Exercises on Wake Fields and Instabilities

Exercise 1:

Show that the impedance of an RLC parallel circuit is that of a resonant mode and relate R, L and C to Q, R_s and ω_r

$$Z_R = R \qquad Z_C = \frac{i}{\omega C} \qquad Z_L = -i\omega L$$

$$\frac{1}{Z} = \frac{1}{R} - i\omega C + i\frac{1}{\omega L} =$$

$$\frac{\omega L - i\omega^2 LCR + iR}{R\omega L} = \frac{1}{R}\left(1 + i\left(\frac{R}{\omega L} - \omega CR\right)\right) =$$

$$\frac{1}{Z} = \frac{1}{R}\left(1 + iR\sqrt{\frac{C}{L}}\left(\frac{1}{\omega\sqrt{CL}} - \omega\sqrt{CL}\right)\right)$$

$$\left(R_s = R \qquad \omega_r = \frac{1}{\sqrt{LC}} \qquad Q = R\sqrt{\frac{C}{L}}\right)$$

$$Z(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

$$\frac{1}{Z(\omega)} = \frac{1}{R_s}\left(1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)\right)$$

Exercise 2:

Calculate the amplitude of the resonator wake field given $R_s = 1\ k\Omega$, $\omega_r = 5\ GHz$, $Q = 10^4$ $(\omega_r R_s /Q = 5*10^8\ V/C)$

Calculate the ratio $|Z(\omega_r)| / |Z(2\omega_r)|$ *for* $Q = 1, 10^3, 10^5$ $|1+i\ 3Q/2|$

$Q=1 \Rightarrow 1.8$

$Q=10^3 \Rightarrow 1.5\times10^3$

$Q=10^5 \Rightarrow 1.5\times10^5$

Exercise 3: Beam Break Up

Consider a beam in a linac at 1 GeV without acceleration. Obtain the growth of the oscillation amplitude after 3 km if:

N = 5e10, $w_\perp$(-1 mm) = 63 V/(pC m), L_w = 3.5 cm, k_y = 0.06 1/m

$$\boxed{\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_o/e)L_w} = 180}$$

To preserve the beam emittance, it is necessary to have

$$\boxed{180\times\hat{y}_2 = \left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max}\hat{y}_2 << \text{ transverse beam size}}$$

This means that the beam must be injected onto the linac axis with an accuracy better than a fraction of a per cent of the beam size, which is difficult to achieve.

Exercise 4: Beam Break Up (2)

Consider the same beam of the previous exercise being now accelerated from 1 GeV with a gradient g =16.7 MeV/m. Obtain the growth of the oscillation amplitude

$$E_f = E_0 + gL_L \approx gL_L = 50\,\mathrm{GeV}$$

$$\boxed{\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_f/e)L_w}\ln\frac{E_f}{E_0} = 14}$$

Acceleration is helpful to reduce the instability

Exercise 5: Haissinski equation with pure inductive impedance

Given the wake field in case of a pure inductive impedance, determine the longitudinal distribution

$$w_{\parallel}(z) = -c^2 L\delta'(z) \longrightarrow \Psi(z) = \frac{\omega_{s0}^2 E_0}{2\alpha_c c^2} z^2 + \frac{c^2 L e^2 N_p}{L_0} \lambda(z)$$

$$\lambda(z) = \bar{\lambda} \exp\left[-\frac{z^2}{2\sigma_{z0}^2} - \frac{c^2 L e^2 N_p}{L_0 E_0 \alpha_c \sigma_{\varepsilon 0}^2} \lambda(z) \right]$$ implicit equation in λ(z)

If we linear expand the exponential

$$\frac{\lambda(z)}{\bar{\lambda}} \cong 1 - \frac{z^2}{2\sigma_{z0}^2} - \frac{c^2 L e^2 N_p}{L_0 E_0 \alpha_c \sigma_{\varepsilon 0}^2} \lambda(z) \longrightarrow \lambda(z)\left(\frac{1}{\bar{\lambda}} + \frac{c^2 L e^2 N_p}{L_0 E_0 \alpha_c \sigma_{\varepsilon 0}^2} \right) = 1 - \frac{z^2}{2\sigma_{z0}^2}$$

$$\lambda(z) = \frac{1}{\left(\frac{1}{\bar{\lambda}} + \frac{c^2 L e^2 N_p}{L_0 E_0 \alpha_c \sigma_{\varepsilon 0}^2} \right)} \left(1 - \frac{z^2}{2\sigma_{z0}^2} \right) = \lambda(0)\left(1 - \frac{z^2}{2\sigma_{z0}^2} \right)$$

parabola

Exercise 6: Microwave instability threshold

Calculate the threshold average current of the microwave instability for an accelerator having the following parameters:

$|Z_{\|} / n| = .5\ \Omega$, $\sigma_z = 1$ cm, $\sigma_\varepsilon = 10^{-3}$, $\alpha_c = 0.027$,
$E_0 = 510$ MeV, $L_0 = 97.69$ m

$$\boxed{\frac{\hat{I}\left|Z_{\|}/n\right|}{2\pi\alpha_c\left(E_0/e\right)\sigma_\varepsilon^2} \leq 1}$$

$$\hat{I} = \frac{2\pi\alpha_c\left(E_0/e\right)\sigma_\varepsilon^2}{\left|Z_{\|}/n\right|} = 173\ \text{A}$$

$$N_p = \frac{\sqrt{2\pi}\sigma_z\hat{I}}{ce} = 9\times10^{10}$$

$$I_{average} = \frac{eN_pc}{L_0} = 44\ \text{mA}$$