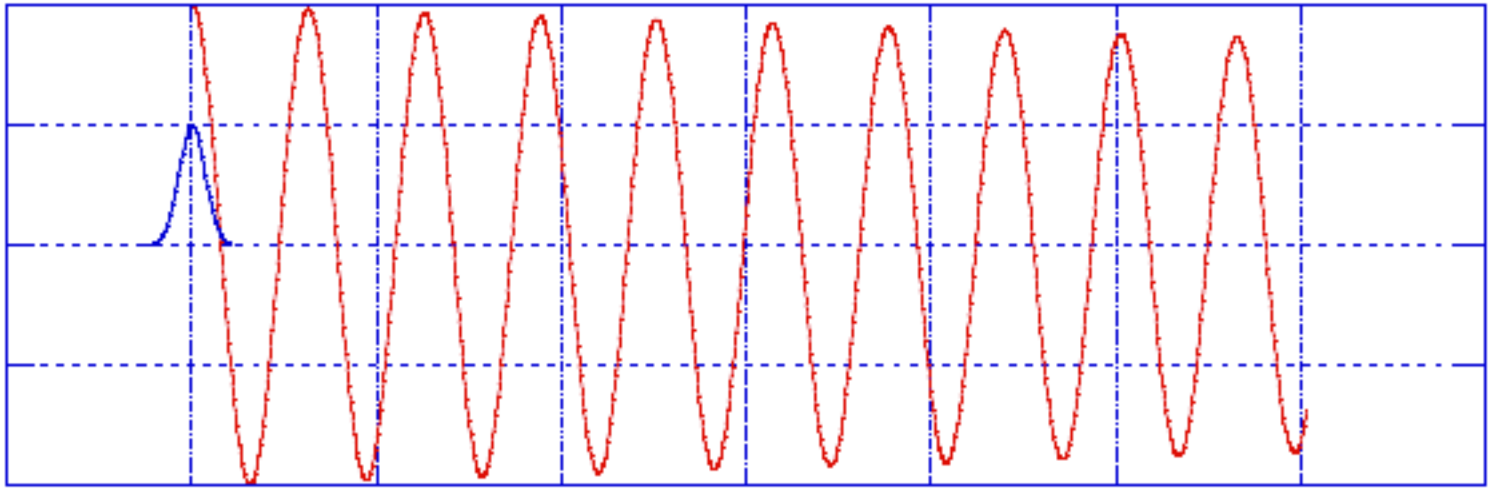
Cures ?

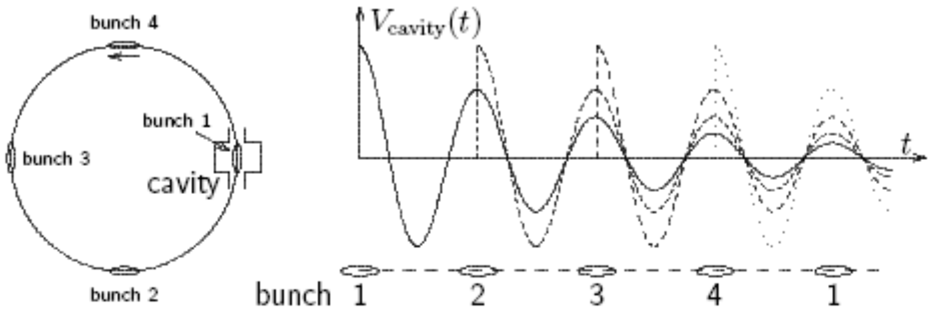
- Reduce parasitic loss, taper discontinuities
- Landau damping

LANDAU DAMPING

- There is a natural stabilising effect against the collective instabilities called “Landau Damping”. The basic mechanism relies on the fact that if the particles in a beam have a spread in their natural frequencies (synchrotron or betatron), their motion can’t be coherent for a long time.
- The mechanism is in general triggered when an infinite set of identical systems oscillates at different frequencies, spread over some range of values. Under these conditions, if any periodic force has its frequency within the considered range, the oscillation amplitude, averaged over all the systems, instead of growing as one should expect, remains constant.
- Even if a periodic force pumps energy into the system, this energy is not converted into an increase of the average oscillation amplitude: the number of particles in resonance with the external force decreases with time, so that the net contribution to the average oscillation amplitude remains constant.

Long range wake fields





A. Hofmann

Interaction with RF fundamental mode: Robinson Instabilities

Longitudinal equations of motion of the bunch centre of mass

$$\dot{z} = -c\alpha_c \varepsilon$$

$$\dot{\varepsilon} = \frac{eV_{RF} - U_0}{T_0 E_0} - \frac{D}{T_0}\varepsilon \qquad D = \frac{2U_0}{E_0}$$ is the damping coefficient

Combined they give a second order differential equation

$$\ddot{z} + \frac{D}{T_0}\dot{z} + \omega_{s0}^2 z = 0 \qquad \text{with} \quad \omega_{s0}^2 = \frac{c^2\alpha_c 2\pi h e\hat{V}\sin(\phi_s)}{L_0^2 E_0}$$

$$\cos(\phi_s) = \frac{U_0}{e\hat{V}} \qquad \left(0 \le \phi_s \le \frac{\pi}{2}\right)$$ synchronous phase

Robinson instability of the fundamental mode

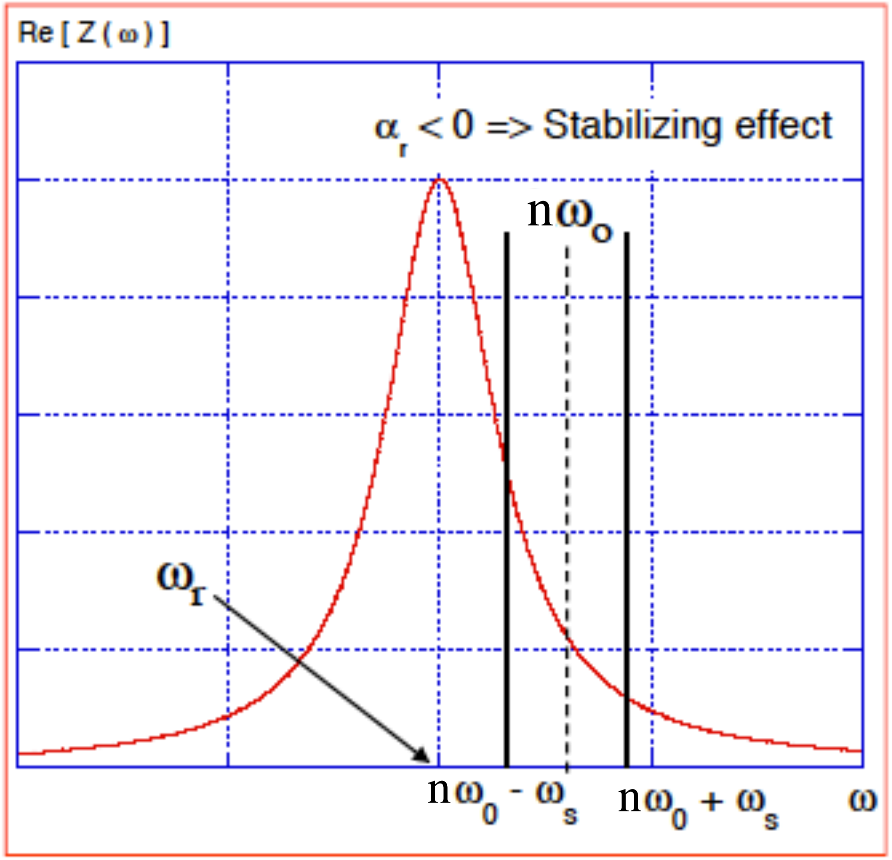
By including also the fundamental mode wakefield (beam loading effect) we have

$$\ddot{z} + \left(\frac{D}{T_0} - \alpha_r\right)\dot{z} + \omega_s^2 z = 0$$

$$\alpha_r = \frac{eN_p\alpha_c h\omega_0}{\omega_s\left(E_0/e\right)T_0^2}\mathrm{Re}\left[\Delta Z\right]$$

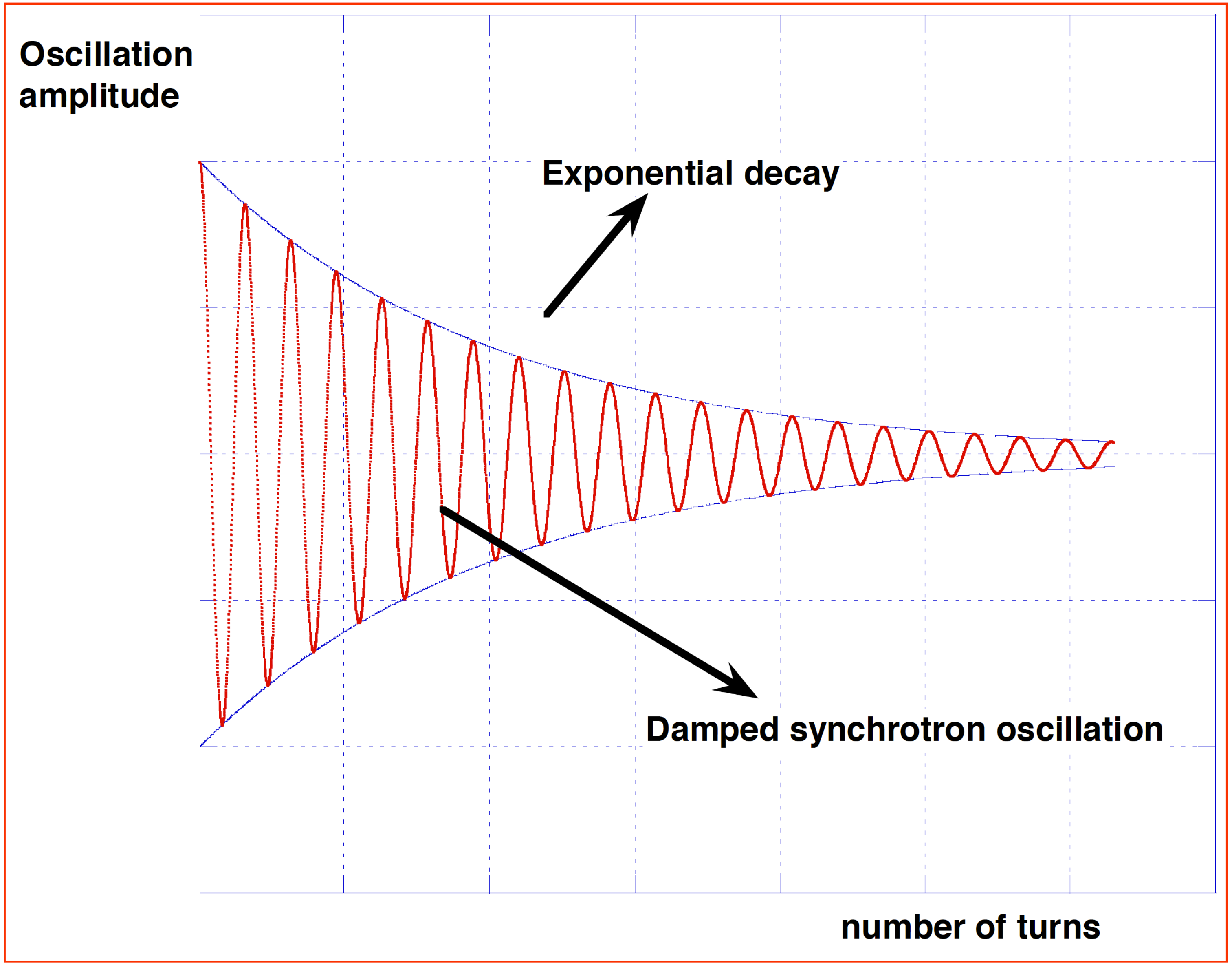
$$\mathrm{Re}\ \left[\Delta Z\right] = \mathrm{Re}\ \left[Z\left(n\omega_0+\omega_s\right) - Z\left(n\omega_0-\omega_s\right)\right]$$

$$z = A_0 \exp\left[-\frac{1}{2}\left(\frac{D}{T_0} - \alpha_r\right)t\right]\cos\left[\omega_s t + \theta_0\right]$$



Robinson instability ....

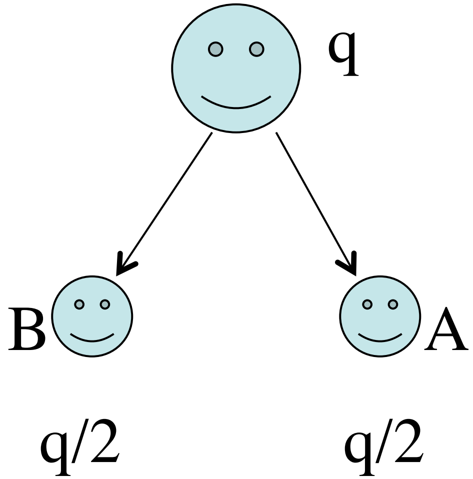
Example
of
stability



Conclusions

- The longitudinal instability mechanisms in circular accelerators are well understood;
- With an accurate model of the machine impedance one can predict the single bunch and multibunch dynamics;
- Longitudinal single bunch instabilities are not destructive but lead to beam heating (increase of energy spread and bunch length)
- Multibunch instabilities are destructive and require the installation of a feedback system on the ring.
- Overall it is very important an accurate design of the vacuum chamber and RF devices

Appendix 1



$$U_A = q_A^2 k = \frac{q^2}{4} k$$

$$U_B = q_B^2 k + q_A q_B w_{//}(z)$$
$$= \frac{q^2}{4} k + \frac{q^2}{4} w_{//}(z)$$

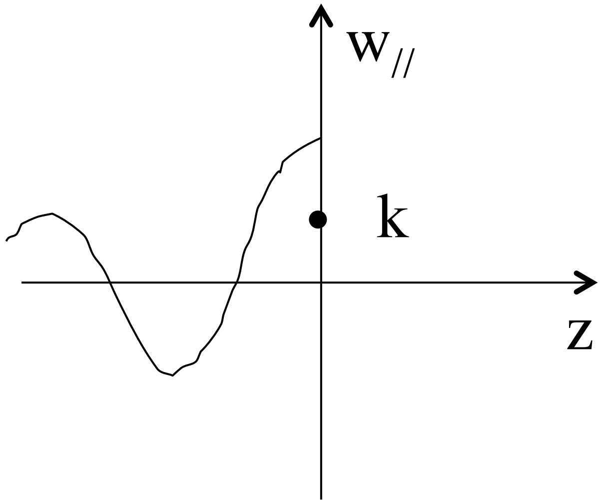
$$\boxed{U_A + U_B = \frac{q^2}{2} k + \frac{q^2}{4} w_{//}(z)}$$

$$\boxed{z \to 0 \quad U_A + U_B = q^2 k}$$

$$\frac{q^2}{2} k + \frac{q^2}{4} w_{//}(0) = q^2 k$$

$$\frac{w_{//}(0)}{4} = \frac{k}{2}$$

$$\boxed{k = \frac{w_{//}(0)}{2}}$$



Appendix 2

Relationship between transverse and longitudinal forces:

The transverse gradient of the longitudinal force is equal to the longitudinal gradient of the transverse force

"**Panofsky-Wenzel theorem**".

$$\boxed{\begin{aligned} \nabla_{\perp} F_{//} &= \frac{\partial}{\partial z} \boldsymbol{F}_{\perp} \\ \nabla_{\perp} w_{//} &= \frac{\partial}{\partial z} \boldsymbol{w}_{\perp} \end{aligned}}$$

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