# Introduction to Transverse Beam Dynamics 

Lecture 4: Dispersion / Errors in fields and gradient / Chromaticity correction

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## Re-Introducing the dispersion

So far we have studied monochromatic beams of particles, but this is slightly unrealistic: We always have some (small) momentum spread among all particles: $\Delta p=p-p_{0} \neq 0$.

Consider three particles with $p$ respectively: less than, greater than, and equal to $p_{0}$, traveling through a dipole. Remembering $B \rho=\frac{p}{q}$ :


The system introduces a correlation of momentum with transverse position. This correlation is known as dispersion (an intrinsic property of the dipole magnets).

## Dispersion function and orbit

We need to study the motion for particles with $\Delta p=p-p_{0} \neq 0$ :

$$
x^{\prime \prime}(s)+K(s) \times(s)=\frac{1}{\rho} \frac{\Delta p}{p_{0}}
$$

The general solution of this equation is:

$$
x(s)=x_{\beta}(s)+x_{D}(s) \quad\left\{\begin{array}{l}
x_{\beta}^{\prime \prime}(s)+K(s) x_{\beta}(s)=0 \\
D^{\prime \prime}(s)+K(s) D(s)=\frac{1}{\rho}
\end{array}\right.
$$

with $x_{D}(s)=D(s) \frac{\Delta p}{p_{0}}$.

## Remarks

- $D(s)$ is that special orbit that a particle would have for $\Delta p / p=1$
- $x_{D}(s)$ describes the deviation from the new closed orbit for an off-momentum particle with a certain $\Delta p$
- the orbit of a generic particle is the sum of the well known $x_{\beta}(s)$ and $x_{D}(s)$


## Understanding the solution $x(s)=x_{\beta}(s)+x_{D}(s)$

 with $x_{D}(s)=D(s) \frac{\Delta p}{p_{0}}$.(a)


Closed orbit for particles with momentum $p \neq p_{0}$ in a weakly (a) and strongly (b) focusing circular accelerator.

- $x_{D}(s)=D(s) \frac{\Delta p}{p_{0}}$ describes the deviation of the closed orbit for off-momentum particles with a fixed $\Delta p$ from the reference orbit
- $x_{\beta}(s)$ describes the betatron oscillation around this dispersive closed orbit


## Dispersion and orbit propagation

Inside a magnet, the dispersion trajectory is solution of $D^{\prime \prime}(s)+K(s) D(s)=\frac{1}{\rho}$ :

$$
D(s)=S(s) \int_{0}^{s} \frac{1}{\rho(t)} C(t) \mathrm{d} t-C(s) \int_{0}^{s} \frac{1}{\rho(t)} S(t) \mathrm{d} t
$$

Now the orbit:

$$
\begin{aligned}
& x(s)=x_{\beta}(s)+x_{D}(s) \\
& x(s)=C(s) x_{0}+S(s) x_{0}^{\prime}+D(s) \frac{\Delta p}{p}
\end{aligned}
$$

In matrix form

$$
\binom{x}{x^{\prime}}_{s}=\left(\begin{array}{cc}
C & S \\
C^{\prime} & S^{\prime}
\end{array}\right)\binom{x}{x^{\prime}}_{0}+\frac{\Delta p}{p}\binom{D}{D^{\prime}}_{0}
$$

We can rewrite the solution in matrix form:

$$
\left(\begin{array}{c}
x \\
x^{\prime} \\
\Delta p / p
\end{array}\right)_{s}=\left(\begin{array}{ccc}
C & S & D \\
C^{\prime} & S^{\prime} & D^{\prime} \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
x \\
x^{\prime} \\
\Delta p / p
\end{array}\right)_{0}
$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_{0}=D_{0}^{\prime}=0$.

## Dispersion examples

Let's study, for different magnetic elements, the solution of:

$$
D(s)=S(s) \int_{0}^{s} \frac{1}{\rho(t)} C(t) \mathrm{d} t-C(s) \int_{0}^{s} \frac{1}{\rho(t)} S(t) \mathrm{d} t
$$

at the exit of the element: that is, $D(s)$ with $s=L_{\text {magnet }}$

- Drift space:

$$
M_{\text {Drift }}=\left(\begin{array}{ll}
1 & L \\
0 & 1
\end{array}\right)
$$

$C(t)=1, S(t)=L, \rho(t)=\infty \quad \Rightarrow$ the integrals cancel

$$
M_{\text {Drift }}=\left(\begin{array}{ccc}
1 & L & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

## Dispersion function: sector dipole

- Sector dipole:

$$
K=\frac{1}{\rho^{2}}:
$$

$$
M_{\text {Dipole }}=\left(\begin{array}{cc}
\cos (\sqrt{K} L) & \frac{1}{\sqrt{K}} \sin (\sqrt{K} L) \\
-\sqrt{K} \sin (\sqrt{K} L) & \cos (\sqrt{K} L)
\end{array}\right)=\left(\begin{array}{cc}
\cos \frac{L}{\rho} & \rho \sin \frac{L}{\rho} \\
-\frac{1}{\rho} \sin \frac{L}{\rho} & \cos \frac{L}{\rho}
\end{array}\right)
$$

which gives

$$
\begin{aligned}
D(s) & =\rho\left(1-\cos \frac{L}{\rho}\right) \\
D^{\prime}(s) & =\sin \frac{L}{\rho}
\end{aligned}
$$

therefore

$$
M_{\text {Dipole }}=\left(\begin{array}{ccc}
\cos \frac{L}{\rho} & \rho \sin \frac{L}{\rho} & \rho\left(1-\cos \frac{L}{\rho}\right) \\
-\frac{1}{\rho} \sin \frac{L}{\rho} & \cos \frac{L}{\rho} & \sin \frac{L}{\rho} \\
0 & 0 & 1
\end{array}\right)
$$

## Dispersion function: quadrupole

- Focusing quadrupole, $K>0$ :

$$
M_{\mathrm{QF}}=\left(\begin{array}{ccc}
\cos (\sqrt{K} L) & \frac{1}{\sqrt{K}} \sin (\sqrt{K} L) & 0 \\
-\sqrt{K} \sin (\sqrt{K} L) & \cos (\sqrt{K} L) & 0 \\
0 & 0 & 1
\end{array}\right) ;
$$

- Defocusing quadrupole, $K<0$ :

$$
M_{\mathrm{QD}}=\left(\begin{array}{ccc}
\cosh (\sqrt{|K|} L) & \frac{1}{\sqrt{|K|}} \sinh (\sqrt{|K|} L) & 0 \\
\sqrt{|K|} \sinh (\sqrt{|K|} L) & \cosh (\sqrt{|K|} L) & 0 \\
0 & 0 & 1
\end{array}\right)
$$

## Dispersion propagation through the machine

- The equation:

$$
D(s)=S(s) \int_{0}^{s} \frac{1}{\rho(t)} C(t) \mathrm{d} t-C(s) \int_{0}^{s} \frac{1}{\rho(t)} S(t) \mathrm{d} t
$$

shows the dispersion inside a magnet, which does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length $L_{m}$ the dispersion reaches the value $D\left(L_{m}\right)$
- ...then it propagates from there on through the rest of the machine, just like any other particle:

$$
\left(\begin{array}{c}
\tilde{D} \\
\tilde{D}^{\prime} \\
1
\end{array}\right)_{s}=\left(\begin{array}{ccc}
C & S & D \\
C^{\prime} & S^{\prime} & D^{\prime} \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\tilde{D} \\
\tilde{D}^{\prime} \\
1
\end{array}\right)_{0}
$$

(to avoid confusion, $\tilde{D}$ and $\tilde{D}^{\prime}$, are often called $\eta$ and $\eta^{\prime}$ )

## Closed orbit for an off-momentum particle

In a periodic lattice, also the dispersion must be periodic.
That is, for $\left(\begin{array}{c}\eta \\ \eta^{\prime} \\ 1\end{array}\right)$ we want:

$$
\left(\begin{array}{c}
\eta \\
\eta^{\prime} \\
1
\end{array}\right)=\left(\begin{array}{ccc}
C & S & D \\
C^{\prime} & S^{\prime} & D^{\prime} \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\eta \\
\eta^{\prime} \\
1
\end{array}\right)
$$

Let's rewrite this in $2 \times 2$ form:

$$
\begin{gathered}
\binom{\eta}{\eta^{\prime}}=\left(\begin{array}{cc}
C & S \\
C^{\prime} & S^{\prime}
\end{array}\right)\binom{\eta}{\eta^{\prime}}+\binom{D}{D^{\prime}} \\
\left(\begin{array}{cc}
1-C & -S \\
-C^{\prime} & 1-S^{\prime}
\end{array}\right)\binom{\eta}{\eta^{\prime}}=\binom{D}{D^{\prime}}
\end{gathered}
$$

The solution is:

$$
\binom{\eta}{\eta^{\prime}}=\frac{1}{(1-C)\left(1-S^{\prime}\right)-C^{\prime} S}\left(\begin{array}{cc}
1-S^{\prime} & S \\
C^{\prime} & 1-C
\end{array}\right)\binom{D}{D^{\prime}}
$$

## Example of dispersion function in a FODO lattice

25 meter $180^{\circ}$ Arc based on $90^{\circ}$-FODO lattice


Aperture radius: $\mathrm{r}=15 \mathrm{~cm}$
$12 \times$ Dipoles:
15 Quads:
field: 3.9 Tesla
gradient: 25 Tesla/m (3.8 Tesla at the pole)
length: 85 cm
length: 50 cm

Dispersion in a FODO cell: $D^{ \pm}=\frac{L \phi\left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin ^{2} \frac{\mu}{2}}$. The dispersion has a maximum at the focusing quadrupoles.

## Example of dispersion function

In this example from the HERA storage ring (DESY) we see the twiss parameters and the dispersion near the interaction point. In the periodic region,

$$
\begin{aligned}
x_{\beta}(s) & =1 \ldots 2 \mathrm{~mm} \\
D(s) & =1 \ldots 2 \mathrm{~m} \\
\Delta p / p & \approx 1 \cdot 10^{-3}
\end{aligned}
$$

Remember:

$$
x(s)=x_{\beta}(s)+D(s) \frac{\Delta p}{p}
$$



Beware: the dispersion contributes to the beam size:

$$
\sigma_{x}=\sqrt{\sigma_{x_{\beta}}^{2}+\left(D \cdot \frac{\Delta p}{p}\right)^{2}}=\sqrt{\epsilon \beta+\left(D \cdot \frac{\Delta p}{p}\right)^{2}}
$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a dispersion suppressor


## The momentum compaction factor

The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$
x(s)=x_{\beta}(s)+D(s) \frac{\Delta p}{p}
$$

The dispersion changes also the length of the offenergy orbit.
 particle with offset $\times$ w.r.t. the design orbit:

$$
\mathrm{d} s^{\prime}=\mathrm{d} s\left(1+\frac{x}{\rho}\right) \quad \frac{\mathrm{d} s^{\prime}}{\mathrm{d} s}=\frac{\rho+x}{\rho} \quad \rightarrow \quad \mathrm{~d} s^{\prime}=\left(1+\frac{x}{\rho}\right) \mathrm{d} s
$$

The circumference change is $\Delta C$, that is $C^{\prime}=\oint\left(1+\frac{x}{\rho}\right) \mathrm{d} s=C+\Delta C$
We define the "momentum compaction factor", $\alpha$, such that:

$$
\frac{\Delta C}{C}=\alpha \frac{\Delta p}{p} \quad \rightarrow \text { a rough estimate is } \alpha=\frac{1}{Q_{x}^{2}}
$$

## Magnetic imperfections

HIgh order multipolar components and misalignments
Taylor expansion of the $B$ field:

$$
B_{y}(x)=\underbrace{B_{y 0}}_{\text {dipole }}+\underbrace{\frac{\partial B_{y}}{\partial x}}_{\text {quad }} x+\frac{1}{2} \underbrace{\frac{\partial^{2} B_{y}}{\partial x^{2}}}_{\text {sextupole }} x^{2}+\frac{1}{3!} \underbrace{\frac{\partial^{3} B_{y}}{\partial x^{3}}}_{\text {octupole }} x^{3}+\ldots \quad \text { divide by } B_{y 0}
$$

There can be undesired multipolar compo-
 nents, due to small fabrication defects

Or also errors in the windings, in the gap $h$, ... remember: $B=\frac{\mu_{0} n l}{h}$


Moreover: "feed-down" effect $\Rightarrow$ a misalign magnet of order $n$, behaves like a magnet of order $n+$ a magnet of order $n-1$ overlapped

## Dipole magnet errors

Let's imagine to have a magnet with $B_{x}=B_{0}+\Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$
F_{x}=e v\left(B_{0}+\Delta B\right) ; \quad \Delta x^{\prime}=\Delta B \mathrm{~d} s / B \rho
$$

A dipole error will cause a distortion of the closed orbit, that will ,,run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

Example: 1 single dipole error

$$
\binom{x}{x^{\prime}}_{s}=M_{\text {lattice }}\binom{0}{\Delta x^{\prime}}_{0}
$$



In order to have bounded motion the tune $Q$ must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum $p_{0}$ an integer $Q$ value is forbidden, since small field errors are always present.

## Orbit distortion for a single dipole field error



We consider a single thin dipole field error at the location $s=s_{0}$, with a kick angle $\Delta x^{\prime}$.

$$
X_{-}=\binom{x_{0}}{x_{0}^{\prime}+\Delta x^{\prime}}, \quad X_{+}=\binom{x_{0}}{x_{0}^{\prime}}
$$

are the phase space coordinates before and after the kick located at $s_{0}$. The closed-orbit condition becomes

$$
M_{\text {Lattice }}\binom{x_{0}}{x_{0}^{\prime}}=\binom{x_{0}}{x_{0}^{\prime}+\Delta x^{\prime}}
$$

The resulting closed orbit at $s_{0}$ is

$$
x_{0}=\frac{\beta_{0} \Delta x^{\prime}}{2 \sin \pi Q} \cos \pi Q ; \quad x_{0}^{\prime}=\frac{\Delta x^{\prime}}{2 \sin \pi Q}\left(\sin \pi Q-\alpha_{0} \cos \pi Q\right)
$$

where $Q$ is the tune. The orbit at any other location $s$ is

$$
x(s)=\frac{\sqrt{\beta_{s} \beta_{0}}}{2 \sin \pi Q} \cos \left(\pi Q-\left|\mu_{s}-\mu_{0}\right|\right) \Delta x^{\prime}
$$

(see the references for a demonstration)

## Orbit distortion for distributed dipole field errors

One single dipole field error

$$
x(s)=\frac{\sqrt{\beta(s) \beta(0)}}{2 \sin \pi Q} \cos (\pi Q-|\mu(s)-\mu(0)|) \Delta x^{\prime}
$$

Distributed dipole field errors

$$
x(s)=\frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \oint \sqrt{\beta(t)} \cos (\pi Q-|\mu(s)-\mu(t)|) \Delta x^{\prime} \mathrm{d} t
$$

- orbit distortion is visible at any position $s$ in the ring, even if the dipole error is located at one single point $s_{0}$
- the $\beta$ function describes the sensitivity of the beam to external fields
- the $\beta$ function acts as amplification factor for the orbit amplitude at the given observation point
- there is a singularity at the denominator when $Q$ integer $\Rightarrow$ it's called resonance


## The resonances

Closed orbit distortion due to distributed dipole field errors:

$$
x(s)=\frac{\sqrt{\beta(s)}}{2 \sin \pi Q} \oint \sqrt{\beta(t)} \cos (\pi Q-|\mu(s)-\mu(t)|) \Delta x^{\prime} \mathrm{d} t
$$

Remember the definition of tune:

$$
Q=\frac{\mu_{L}}{2 \pi}
$$

$\Rightarrow$ it is the phase advance for a revolution $\mu_{L}$ in units of $2 \pi$.
Extremely important:

- In case of imperfections the orbit becomes unstable for $Q$ integer
- Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error!


## Tunes and resonances

The particles - oscillating under the influence of the external magnetic fields - can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.


The cure:

1. avoid large magnet errors
2. avoid forbidden tune values in both planes

$$
\mathrm{m} \cdot Q_{x}+\mathrm{n} \cdot Q_{y} \neq \mathrm{p}
$$

with $m, n, p$ integer numbers

## Resonance diagram



A resonance diagram for the Diamond light source. The lines shown are the resonances and the black dot shows a suitable place where the machine could be operated.

## Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$
M_{\text {Perturbed }}=\underbrace{\left(\begin{array}{cc}
1 & 0 \\
\Delta k \mathrm{~d} s & 1
\end{array}\right)}_{\text {perturbation }} \underbrace{\left(\begin{array}{cc}
\cos \mu_{0}+\alpha \sin \mu_{0} & \beta \sin \mu_{0} \\
-\gamma \sin \mu_{0} & \cos \mu_{0}-\alpha \sin \mu_{0}
\end{array}\right)}_{\text {ideal ring }}
$$

Let's see how the tunes changes: one-turn map

$$
M_{\text {Perturbed }}=\left(\begin{array}{cc}
\cos \mu_{0}+\alpha \sin \mu_{0} & \beta \sin \mu_{0} \\
\Delta k \mathrm{~d} s\left(\cos \mu_{0}+\alpha \sin \mu_{0}\right)-\gamma \sin \mu_{0} & \Delta k \mathrm{~d} \boldsymbol{\beta} \beta \sin \mu_{0}+\cos \mu_{0}-\alpha \sin \mu_{0}
\end{array}\right)
$$

Remember the rule for computing the tune:

$$
2 \cos \mu=\operatorname{trace}(M)=2 \cos \mu_{0}+\Delta k \mathrm{~d} \boldsymbol{s} \beta \sin \mu_{0}
$$

## Quadrupole errors: tune shift (cont.)

We rewrite $\cos \mu=\cos \left(\mu_{0}+\Delta \mu\right)$

$$
\cos \left(\mu_{0}+\Delta \mu\right)=\cos \mu_{0}+\frac{1}{2} \Delta k \mathrm{~d} s \beta \sin \mu_{0}
$$

from which we can compute that

$$
\begin{aligned}
& \Delta \mu=\frac{\Delta k \mathrm{~d} s \beta}{2} \text { shift in the phase advance } \\
& \Delta Q=\int_{s_{0}}^{s_{0}+L} \frac{\Delta k(s) \beta(s) \mathrm{d} s}{4 \pi} \text { tune shift }
\end{aligned}
$$

Important remarks:

- the tune shift if proportional to the $\beta$-function at the quadrupole
- field quality, power supply tolerances etc. are much tighter at places where $\beta$ is large
- $\beta$ is a measurement of the sensitivity of the beam


## Quadrupole errors: tune shift example

Deliberate change of a quadrupole strength in a synchrotron:

$$
\Delta Q=\int_{s_{0}}^{s_{0}+L} \frac{\Delta K(s) \beta(s) \mathrm{d} s}{4 \pi} \approx \frac{\Delta K(s) L_{\text {quad }} \bar{\beta}}{4 \pi}
$$


the tune is measured permanently


After changing the strength of a quad: we get a second peak

G106 NR


## Quadrupole errors: beta beat

A quadrupole error at $s_{0}$ causes distortion of $\beta$-function at $s: \Delta \beta(s)$ due to the errors of all quadrupoles:

$$
\frac{\Delta \beta(s)}{\beta(s)}=\frac{1}{2 \sin 2 \pi Q} \oint \beta(t) \Delta k(t) \cos (2 \pi Q-2(\mu(t)-\mu(s))) \mathrm{d} t
$$

affects the element $M_{12}$ of the $M$ matrix.


## Quadrupole errors: chromaticity, $\xi$

Is an error (optical aberration) that happens in quadrupoles when $\Delta p / p \neq 0$ :


The chromaticity $\xi$ is the variation of tune $\Delta Q$ with the relative momentum error:

$$
\Delta Q=\xi \frac{\Delta p}{p_{0}} \quad \Rightarrow \quad \xi=\frac{\Delta Q}{\Delta p / p}
$$

Remember the quadrupole strength:

$$
k=\frac{g}{p / e} \quad \text { with } p=p_{0}+\Delta p
$$

then

$$
\begin{gathered}
k=\frac{e g}{p_{0}+\Delta p} \approx \frac{e}{p_{0}}\left(1-\frac{\Delta p}{p_{0}}\right) g=k_{0}+\Delta k \\
\Delta k=-\frac{\Delta p}{p_{0}} k_{0}
\end{gathered}
$$

## Quadrupole errors: chromaticity (cont.)

$$
\Delta k=-\frac{\Delta p}{p_{0}} k_{0}
$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a tune spread:

$$
\Delta Q_{\text {one quad }}=-\frac{1}{4 \pi} \frac{\Delta p}{p_{0}} k_{0} \beta(s) \mathrm{d} s \quad \Rightarrow \Delta Q_{\text {all quads }}=-\frac{1}{4 \pi} \frac{\Delta p}{p_{0}} \oint k(s) \beta(s) \mathrm{d} s
$$

Therefore the definition of chromaticity $\xi$ is

$$
\xi=-\frac{1}{4 \pi} \oint_{\text {quads }} k(s) \beta(s) \mathrm{d} s
$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

Remarks:

- $\xi$ is a number indicating the size of the tune spot in the working diagram
- $\xi$ is always created by the focusing strength $k$ of all quadrupoles In other words, because of chromaticity the tune is not a point, but it is pancake


## Example: Chromaticity of the FODO cell

Consider a ring composed by $N_{\text {cell }}$ FODO cells like in figure, with two thin quads separated by length $L / 2$,


The natural chromaticity $\xi_{N}$ for the $N_{\text {cell }}$ cells is:

$$
\begin{array}{rlrl}
\xi_{N} & =-\frac{1}{4 \pi} \oint \beta(s) k(s) d s & & =-\frac{1}{4 \pi \sin \mu} N_{\text {cell }}\left[\left(L+\frac{L^{2}}{4 f_{D}}\right) \frac{1}{f_{F}}-\left(L-\frac{L^{2}}{4 f_{F}}\right) \frac{1}{f_{D}}\right] \\
& =-\frac{1}{4 \pi} N_{\text {cell }} \int_{\text {cell }} \beta(s) \underbrace{k(s) d}_{\frac{1}{f}} & =-\frac{1}{4 \pi \sin \mu} N_{\text {cell }}\left[\frac{L}{f_{F}}-\frac{L}{f_{D}}+\frac{L^{2}}{2 f_{F} f_{D}}\right] \\
& \left.=-\frac{1}{4 \pi} N_{\text {cell }}\left[\beta^{+}\left(\frac{1}{f_{F}}\right)-\beta^{-}\left(\frac{1}{f_{D}}\right)\right] \right\rvert\, & \simeq-\frac{1}{8 \pi \sin \mu} N_{\text {cell }} \frac{L^{2}}{f_{F} f_{D}}
\end{array}
$$

The chromaticity of the ring is $N_{\text {cell }}$ times the chromaticity of each cell.

## Quadrupole errors: chromaticity



Tune signal for a nearly uncompensated cromaticity ( $Q^{\prime}$ * 20)

Ideal situation: cromaticity well corrected, ( $Q^{\prime}$ ~1)

## Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$
\Delta Q=\oint_{\text {quads }} \frac{\Delta k(s) \beta(s) \mathrm{d} s}{4 \pi}
$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1 / f=\Delta k_{1} L$
- it changes both $Q_{x}$ and $Q_{y}$ :

$$
\Delta Q_{x}=\frac{\beta_{1 x}}{4 \pi f_{1}} \quad \text { and } \quad \Delta Q_{y}=-\frac{\beta_{1 y}}{4 \pi f_{1}}
$$

- We need to use two independent correcting quadrupoles:

$$
\begin{aligned}
& \Delta Q_{x}=\frac{\beta_{1 x}}{4 \pi f_{1}}+\frac{\beta_{2 x}}{4 \pi f_{2}} \\
& \Delta Q_{y}=-\frac{\beta_{1 y}}{4 \pi f_{1}}-\frac{\beta_{2 y}}{4 \pi f_{2}}
\end{aligned} \quad\binom{\Delta Q_{x}}{\Delta Q_{y}}=\frac{1}{4 \pi}\left(\begin{array}{ll}
\beta_{1 x} & \beta_{2 x} \\
\beta_{1 y} & \beta_{2 y}
\end{array}\right)\binom{1 / f_{1}}{1 / f_{2}}
$$

- Solve by inversion:

$$
\binom{1 / f_{1}}{1 / f_{2}}=\frac{4 \pi}{\beta_{1 x} \beta_{2 y}-\beta_{2 x} \beta_{1 y}}\left(\begin{array}{cc}
\beta_{2 y} & -\beta_{2 x} \\
-\beta_{1 y} & \beta_{1 x}
\end{array}\right)\binom{\Delta Q_{x}}{\Delta Q_{y}}
$$

## Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances Cure: we need additional energy dependent focusing. This is given by sextupoles

- The sextupole magnetic field rises quadratically:

$$
\begin{aligned}
& B_{x}=\tilde{g} x y \\
& B_{y}=\frac{1}{2} \tilde{g}\left(x^{2}-y^{2}\right)
\end{aligned} \quad \Rightarrow \frac{\partial B_{x}}{\partial y}=\frac{\partial B_{y}}{\partial x}=\tilde{g} x \quad \text { a "gradient" }
$$

it provides a linearly increasing quadrupole gradient

## Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$
k=\frac{g}{p / e}\left[m^{-2}\right]
$$

- Sextupoles are characterised by a normalised sextupole strength $k_{2}$, which carries a focusing quadrupolar component $k_{1}$ :

$$
k_{2}=\frac{\tilde{g}}{p / e}\left[\mathrm{~m}^{-3}\right] ; \quad \tilde{k}_{1}=\frac{\tilde{g} x}{p / e}\left[\mathrm{~m}^{-2}\right]
$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x \rightarrow x+D \cdot \frac{\Delta p}{p}$

$$
\tilde{k}_{1}=\frac{\tilde{g}\left(x+D \frac{\Delta p}{p}\right)}{p / e}\left[\mathrm{~m}^{-2}\right]
$$

- for $x=0$ it corresponds to an energy-dependent focal length

$$
\frac{1}{f_{\text {sext }}}=\tilde{k}_{1} L_{\text {sext }}=\overbrace{\underbrace{\frac{\tilde{g}}{p / e}}_{k_{2}} D \frac{\tilde{k}_{1}}{\tilde{k}_{1}} \frac{\Delta p}{p}} \cdot L_{\text {sext }}=k_{2} D \cdot \frac{\Delta p}{p} \cdot L_{\text {sext }}
$$

Now the formula for the chromaticity rewrites:

$$
\xi=\underbrace{-\frac{1}{4 \pi} \oint k(s) \beta(s) \mathrm{d} s}_{\text {chromaticity due to quadrupoles }}+\underbrace{\frac{1}{4 \pi} \oint k_{2}(s) D \beta(s) \mathrm{d} s}_{\text {chromaticity due to sextupoles }}
$$

## Design rules for sextupole scheme

- Chromatic aberrations must be corrected in both planes $\Rightarrow$ you need at least two sextupoles
- In each plane the sextupole fields contribute with different signs to the chromaticity $\xi_{x}$ and $\xi_{y}$ :

$$
\begin{aligned}
& \xi_{x}=-\frac{1}{4 \pi} \oint\left[k-S_{F} D_{x}+S_{D} D_{x}\right] \beta_{x}(s) \mathrm{d} s \\
& \xi_{y}=\frac{1}{4 \pi} \oint\left[k-S_{F} D_{x}+S_{D} D_{x}\right] \beta_{y}(s) \mathrm{d} s
\end{aligned}
$$

- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_{x} D_{x}$ and $\beta_{y} D_{x}$ are maximum.
- Important remark: for offset orbits, the sextupoles introduce geometric aberrations. They can be reduced by adopting a - I transformation scheme:
- put sextupoles in $(2 n+1) \pi$ phase advance apart in a periodic lattice to compensate the effect


## Summary

orbit for an off-momentum particle $\quad x(s)=x_{\beta}(s)+D(s) \frac{\Delta p}{p}$
dispersion trajectory $\quad D(s)=S(s) \int_{0}^{s} \frac{1}{\rho(t)} C(t) \mathrm{d} t-C(s) \int_{0}^{s} \frac{1}{\rho(t)} S(t) \mathrm{d} t$
equations of motion with dispersion $\left(\begin{array}{c}x \\ x^{\prime} \\ \Delta p / p\end{array}\right)_{s}=\left(\begin{array}{ccc}C & S & D \\ C^{\prime} & S^{\prime} & D^{\prime} \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{c}x \\ x^{\prime} \\ \Delta p / p\end{array}\right)_{0}$
definition of momentum compaction $\quad \frac{\Delta C}{C}=\alpha \frac{\Delta p}{p}$
stability condition $\quad m \cdot Q_{x}+n \cdot Q_{y} \neq p \quad$ with $n, m, p$ integers

$$
\text { tune shift } \quad \Delta Q=\frac{1}{4 \pi} \oint_{\text {quads }} \Delta k(s) \beta(s) \mathrm{d} s
$$

beta beat

$$
\begin{aligned}
\frac{\Delta \beta(s)}{\beta(s)}= & \frac{1}{2 \sin 2 \pi Q} \\
& \oint \beta(t) \Delta k(t) \cos (2 \pi Q-2(\mu(t)-\mu(s))) d t
\end{aligned}
$$

$$
\text { chromaticity } \quad \xi=\frac{\Delta Q}{\Delta P / p}=-\frac{1}{4 \pi} \oint_{\text {quads }} k(s)^{\prime} \beta(s) \mathrm{d} s
$$

## References

Derivation of the equation of the orbit distortion for a dipole field errors:

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